Making the Transition to Algebraic Thinking: Taking Students' Arithmetic Modes of Reasoning into Account

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The purpose of this article is to trigger reflection and discussion on the transition from arithmetic to algebraic problem solving and its teaching. When students are introduced to algebraic problem solving in their first years of secondary schooling, they have already acquired arithmetic procedures, experiences and tools. These arithmetic modes of reasoning significantly differ from the ones we teachers are expected to teach in algebra. Students arrive with at least seven years of arithmetic operations and problem solving. These procedures and ways of doing mathematics are rooted in operations on known quantities or givens, whereas algebra requires operations on unknown quantities.

The conceptual step of accepting and understanding what it means to operate on unknown values in the same way that we operate on known and given values was an important historical difficulty for mathematicians as well. It should not then be a surprise to see students experiencing difficulties in that domain. Therefore, the transitional step to algebraic thinking is one of the most difficult steps experienced in a student's mathematical life[1].

To ease this transition, teachers must be sensitized to students' arithmetic procedures for solving problems and must consider these ways of thinking in teaching. To set aside all students' prior knowledge construed in the elementary years of mathematics schooling would be nonsense.

My argument underlies this conceptual umbrella. I intend to raise sensibility toward prealgebraic students' ways of solving problems to make better sense of (1) students' skills and knowledge with which they enter introduction-to-algebra classrooms, and (2) how these strategies can be accounted for in teaching to ease this important transition in school mathematics.

With this in mind, I will offer some traditional algebraic problems and how students with no background in algebraic problem solving make sense of and solve these problems. With these solutions in hand, one intent will be (1) to see similarities and differences between arithmetic and algebraic ways of solving, (2) to see possible usage and avenues these similarities and differences give to ease the transition, and (3) to realize the strength and the limits of these arithmetic solutions to better understand how to promote the power and relevance of algebraic reasoning to solve problems.

Teaching Algebra or Solving Problems with Algebra

The traditional algebraic word problems that I will offer represent what is normally given in the introduction to algebraic problem solving in junior high. However, as will be shown later, these problems are not algebraic in themselves, because they can be and are solved without using algebra.

This is no small point, because it flags the purpose of algebraic problem solving in school mathematics. Algebra represents a tool to solve problems as much as geometric or arithmetic skills do. Seeing algebra as a problem-solving tool brings us to question deeply our assumptions about algebra. Algebra, as powerful as it is for solving particular word problems, should not be seen as an end in itself; solving the word problems represents the end in itself. When we want students to solve a problem, the fact that they use different or nonalgebraic methods and strategies should not be seen as problematic. The goal is to solve the word problem and not simply to use a specific predetermined strategy. In other words, imposing on and demanding that students only use algebra to solve word problems is nonsense, because algebra becomes the goal of instruction and solving word problems becomes secondary. This is important because algebra has become so prominent in the school curriculum at the secondary level that it is almost seen as a subject in itself, not as a mathematical tool invented to solve problems[2]. I am not saying that algebra is not important; however, the status and utility of algebra in school mathematics must be understood.

In fact, seeing algebra as a powerful problem-solving tool makes it more relevant in the school mathematics curriculum. This perception enables teachers to present and offer algebraic thinking and solving to students as a powerful problem-solving tool, which permits the solving of problems that other methods cannot solve.

By showing the unparalleled and unmatched capacity of the algebraic tool to solve certain types of problems, algebra becomes relevant and makes sense as a tool. If algebra is imposed in places where other methods are as efficient, as fast and even more economical, algebra loses its significance and becomes an unnecessary action or even a burden on the learner.

The Problems and an Analysis of Their Solutions

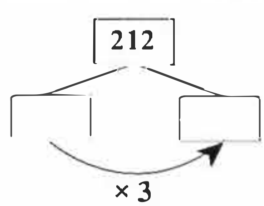
To introduce the ideas, I will first present some problems and their underlying structure[3]. To some extent, these types of problems are offered when algebra is introduced. For each problem, a possible algebraic solution will be offered, followed by some students' possible arithmetic solutions.

Problem 1

The first problem contains a multiplicative structure:

The school cafeteria offered two different meals at lunch. Three times more hamburgers than pizzas were served. If 212 meals in total were served, how many hamburgers and how many pizzas were served?

Structure of problem 1



A possible algebraic solution for this problem follows:

Algebraic solution

x = number of pizzas, $3x$ = number of hamburgers

$$3x + x = 212$$

$$4x = 212$$

$$\frac{4x}{4} = \frac{212}{4}$$

$$x = 53$$

$$3 \times 53 = 159$$

→ 53 pizzas and 159 hamburgers

This solution represents an expected algebraic solution in school mathematics. Interestingly, many students could answer that problem without knowing algebra. For example, three prominent *adequate*[4] arithmetic solutions could be the guess-and-check number trial, whereby students randomly try answers until they reach a possible result. Then they keep trying other answers until the problem is solved.

Another type of arithmetic answer is the *control* solution, whereby students methodically try numbers and constantly consider the relation between the data; for example, "one is three times more than the other one," and the total number of meals is 212. The following example of a table is often drawn by students using this strategy:

Control solution

× 3

30	90	= 120
40	120	= 160
50	150	= 200
55	165	= 220
52	156	= 208
53	159	= 212

Finally, another possible answer is the *structure* solution, whereby students work with the relations between the parts and the data of the problem. For example:

Structure solution

212 ÷ 4 = 53 [÷ 4 because I count 3 times more hamburgers and 1 times the pizzas]

53 × 3 = 159 [3 times the number of pizzas = the number of hamburgers]

53 + 159 = 212

→ 53 pizzas and 159 hamburgers

In this strategy, students see a little part, the pizza, and see this same little part repeated three times for the hamburgers (three times the number of pizzas = the number of hamburgers). Students see four parts for the whole problem and for the number of meals served in total (three for the hamburgers and one for the pizzas). So, the students divide the total number of meals into four parts (212 ÷ 4 = 53). The value obtained for this part represents the number of pizza meals, so they multiply it by three to obtain the number of hamburgers served (53 × 3 = 159).

Reflecting on These Arithmetic Strategies

What stands out in each of these last two strategies[5] (control and structure) is the students' demonstration of control over the data. In fact, Bednarz and Janvier's (1996) research shows that this skill, controlling the relations between the data in the problem, is of major importance to solving problems in algebra. Each solution has its strength for two important parts of algebraic solving: (1) the creation of an algebraic equation, and (2) the algebraic operations to find (isolate) the value of x in the problem. The control solution shows how this student is comfortable with the relation between data (one for three) and the knowledge that the total must stay at 212. Even if this seems obvious or easy, this is exactly what is needed to create the algebraic equation $3x + x = 212$. This solution shows a double-control: the control over the relation between the givens (three times the other) and the fact that the addition of both gives 212. Thus, inherent understandings are present in the control solution that could be worked on and used to help students create and understand the algebraic equation, which is often the hardest part of solving algebraic word problems.

The similarities between the structure solution and the expected algebraic solution are important regarding the operations to arrive at a value for x. Each algebraic step is mimicked by each arithmetic step (or should we say the opposite?). In the algebraic solution, the student "adds its x" and obtains $4x$. In the arithmetic solution, the student adds its parts: "4 because I count 3 times more hamburgers and 1 time the pizzas." Afterward, the algebraic student divides 212 by 4 to obtain the value of x, whereas the arithmetic student divides its total by 4 to obtain the value of one part. After obtaining 53, both students find the quantity of hamburgers by reapplying the relation that links both data (three times more). The link between both solutions is strong and represents important insights that show some possible links (even if there are obviously some differences) between the students' arithmetic solution and the algebraic solution expected.

It must be emphasized that these solutions are different. The major difference resides in the presence of the context. In the arithmetic solutions, the presence of the context is present at each step. The operations are made on hamburgers and pizzas and on the number of meals. In the algebraic solution, the operations conducted to arrive at a value of x are made in a decontextualized fashion; that is, at a mechanical level. Isolating the x does not require the solver to keep a grasp on the context (that is, on the meals). It is a set of procedures and steps to arrive at isolating x. This represents a major difference between an algebraic and an arithmetic solution. Although the link to the context is strong in an arithmetic solution, it is unnecessary (and sometimes does not even make sense) in the algebraic solution. There is a need to step out of the context in an algebraic solution to conduct the operations. In that sense, even though arithmetic solutions seem similar to algebraic ones, they are conceptually different. However, despite this conceptual difference, the similarities highlighted hint at some important insights into how to ease the transition from one to the other.

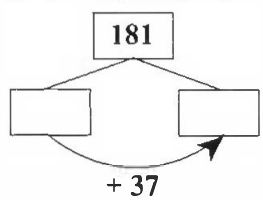
With these differences and similarities in mind, I will present two more problems and possible solutions: one with an additive structure and the other with a combination additive and multiplicative structure. After an analysis of the solutions, I will introduce an approach based on the insights drawn from the first three problems offered.

Problem 2

This problem contains an additive structure:

Two children have a stamp collection. Alex has 37 more stamps than Josie. If they have 181 stamps altogether, how many stamps do they each have?

Structure of problem 2



A possible algebraic solution for this problem could be:

Algebraic solution

x = number of stamps of Josie, $x + 37$ = number of stamps of Alex

$x + x + 37 = 181$

$2x + 37 = 181$

$2x = 144$

$x = 72$

$72 + 37 = 109$

→72 for Jose and109 for Alex

In the same line of thought as the previous problem, an example of a control solution could be:

Control solution

+ 37

50		87	= 137
60		97	= 157
70		107	= 177
71		108	= 179
72		119	= 181

Here, the student controls the relation between the givens by knowing simultaneously that they must always have a difference of 37 between them (or that the second one has 37 more) and that their addition gives 181. These two relations (37 more and 181 as a total) are regarded throughout the whole solving process.

Here is what a possible structure solution would look like:

Structure solution

181 – 37 = 144 [Alex has 37 more]

144 ÷ 2 = 72 [they now have the same amount so I can divide in two]

72 + 37 = 109

→ 72 for Jose and 109 for Alex

In this structure solution, the student sees two quantities: Josie's and Alex's. Because the student knows that the two quantities are not equivalent (Alex has 37 more), the student reorganizes the problem by taking out the quantity (the surplus) (181 – 37 = 144). By subtracting the surplus, the student obtains two equivalent quantities. The student's new amount represents the total that Josie and Alex would have if they had the same amount. Then, the student divides the result into two parts (144 ÷ 2 = 72). This new quantity (72) represents the number of Josie's stamps. The student then adds 37 to 72 to obtain Alex's number of stamps (72 + 37 = 109).

Reflecting on These Arithmetic Strategies

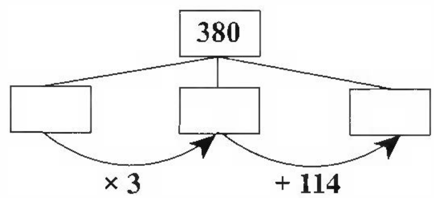
Again, the possible links and similarities between the control and structure solutions are worth mentioning. As I underlined before, the control solution, with its control over the relations between the data (the +37 and the total of 181), hints very well at an understanding of the algebraic equation ($x + x + 37 = 181$). As for the structure solution, its steps toward arriving at Josie's number of stamps (subtracting 37, dividing by 2) are representative of the operations needed to isolate the x. This arithmetic solution again hints directly at the processes of algebraic problem solving, because it enables us to create links and make sense of (1) the creation of the algebraic equation, and (2) the steps of algebraic operations. The last example is a multiplicative and additive structure.

Problem 3

Here is a third problem and its structure:

380 students are registered in three sports activities offered during the semester. Basketball has 3 times more students than skating, and swimming has 114 more students than basketball. How many students are registered in each activity?

Structure of problem 3



A possible algebraic solution would be:

Algebraic solution

x = number of students registered in skating,

$3x$ = number of students registered in basketball,

$3x + 114$ = number of students registered in swimming.

$x + 3x + 3x + 114 = 380$

$7x + 114 = 380$

$7x = 266$

$x = 38$

$3 \times 38 = 114$

$3 \times 38 + 114 = 228$

→ 38 students are registered in skating, 114 in basketball and 228 in swimming

Now, with three unknown quantities to consider, the strategies of control and structure become more complicated, but they still follow the same thinking as before. In the control solution, one difference is that because of the three unknowns, more control can be exerted on the whole problem as the following solution shows:

Control solution

	×3 →		+114 →	
50		150	264	= 464
40		120	234	= 394
35		105	219	= 359
37		111	225	= 373
38		114	228	= 380

In effect, the student needs to consider that basketball has 3 times more students than skating, and swimming has 114 more students than basketball. After considering these relations, the student needs to control the fact that these three numbers added together total 380. In the previous problems, there are only two relations to consider, but here there are three (3 times more, +114 and a total of 380).

Here is an example of a structure solution:

Structure solution

To make swimming and basketball equivalent:
380 – 114 = 266

For the skating: 266 ÷ (3 + 3 + 1) = 38 [there are now seven parts]
Basketball: 38 × 3 = 114
Swimming: 114 + 114 = 228

→ 38 students are registered in skating, 114 in basketball and 228 in swimming

In this case, the student attempts to get equal parts or the same number of students for each sport, but swimming has 114 more students than basketball. The student subtracts this surplus of 114 from 380 (which gives him 266) and ends up with a possibility of expressing the problem in equal parts. If skating is one part, then basketball is three parts, and because the difference between swimming and basketball was "erased," swimming is also three parts. Altogether, it adds up to seven parts. Therefore, 266 is divided by 7, and 38 represents the number of students in skating. Three times 38 is the number of students in basketball, which is 114, and swimming is 114 more than the number of students in basketball. Swimming has 228 students.

Reflecting on These Arithmetic Strategies

Again, similar links can be seen between the control solution and the establishment of the algebraic equation ($x + 3x + 3x + 114 = 380$), as well as between the structure solution and all algebraic operations done to isolate the x (–114, dividing by 7 and so on). Even with three unknown quantities, the same links are present, which means that the links between arithmetic and algebraic solutions exist for simple two-data problems and for more complicated problems.

However, in complicated arithmetic solutions in which many things have to be considered and remembered, it is possible to see a limit to the arithmetic solutions for solving these types of problems. As for algebra, the solution is not affected by the amount of unknown quantities or data to work with (except that there are more operations to do), which highlights an important strength of algebraic thinking and solving. Building on this thought, the next section will outline a possible approach emerging from the analysis and reflections on these solutions.

An Alternative for Teaching Based on an Analysis of These Solutions

Paralleling Both Types of Solutions

The many similarities between arithmetic and algebraic solutions must be highlighted to help students clarify their understanding of algebraic solutions. Introducing algebraic solutions on the basis of these resemblances will create an explanatory bridge between the two. Creating this parallel can bring meaning to algebraic solutions and ways of solving.

The basis for my idea resides in the exposition of both types of solving to enable students to understand the algebraic solutions on the basis of arithmetic solutions they already understand and use. Specifically, emphasis should be placed on the links between the control arithmetic solution and the algebraic equation, and on the structure arithmetic solution to give meaning to the algebraic operations to isolate the x of the equation. The idea is to show the use of this new tool of algebra by creating links between it and the previous arithmetic solutions.

Of course, the students will provide answers to the arithmetic solutions and problems, and the teacher will provide the algebraic solution. This allows students to make sense of another "expert" solution for solving a problem, which is prominently used in high school mathematics.

Although this contradicts the philosophies of having the solutions emerge from the students' ways of solving, in the case of algebra this type of solution will rarely emerge from the students. And because there is a need for these algebraic solutions, students must be introduced to these types of solutions. In that sense, because students will not think of these approaches and because the approaches are linked to what they already know, to explicitly initiate students into these expert ways of solving is not a problem in itself.

Going Beyond Arithmetic to Show the Power of Algebra to Solve Problems

As well as exposing the arithmetic and algebraic solutions in parallel, showing the clear advantage or power of algebra to solve problems is important. Paralleling these solutions familiarizes students with these high-level strategies of algebra and shows the limits of arithmetic thinking.

The three problems above can be solved with arithmetic skills, and algebra is not even needed to solve them. This is quite important, because students do not see how powerful algebra is. We need to show them the relevance of algebra, so the challenge here is for teachers to offer more and more difficult problems to show students the advantages of using algebra in comparison to arithmetic. In doing this, algebraic solutions gain more power and relevance because they help students succeed in solving more complicated problems.

Marchand and Bednarz (1999, 40) argue:

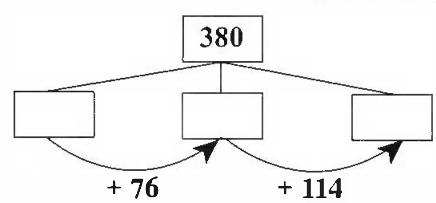
> In effect, the choice of the situations is not haphazard, since it is determining the way in which the students will see or not see the relevance of a passage to the algebraic reasoning, and will seize the eventual power of algebra to solve a class of problems for which the arithmetic reasoning becomes insufficient. (my translation)

Showing the limits of arithmetic and the power of algebra is important because students begin to use algebra to solve problems and to opt for this algebraic reasoning. This is also important, because algebra can succeed in solving problems that other skills cannot.

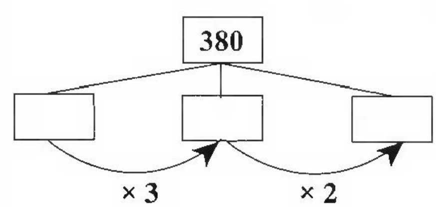
Following is a list of problems (and their structures) that could extend the previous problems. The level of difficulty becomes more and more important.

Problem — **Structure**

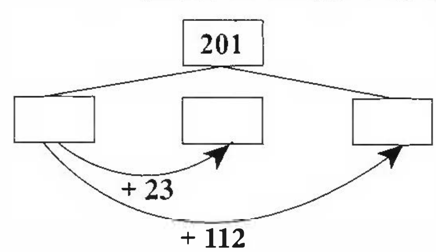
4 380 CDs are placed in three different rooms in the house. There are 76 more CDs in the living room than in the bedroom, and there are 114 more CDs in the kitchen than in the living room. How many CDs are there in each room?



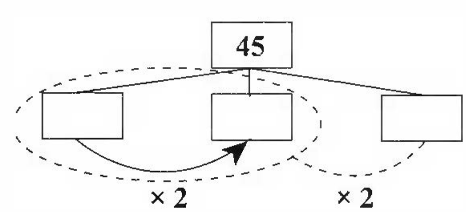
5 380 students are registered in 3 sports activities offered at school. Basketball has 3 times more students than skating, and swimming has twice as many students as basketball. How many students are registered in each activity?

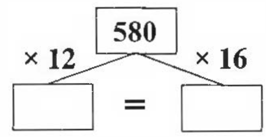
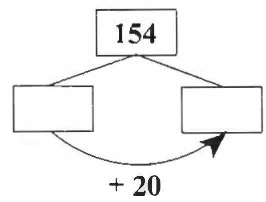
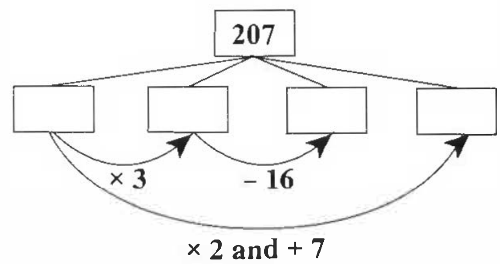
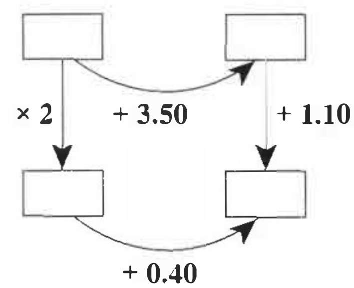


6 Three kids are playing marbles. Altogether they have 201 marbles. Claude has 23 more marbles than Andrew, and Luis has 112 more marbles than Andrew. How many marbles does each kid have?



7 Electricians use black, red and white wires. On a construction site, they have used twice as much white wire than black wire and red wire altogether. They have used 45 metres of wire in total. They have used twice less red wire than black wire. How much wire of each colour was used?



8	Two trains have to carry a total of 588 travellers. The first train has cars that contain 12 seats each, and the second train has cars that contain 16 seats each. If both trains have the same number of cars, how many travellers can ride in each train?	
9	Lynn and Mary have $154 altogether. Mary gets $20 more. Now they have the same amount of money. How much did they have in the beginning?	
10	207 members worldwide were present at the last sports drug-testing meeting held in Canada. There were 3 times more American representatives than Asian, and 16 fewer European representatives than Americans. There were 7 more African representatives than the double of the Asian representatives. How many representatives were there for each group?	
11	Luc has $3.50 less than Michael. Luc doubles his amount, whereas Michael adds $1.10 to his. Now Luc has $0.40 less than Michael. How much did each have in the beginning? How much do they each have now?	

As shown above, using arithmetic skills to solve complicated problems is difficult, although, it is not impossible. For example, by the seventh problem, arithmetic thinking becomes quite difficult. This list is also limited in itself; much more difficult problems could be highlighted, and the limit of arithmetic skills would become even more obvious. However, the fact that the difficulty level of the problems slowly and gradually increases is important because the idea is not to create a break but to facilitate the transition. So by gradually upgrading the difficulty level of the problems, algebra slowly obtains a greater status of relevance. In that sense, the transition from arithmetic to algebra is eased.

However, the limit of arithmetic thinking may not be the same for all students. Although some students will experience it as early as the third problem, others well rooted in and comfortable with arithmetic thinking may need more problems to find a limit to their thinking and give relevance to the algebraic approach.

Historical Account on Algebra Teaching

Historically, algebra teaching was strongly linked to what I offer here. In the beginning of the 20th century, algebra was introduced and taught in schools by creating parallels between arithmetic and algebraic solutions (Chevallard 1985). These steps were aimed at showing the power of algebra to solve a class of problems. Schmidt (1994, 71) highlights the same historical event:

> Arithmetic and algebraic ways of solving were offered, and an emphasis was placed on the power of algebra to solve other problems of the same type. In this approach, algebra was offered as a new tool that, while rooted in arithmetic traditions and knowledge, enabled the solving of problems that arithmetic was not able to solve locally. (my translation)

Conclusion

This approach offered attempts to facilitate the transition from arithmetic to algebraic thinking by clarifying the links and differences of arithmetic thinking (where students are at) to algebraic thinking (where they are expected to be). As we have seen, although these solutions are different, they do have similarities. By paralleling and comparing them, a better sense of each can emerge.

In brief, this approach is twofold: (1) to connect the arithmetic and algebraic solutions and introduce students to algebra by establishing links with their already known strategies, and (2) to create a parallel between solutions to gradually show how the algebraic solutions are more advantageous, thus creating relevance for using algebra. The fact that solutions will be placed in parallel will highlight the power of algebra quite clearly. This paralleled exposure introduces students to algebra as a new tool, and its relevance is shown and put forward because it can solve problems that can't be solved with other methods.

A key aspect here is the idea of gradually augmenting the difficulty level of the problems. This is central to easing the transition to algebraic thinking because it slowly demonstrates the limits or complexity of arithmetic solutions. Simultaneously, it shows how algebraic solutions can continue to solve more complex problems. By explicitly showing the limits of arithmetic solving, algebraic solving will gain strength and relevance for the student.

The same can be said about the introduction of solving algebraic problems using two variables. Unfortunately, in many textbooks, most of the problems offered for systems of equations are easily solved by using only one variable and even sometimes by using arithmetic procedures. In fact, the list of problems presented above often represents the type of problems offered in chapters on systems of equations. In that case, the relevance of now opting for two variables is definitely absent, and this becomes problematic and unfortunate because using two variables becomes an imposition and not a powerful strategy to opt for.

Finally, it should not be surprising to see students struggle with the idea of operating on unknowns. In effect, as I have mentioned before, it represents an important step to accept and understand, and the history of mathematics shows how it is difficult. However, it seems important to flag and explain that it is indeed possible to operate on unknowns (algebraic letters) in the same way that we operate on known quantities, precisely because the letters are not simply unknowns, but are unknown quantities. This is a nuance, but an important one: the letter x is not a thing in itself but represents a number of; x is not an unknown, it is an unknown quantity.

Again, these subtleties can appear to be of small importance for the expert solver, but for teachers they are important because they relate to students' difficulties with algebra. Being sensitive to these difficulties is key to facilitating and improving students' transition from an arithmetic mode of thought to an algebraic one.

Notes

1. As the anonymous reviewers have highlighted, students do some algebraic thinking in the elementary grades when they explore patterns and generalizations. However, this is not the same domain as solving word problems, in which the letters are used to represent unknown quantities and not to establish a rule or describe a pattern. Students will indeed come to postsecondary education with some experience using algebraic letters, but not for solving word problems. They also arrive with many previously acquired tools and concepts (for example, letters, equality sign and structured arithmetic strategies) that will possibly help them in solving algebraic word problems. This article mainly focuses on students' previously developed arithmetic strategies for solving word problems.

2. Note here that I am referring to school algebra, not abstract algebra that pure mathematicians work on. Abstract algebra does represent a discrete topic in research in pure mathematics, but school algebra does not reside in that sphere.

3. These problems, their structures and some student solutions are inspired from the work of Nadine Bednarz, from the Université du Québec à Montréal, and her colleagues (see Bednarz and Janvier 1996; Schmidt 1994; Marchand and Bednarz 1999, 2000). I am grateful to her for having introduced me to her work.

4. I italicize "adequate" because there are probably a lot of inadequate solutions that students would and could use. Here, I want to focus on efficient solutions (in the sense that the students arrive at adequately solving the problem), look at them and try to analyze them in relation to algebraic solving.

5. Here, I pay more attention to the control and structure strategies. In fact, guess-and-check is closely linked to the control strategy, whereby the difference is situated in the systematic trials present in the control one. Therefore, because it is linked to the control strategy, no specific attention will be paid to the guess-and-check one.

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