Problems requiring Short Answers

1. Ace must receive at least $20 + (\frac{500-20}{2} + 1) = 261$ of the remaining 500 votes. **Answer: 261**

2. When Peter runs 5 laps, Steven runs 3 laps for a difference of 2 laps. For a difference of 4 laps, Peter must run 10 laps and Steven 6. **Answer: 6**

3. In order for $n + 13$ to be a multiple of 5, we must have $n = 2, 7, 12, \ldots$. Now $2 - 13$ is not divisible by 6, and while $7 - 13$ is, it is not a positive multiple of 6. The least common multiple of 5 and 6 is 30, and 30+7=37 is the desired minimum. **Answer: 37.**

4. We have $\frac{w}{z} = \frac{5^3}{8^3}$. Hence the smallest positive integers satisfying the given relation are $w = 5^3$, $x = 5^2 \times 8$, $y = 5 \times 8^2$ and $z = 8^3$, yielding $w + x + y + z = 1157$. **Answer: 1157**

5. The last digit of $1^3 - 2^3 + \cdots - 10^3$ is the same as that of $11^3 - 12^3 + \cdots - 20^3$, and so on. It follows that the last digit of $1^3 - 2^3 + \cdots - 2000^3$ must be 0, so that the last digit of $1^3 - 2^3 + 3^3 - \cdots + 2001^3$ is 1. **Answer: 1**

6. Let v be the amount of viruses produced per minute and a be the amount of viruses destroyed by each antivirus per minute. Then $40(2a - v) = 16(4a - v)$, which simplifies to $\frac{a}{v} = \frac{3}{2}$. Let n be the desired number of antiviruses. Then $40(2a - v) = 10(na - v)$ which is equivalent to $8v = (\frac{3n}{2} - 1)v$. Hence $n = 6$. **Answer: 6**

7. In order for the four-digit number to be divisible by 11, the sum of the first and the third digits must be 5. They may either be 1 and 4 or 2 and 3. Moreover, each pair can be permuted internally, yielding a total of $2 \times 2 \times 2 = 8$ such numbers. **Answer: 8**

8. We have $217 = (m - n)^2 + 13n^2$. If $n \geq 5$, then $(m - n)^2$ will be negative, which is impossible. If $n = 1$, then $(m - n)^2 = 204$ has no integral solutions. Hence $n = 3$ so that $(m - 3)^2 = 100$, which yields $m = 13$. **Answer: 13**

9. We have $xy = \frac{1}{(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})} = 1$ while $x + y = \frac{\sqrt{3}-\sqrt{2}+\sqrt{3}+\sqrt{2}}{(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})} = 2\sqrt{3}$. It follows that $x^2 + y^2 = (x + y)^2 - 2xy = 10$. **Answer: 10**

10. We have $x^4 + x^3 + xy^3 + y^4 = x^3(x + y) + y^3(x + y) = (x + y)^2(x^2 - xy + y^2) = 36$. **Answer: 36**

11. The area of triangle GAD is $\frac{1}{4} \times 24 = 6$. The area of triangle AGE is $\frac{1}{2} \times \frac{1}{4} \times 24 = 3$. The area of triangle BEF is $\frac{1}{2} \times \frac{3}{4} \times \frac{1}{3} \times 24 = 3$. The area of triangle CFG is $\frac{1}{2} \times \frac{1}{2} \times \frac{2}{3} \times 24 = 4$. Hence the area of triangle EFG is $24 - 6 - 3 - 3 - 4 = 8$. **Answer: 8**

12. Let BC intersect DE at H. We have $\angle BGE = 75°$ and $75° + \angle E = \angle CHD = 180° - \angle C - \angle D$. Hence $\angle C + \angle D + \angle E = 105°$. Similarly, $\angle A + \angle B + \angle F = 105°$, so that the sum of these six angles is $210°$. **Answer:** $210°$

13. By symmetry, $AP = CP$. By Pythagoras' Theorem, $CP^2 = AP^2 = AD^2 + PD^2 = 16 + (8 - CP)^2$. This yields $CP = 5$. Hence the area of triangle CAP is $\frac{1}{2} CP \cdot AD = 10$. **Answer: 10**

14. The ratio of the areas of triangles BPD and CPD is 2:3. By symmetry, triangles CPD and CPE have the same area. Hence the ratio of the areas of triangles PCB and PCE is 5:3. Since the area of triangle ABC is $\frac{25}{2}$, the area of triangle BAP is $\frac{25}{2} \times \frac{2}{5} \times \frac{5}{8} = \frac{25}{8}$. **Answer:** $\frac{25}{8}$

15. Let the extensions of DM and CB meet at N, and let the foot of perpendicular from D to BC be P. Since M is the midpoint of AB, we have $BN = AD = 2$. Also, $CP = \frac{BC-AD}{2} = 3$. Note that triangles CDP and DNP are similar to each other, so that $\frac{CP}{DP} = \frac{DP}{NP}$. This yields $DP = \sqrt{21}$ so that the area of $ABCD$ is $\frac{2+8}{2}\sqrt{21} = 5\sqrt{21}$. **Answer:** $5\sqrt{21}$