delta-k

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GUIDELINES FOR MANUSCRIPTS

delta-K is a professional journal for mathematics teachers in Alberta. It is published twice a year to

- promote the professional development of mathematics educators, and
- stimulate thinking, explore new ideas and offer various viewpoints.

Submissions are requested that have a classroom as well as a scholarly focus. They may include

- personal explorations of significant classroom experiences;
- descriptions of innovative classroom and school practices;
- reviews or evaluations of instructional and curricular methods, programs or materials;
- discussions of trends, issues or policies;
- a specific focus on technology in the classroom; or
- a focus on the curriculum, professional and assessment standards of the NCTM.

Suggestions for Writers

1. *delta-K* is a refereed journal. Manuscripts submitted to *delta-K* should be original material. Articles currently under consideration by other journals will not be reviewed.
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5. Pictures or illustrations should be clearly labelled and placed where you want them to appear in the article. Please also include all graphics as separate files (JPEG, GIF, TIF). A caption and photo credit should accompany each photograph.
6. References should be formatted using *The Chicago Manual of Style*'s author-date system.
7. If any student work is included, please provide a release letter from the student's parent/guardian allowing publication in the journal.
8. Limit your manuscript to no more than eight pages double-spaced.
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MCATA Mission Statement

Providing leadership to encourage the continuing enhancement of teaching, learning and understanding mathematics.

delta-K

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EDITORIAL

Professional development continues to be an exciting endeavour in teaching mathematics. This is evident by the overwhelming number of teachers who participated in the annual conference of the Mathematics Council of the Alberta Teachers' Association (MCATA) in October (photographs and report to follow in our next publication). I suspect that the next few years will bring many opportunities to expand our understanding of how math is learned and taught with the introduction of the new provincial curriculum. I encourage you to be involved in this process of curriculum revision (see Jennifer Dolecki's report in this issue).

As always, it is important that teachers share their professional knowledge with the community. This issue contains articles that may help us think differently about our pedagogical approaches and the mathematics that we teach. The feature articles focus on unique ways of interacting with mathematical ideas from elementary to high school. The teaching ideas include several examples of lesson plans that have been taught and innovative approaches to integrating technology, factoring trinomials and reducing fractions. I welcome your reactions and responses to any of the ideas you encounter in these articles. I am thankful to these authors for their contributions. I hope their ideas provoke your thinking.

I wish to thank the reviewers who work diligently to provide feedback to our authors. It is their commitment to our profession that makes this publication possible. The work that the MCATA executive does to facilitate professional development is sometimes an invisible activity; their support in the process of publishing this journal is significant and much appreciated.

I wish you all the best as you return to school, renewed and inspired in anticipation of what 2007 will bring.

Gladys Sterenberg

From the President's Pen

Those who love math can relax, reduce stress and combat boredom through mathematical pursuits. I can remember sitting in a high school history class analyzing the pattern of holes in ceiling tiles, determining the ratio of small holes to large holes, counting the holes in one tile and calculating the number of holes in one classroom and so on.

I estimate the number of people at meetings and calculate the cost per hour. I calculate the number of board feet in ceilings. I doodle patterns created without retracing lines. I determine how much paint would be needed to paint a room. I calculate probabilities. Ah, just listing a few possibilities relaxes me.

My favourite mathematical pursuit of all is knitting. Knitting is inherently mathematical. How do you calculate the area of a sweater? What is the ratio of body circumference to sleeve circumference? If a sweater has 22 stitches and 30 rows to 10 centimetres, how many metres of yarn will the sweater take? Expert knitters regularly indulge in these activities. When asked how many balls of yarn a certain sweater will take, apparently we gaze off into space and pull a number out of thin air. Actually, we perform a rapid mental calculation based on several of the above factors.

Knitting design is heavily based on the golden mean and Fibonacci numbers. If you examine a striped sweater that is pleasing to the eye, inevitably the number of rows in each stripe is a Fibonacci number. If you measure the width of the pattern units in a fisherman knit sweater, they also will be Fibonacci numbers. *Threads* magazine published a very interesting article on knitwear design using the golden mean and Fibonacci numbers (Korach 1990). Kaffe Fassett, perhaps the most famous designer of creative knitted items, uses tessellations and Escher type drawings as the basis for many of his designs.

Knitting is only one area of recreation that is highly mathematical. I could have talked about woodworking, art or many others. People who enjoy school mathematics often enjoy recreational activities that involve the use of math, yet the opposite is not equally true. I know many excellent knitters and woodworkers who say that they were never good at math in school. What a shame that these people did not see themselves as part of a mathematical community. As we continue to focus on building a deep and broad understanding in our students through the connections we forge to life outside of school, I hope that we will continue to find fewer mathematical thinkers who are "not good at math."

Reference

Korach, A. 1990. "A Balancing Act." *Threads* 30: 57–61.

Janis Kristjansson

The Right Angle: Report from Alberta Education

Jennifer Dolecki

The Process of Curriculum Revision

Alberta Education uses a six-phase cycle for curriculum review, development and implementation. Although the details of this cycle vary between programs, all programs follow the general phases. The next installment of The Right Angle will examine the development and implementation of the mathematics program of studies. For now, we'll look at the general phases in the curriculum review, development and implementation cycle.

The key questions that must be addressed in each phase of the cycle are provided below. The answers to these questions determine whether a program will move from one phase of the cycle to the next and when.

1. **Review:** Gather and review information
   What is working well?
   Are there issues or concerns to be addressed?
   Should a needs assessment be conducted?
   What research and background information are needed?
2. **Initiate:** Develop initial proposal
   What will the changes be?
   What strategies should be used?
   What are the implications for students, teachers and school authorities?
   How will the changes be communicated?
3. **Plan:** Develop project plan
   Who will be our partners and provide support?
   What are the timelines?
   What processes will be used for consultation?
   What learning and teaching resources will support the program change?
4. **Develop:** Prepare programs of study
   What are the philosophy and rationale for the program?
   What are the program outcomes?
   How is feedback from consultations on the program gathered and included in the revisions?
5. **Implement:** Authorize program and resources, support implementation
   Has there been a final quality check to ensure that all components are in place for implementation?
   Do clients, partners and stakeholders have the information needed for implementation?
   Are teachers, administrators and school authorities knowledgeable about the program and implementation requirements?
   Have teachers received support for implementation?
6. **Maintain:** Support and sustain
   Do teachers have access to ongoing guidance and support?
   Is feedback from the field regarding curriculum implementation and maintenance being monitored and collated?

Each Alberta Education program of study can be placed somewhere in this cycle, and often parts of a program can be split between phases. This emphasizes the continuous nature of curriculum development and implementation.

Noticing as a Form of Professional Development in Teaching Mathematics

Julie S Long

As a teacher, I constantly change my practice. I make changes not to correct something but to respond to students and to answer my own questions about teaching mathematics. These changes are a form of professional development because they are often derived from readings or working with other teachers. This article focuses on how John Mason's book *Researching Your Own Practice: The Discipline of Noticing* (2002) can enrich professional development. In particular, I will look at *accounts-of* and *accounts-for* experience, professional development and connections to mathematics.

Accounts-of and Accounts-for

Mason's (2002) work centres on developing sensitivities for attending to, or noticing, aspects of unexamined and habitual practice, so that choices in moments of teaching practice might be better informed. Mason's research has shown the importance of reflection in developing professional practice by offering a "detailed, structured, systematic" (p 25) way to record and act on reflections.

Mason differentiated between an account-of and an account-for an experience. An account-for an experience includes explanations, judgments and evaluations surrounding an event; an account-of an experience minimizes these aspects. The idea is to write up the account so that others recognize the experience. Mason (2002, 41) wrote that collecting these accounts-of "is one step towards … identifying a type of situation, tension, issue or interaction which is exemplified in several different incidents or experiences." I decided to try it by writing an account-of a teaching moment.

Account-of Fractions

While James presented his ideas about dividing fractions, he drew circles on the whiteboard. He explained his method for dividing one-half by one-quarter, and his classmates asked him questions. James re-explained his method using different words and the same drawings. When he stopped talking, students turned their heads from James and looked at me. James sat down and I elaborated on his strategy. Students then discussed fractions in groups and made drawings of the fractions.

In this account-of a teaching moment, I distilled the experience into a short paragraph. I avoided emotional words and explanations of my thinking, which was difficult because emotions were important in my decisions that day. I also had difficulty identifying the essence of the experience. I had to work at stressing certain aspects, such as what I could remember about the physical situation, and ignoring others, such as my emotions. At first I thought that I was writing about listening to students, but I realized that the essence in this account-of was a moment of taking authority in the classroom. This is different from my original account-for.

Account-for Fractions

One day I talked about dividing fractions and how simply knowing the procedure is not helpful. It's easy to forget what to do to which fraction. I explained that if you understand it, the procedure is meaningful. I drew an example on the board.

While students were working on a problem, I circulated and chatted. James explained his thinking about the division of fractions to me a couple of times, and I had difficulty understanding him. The students at his table were also confused. We all asked many questions until some students began to lose interest. We were off task, but I thought that the exploration was important.

I asked James to record his thinking for me so that I could consider it some more. I puzzled over his ideas and fraction circles until next class and then asked him for more clarification. When James presented his ideas to the class, his classmates had lots of questions. Students were getting annoyed because the method made no sense to them. Although James was good at explanations and answering questions, I needed to intervene. I told him that I didn't understand his thinking. James sat down and I presented a similar strategy. The class was focused and silent as I spoke and wrote. I was nervous about adding to the confusion, but I could almost hear that audible aha from students. They began to excitedly talk in groups. I hoped that meant that they were sharing their understanding and not getting bored or confused.

A few days later a student mentioned that she had never understood the division of fractions until that class. I wondered how those teaching moments came about. A lot had led up to that moment, including positive and negative feelings. I'm not sure that the moment would have been as meaningful if the students hadn't struggled to understand a classmate's unfamiliar idea, if there hadn't been time to think and discuss, if they hadn't been emotionally involved or if they hadn't already spent time listening to each other's ideas.

In this account-for, I skipped ahead of simply describing the incident to explaining my actions and trying to draw a lesson out of the experience. If I shared this account-for with others, it might be difficult for them to support me in my re-examination of the experience because I have already explained it and there are no alternatives to explore. Sharing accounts-of (not accounts-for, though, the line between the two is unclear) experiences might be "used explicitly to foster and sustain professional development in others" (Mason 2002, 139).

Professional Development

Although Mason (2002) mainly focused on how to use noticing for one's own practice, he also wrote about how to use accounts-of to support professional development in others.

> A good way to expose people to alternative practices without pressuring them to suddenly adopt one and to act differently is to arrange that one person gives a brief-but-vivid account of some problematic situation, and then others recount situations which they think have some similarities. In the process, different practices will be revealed, but in a non-threatening manner... It is a matter of offering a brief-but-vivid account without the intention of "offering a solution." (p 146)

Following this suggestion, preservice and inservice teachers could write accounts-of their teaching and share them with others, but they must be open to this sort of inquiry and look to change their own practice. Developing trusting and collegial relationships would also be important, though, Mason did not write about this explicitly. Assuming that these conditions are met—which is no small feat—this could be a fruitful way of interacting with teachers. Blending teachers' practical concerns with professional development is possible. These concerns might also be used as a basis for research, whereby theory and practice overlaps. This research might be done by teachers from the inside of practice or by researchers in conjunction with teachers from the outside of practice.

Connections to Mathematics

The discipline of noticing and Mason's previous work *Thinking Mathematically* (Mason, Burton and Stacey 1985) are parallel. This helps me to better understand how professional development of teaching in general is connected to teaching and learning mathematics.

The acts of stressing and ignoring are part of both mathematical thinking and professional development through the discipline of noticing. When thinking mathematically, I often stress one part of the question while ignoring other parts; for example, looking at a geometric shape and stressing the global characteristics (it looks like a diamond), while ignoring the specific characteristics, such as the angle measures. What I stress and what I ignore can be described as habitual and depends on the situation's context. By stressing and ignoring, I can first specialize and then generalize; both are essential features of mathematical thinking (Mason, Burton and Stacey 1985). The discipline of noticing calls me to attend to what is stressed and ignored in my own mathematical work as well as in my teaching practice. In addition, I am invited to stress the essence and ignore the emotions in writing accounts-of experience. Stressing and ignoring are mathematical ways of examining teaching practice.

Accounts-of experiences are described as "brief-but-vivid" (Mason 2002, 47). A problem of teaching practice is distilled into a few sentences. In this discipline of noticing, a number of these accounts are examined for relationships and inconsistencies. This is similar to the work of mathematicians, which might be characterized as compressing information, and of teachers, which can be thought of as unpacking this condensed knowledge (Ball and Bass 2003). Through noticing as a form of professional development, teachers condense experiences and then reconceptualize their accounts to transform practice.

Questions and Issues

This article explains how to consciously notice. Practices can be transformed by attending to experiences, recording them systematically, questioning the accounts and then acting deliberately. Though Mason's description of the discipline of noticing resonates with my own process of reflecting, I also wonder about the power of accounts-for in this discipline. I have used accounts-for in my writing, talking and thinking with meaningful results. Though I resist the focus on the accounts-of (as opposed to the accounts-for), part of this resistance comes from the difficulty and work involved in writing a compelling account-of an experience. Writing and using both accounts-for and accounts-of have been fruitful ways to engage in professional development in my teaching of mathematics.

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Making the Transition to Algebraic Thinking: Taking Students' Arithmetic Modes of Reasoning into Account

Jérôme Proulx

The purpose of this article is to trigger reflection and discussion on the transition from arithmetic to algebraic problem solving and its teaching. When students are introduced to algebraic problem solving in their first years of secondary schooling, they have already acquired arithmetic procedures, experiences and tools. These arithmetic modes of reasoning significantly differ from the ones we teachers are expected to teach in algebra. Students arrive with at least seven years of arithmetic operations and problem solving. These procedures and ways of doing mathematics are rooted in operations on known quantities or givens, whereas algebra requires operations on unknown quantities.

The conceptual step of accepting and understanding what it means to operate on unknown values in the same way that we operate on known and given values was an important historical difficulty for mathematicians as well. It should not then be a surprise to see students experiencing difficulties in that domain. Therefore, the transitional step to algebraic thinking is one of the most difficult steps experienced in a student's mathematical life[1].

To ease this transition, teachers must be sensitized to students' arithmetic procedures for solving problems and must consider these ways of thinking in teaching. To set aside all students' prior knowledge construed in the elementary years of mathematics schooling would be nonsense.

My argument underlies this conceptual umbrella. I intend to raise sensibility toward prealgebraic students' ways of solving problems to make better sense of (1) students' skills and knowledge with which they enter introduction-to-algebra classrooms, and (2) how these strategies can be accounted for in teaching to ease this important transition in school mathematics.

With this in mind, I will offer some traditional algebraic problems and how students with no background in algebraic problem solving make sense of and solve these problems. With these solutions in hand, one intent will be (1) to see similarities and differences between arithmetic and algebraic ways of solving, (2) to see possible usage and avenues these similarities and differences give to ease the transition, and (3) to realize the strength and the limits of these arithmetic solutions to better understand how to promote the power and relevance of algebraic reasoning to solve problems.

Teaching Algebra or Solving Problems with Algebra

The traditional algebraic word problems that I will offer represent what is normally given in the introduction to algebraic problem solving in junior high. However, as will be shown later, these problems are not algebraic in themselves, because they can be and are solved without using algebra.

This is no small point, because it flags the purpose of algebraic problem solving in school mathematics. Algebra represents a tool to solve problems as much as geometric or arithmetic skills do. Seeing algebra as a problem-solving tool brings us to question deeply our assumptions about algebra. Algebra, as powerful as it is for solving particular word problems, should not be seen as an end in itself; solving the word problems represents the end in itself. When we want students to solve a problem, the fact that they use different or nonalgebraic methods and strategies should not be seen as problematic. The goal is to solve the word problem and not simply to use a specific predetermined strategy. In other words, imposing on and demanding that students only use algebra to solve word problems is nonsense, because algebra becomes the goal of instruction and solving word problems becomes secondary. This is important because algebra has become so prominent in the school curriculum at the secondary level that it is almost seen as a subject in itself, not as a mathematical tool invented to solve problems[2]. I am not saying that algebra is not important; however, the status and utility of algebra in school mathematics must be understood.

In fact, seeing algebra as a powerful problem-solving tool makes it more relevant in the school mathematics curriculum. This perception enables teachers to present and offer algebraic thinking and solving to students as a powerful problem-solving tool, which permits the solving of problems that other methods cannot solve.

By showing the unparalleled and unmatched capacity of the algebraic tool to solve certain types of problems, algebra becomes relevant and makes sense as a tool. If algebra is imposed in places where other methods are as efficient, as fast and even more economical, algebra loses its significance and becomes an unnecessary action or even a burden on the learner.

The Problems and an Analysis of Their Solutions

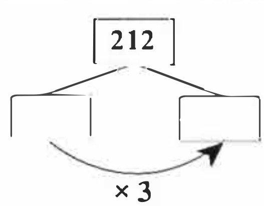
To introduce the ideas, I will first present some problems and their underlying structure[3]. To some extent, these types of problems are offered when algebra is introduced. For each problem, a possible algebraic solution will be offered, followed by some students' possible arithmetic solutions.

Problem 1

The first problem contains a multiplicative structure:

The school cafeteria offered two different meals at lunch. Three times more hamburgers than pizzas were served. If 212 meals in total were served, how many hamburgers and how many pizzas were served?

Structure of problem 1



A possible algebraic solution for this problem follows:

Algebraic solution

x = number of pizzas, $3x$ = number of hamburgers

$$3x + x = 212$$
$$4x = 212$$
$$\frac{4x}{4} = \frac{212}{4}$$
$$x = 53$$
$$3 \times 53 = 159$$

➛53 pizzas and 159 hamburgers

This solution represents an expected algebraic solution in school mathematics. Interestingly, many students could answer that problem without knowing algebra. For example, three prominent *adequate*[4] arithmetic solutions could be the guess-and-check number trial, whereby students randomly try answers until they reach a possible result. Then they keep trying other answers until the problem is solved.

Another type of arithmetic answer is the *control* solution, whereby students methodically try numbers and constantly consider the relation between the data; for example, "one is three times more than the other one," and the total number of meals is 212. The following example of a table is often drawn by students using this strategy:

Control solution

× 3

30	90	= 120
40	120	= 160
50	150	= 200
55	165	= 220
52	156	= 208
53	159	= 212

Finally, another possible answer is the *structure* solution, whereby students work with the relations between the parts and the data of the problem. For example:

Structure solution

$212 \div 4 = 53$ [÷ 4 because I count 3 times more hamburgers and 1 times the pizzas]

$53 \times 3 = 159$ [3 times the number of pizzas = the number of hamburgers]

$53 + 159 = 212$

➛53 pizzas and 159 hamburgers

In this strategy, students see a little part, the pizza, and see this same little part repeated three times for the hamburgers (three times the number of pizzas = the number of hamburgers). Students see four parts for the whole problem and for the number of meals served in total (three for the hamburgers and one for the pizzas). So, the students divide the total number of meals into four parts ($212 \div 4 = 53$). The value obtained for this part represents the number of pizza meals, so they multiply it by three to obtain the number of hamburgers served ($53 \times 3 = 159$).

Reflecting on These Arithmetic Strategies

What stands out in each of these last two strategies[5] (control and structure) is the students' demonstration of control over the data. In fact, Bednarz and Janvier's (1996) research shows that this skill, controlling the relations between the data in the problem, is of major importance to solving problems in algebra. Each solution has its strength for two important parts of algebraic solving: (1) the creation of an algebraic equation, and (2) the algebraic operations to find (isolate) the value of x in the problem. The control solution shows how this student is comfortable with the relation between data (one for three) and the knowledge that the total must stay at 212. Even if this seems obvious or easy, this is exactly what is needed to create the algebraic equation $3x + x = 212$. This solution shows a double-control: the control over the relation between the givens (three times the other) and the fact that the addition of both gives 212. Thus, inherent understandings are present in the control solution that could be worked on and used to help students create and understand the algebraic equation, which is often the hardest part of solving algebraic word problems.

The similarities between the structure solution and the expected algebraic solution are important regarding the operations to arrive at a value for x. Each algebraic step is mimicked by each arithmetic step (or should we say the opposite?). In the algebraic solution, the student "adds its x" and obtains $4x$. In the arithmetic solution, the student adds its parts: "4 because I count 3 times more hamburgers and 1 time the pizzas." Afterward, the algebraic student divides 212 by 4 to obtain the value of x, whereas the arithmetic student divides its total by 4 to obtain the value of one part. After obtaining 53, both students find the quantity of hamburgers by reapplying the relation that links both data (three times more). The link between both solutions is strong and represents important insights that show some possible links (even if there are obviously some differences) between the students' arithmetic solution and the algebraic solution expected.

It must be emphasized that these solutions are different. The major difference resides in the presence of the context. In the arithmetic solutions, the presence of the context is present at each step. The operations are made on hamburgers and pizzas and on the number of meals. In the algebraic solution, the operations conducted to arrive at a value of x are made in a decontextualized fashion; that is, at a mechanical level. Isolating the x does not require the solver to keep a grasp on the context (that is, on the meals). It is a set of procedures and steps to arrive at isolating x. This represents a major difference between an algebraic and an arithmetic solution. Although the link to the context is strong in an arithmetic solution, it is unnecessary (and sometimes does not even make sense) in the algebraic solution. There is a need to step out of the context in an algebraic solution to conduct the operations. In that sense, even though arithmetic solutions seem similar to algebraic ones, they are conceptually different. However, despite this conceptual difference, the similarities highlighted hint at some important insights into how to ease the transition from one to the other.

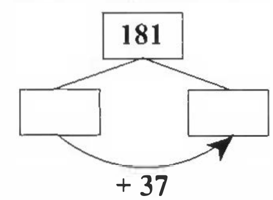
With these differences and similarities in mind, I will present two more problems and possible solutions: one with an additive structure and the other with a combination additive and multiplicative structure. After an analysis of the solutions, I will introduce an approach based on the insights drawn from the first three problems offered.

Problem 2

This problem contains an additive structure:

Two children have a stamp collection. Alex has 37 more stamps than Josie. If they have 181 stamps altogether, how many stamps do they each have?

Structure of problem 2



A possible algebraic solution for this problem could be:

Algebraic solution

x = number of stamps of Josie, $x + 37$ = number of stamps of Alex

$x + x + 37 = 181$

$2x + 37 = 181$

$2x = 144$

$x = 72$

$72 + 37 = 109$

→72 for Jose and109 for Alex

In the same line of thought as the previous problem, an example of a control solution could be:

Control solution

+ 37

50		87	= 137
60		97	= 157
70		107	= 177
71		108	= 179
72		119	= 181

Here, the student controls the relation between the givens by knowing simultaneously that they must always have a difference of 37 between them (or that the second one has 37 more) and that their addition gives 181. These two relations (37 more and 181 as a total) are regarded throughout the whole solving process.

Here is what a possible structure solution would look like:

Structure solution

181 – 37 = 144 [Alex has 37 more]

144 ÷ 2 = 72 [they now have the same amount so I can divide in two]

72 + 37 = 109

→ 72 for Jose and 109 for Alex

In this structure solution, the student sees two quantities: Josie's and Alex's. Because the student knows that the two quantities are not equivalent (Alex has 37 more), the student reorganizes the problem by taking out the quantity (the surplus) (181 – 37 = 144). By subtracting the surplus, the student obtains two equivalent quantities. The student's new amount represents the total that Josie and Alex would have if they had the same amount. Then, the student divides the result into two parts (144 ÷ 2 = 72). This new quantity (72) represents the number of Josie's stamps. The student then adds 37 to 72 to obtain Alex's number of stamps (72 + 37 = 109).

Reflecting on These Arithmetic Strategies

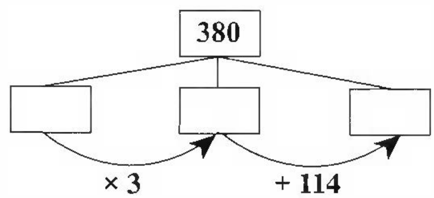
Again, the possible links and similarities between the control and structure solutions are worth mentioning. As I underlined before, the control solution, with its control over the relations between the data (the +37 and the total of 181), hints very well at an understanding of the algebraic equation ($x + x + 37 = 181$). As for the structure solution, its steps toward arriving at Josie's number of stamps (subtracting 37, dividing by 2) are representative of the operations needed to isolate the x. This arithmetic solution again hints directly at the processes of algebraic problem solving, because it enables us to create links and make sense of (1) the creation of the algebraic equation, and (2) the steps of algebraic operations. The last example is a multiplicative and additive structure.

Problem 3

Here is a third problem and its structure:

380 students are registered in three sports activities offered during the semester. Basketball has 3 times more students than skating, and swimming has 114 more students than basketball. How many students are registered in each activity?

Structure of problem 3



A possible algebraic solution would be:

Algebraic solution

x = number of students registered in skating,

$3x$ = number of students registered in basketball,

$3x + 114$ = number of students registered in swimming.

$x + 3x + 3x + 114 = 380$

$7x + 114 = 380$

$7x = 266$

$x = 38$

$3 \times 38 = 114$

$3 \times 38 + 114 = 228$

→ 38 students are registered in skating, 114 in basketball and 228 in swimming

Now, with three unknown quantities to consider, the strategies of control and structure become more complicated, but they still follow the same thinking as before. In the control solution, one difference is that because of the three unknowns, more control can be exerted on the whole problem as the following solution shows:

Control solution

× 3 → + 114 →

50	150	264	= 464
40	120	234	= 394
35	105	219	= 359
37	111	225	= 373
38	114	228	= 380

In effect, the student needs to consider that basketball has 3 times more students than skating, and swimming has 114 more students than basketball. After considering these relations, the student needs to control the fact that these three numbers added together total 380. In the previous problems, there are only two relations to consider, but here there are three (3 times more, +114 and a total of 380).

Here is an example of a structure solution:

Structure solution

To make swimming and basketball equivalent:
380 – 114 = 266

For the skating: 266 ÷ (3 + 3 + 1) = 38
[there are now seven parts]
Basketball: 38 × 3 = 114
Swimming: 114 + 114 = 228

→ 38 students are registered in skating, 114 in basketball and 228 in swimming

In this case, the student attempts to get equal parts or the same number of students for each sport, but swimming has 114 more students than basketball. The student subtracts this surplus of 114 from 380 (which gives him 266) and ends up with a possibility of expressing the problem in equal parts. If skating is one part, then basketball is three parts, and because the difference between swimming and basketball was "erased," swimming is also three parts. Altogether, it adds up to seven parts. Therefore, 266 is divided by 7, and 38 represents the number of students in skating. Three times 38 is the number of students in basketball, which is 114, and swimming is 114 more than the number of students in basketball. Swimming has 228 students.

Reflecting on These Arithmetic Strategies

Again, similar links can be seen between the control solution and the establishment of the algebraic equation ($x + 3x + 3x + 114 = 380$), as well as between the structure solution and all algebraic operations done to isolate the x (–114, dividing by 7 and so on). Even with three unknown quantities, the same links are present, which means that the links between arithmetic and algebraic solutions exist for simple two-data problems and for more complicated problems.

However, in complicated arithmetic solutions in which many things have to be considered and remembered, it is possible to see a limit to the arithmetic solutions for solving these types of problems. As for algebra, the solution is not affected by the amount of unknown quantities or data to work with (except that there are more operations to do), which highlights an important strength of algebraic thinking and solving. Building on this thought, the next section will outline a possible approach emerging from the analysis and reflections on these solutions.

An Alternative for Teaching Based on an Analysis of These Solutions

Paralleling Both Types of Solutions

The many similarities between arithmetic and algebraic solutions must be highlighted to help students clarify their understanding of algebraic solutions. Introducing algebraic solutions on the basis of these resemblances will create an explanatory bridge between the two. Creating this parallel can bring meaning to algebraic solutions and ways of solving.

The basis for my idea resides in the exposition of both types of solving to enable students to understand the algebraic solutions on the basis of arithmetic solutions they already understand and use. Specifically, emphasis should be placed on the links between the control arithmetic solution and the algebraic equation, and on the structure arithmetic solution to give meaning to the algebraic operations to isolate the x of the equation. The idea is to show the use of this new tool of algebra by creating links between it and the previous arithmetic solutions.

Of course, the students will provide answers to the arithmetic solutions and problems, and the teacher will provide the algebraic solution. This allows students to make sense of another "expert" solution for solving a problem, which is prominently used in high school mathematics.

Although this contradicts the philosophies of having the solutions emerge from the students' ways of solving, in the case of algebra this type of solution will rarely emerge from the students. And because there is a need for these algebraic solutions, students must be introduced to these types of solutions. In that sense, because students will not think of these approaches and because the approaches are linked to what they already know, to explicitly initiate students into these expert ways of solving is not a problem in itself.

Going Beyond Arithmetic to Show the Power of Algebra to Solve Problems

As well as exposing the arithmetic and algebraic solutions in parallel, showing the clear advantage or power of algebra to solve problems is important. Paralleling these solutions familiarizes students with these high-level strategies of algebra and shows the limits of arithmetic thinking.

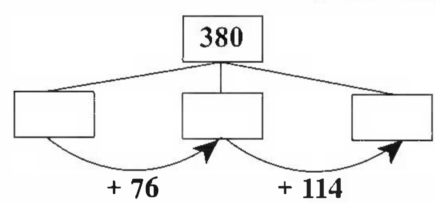
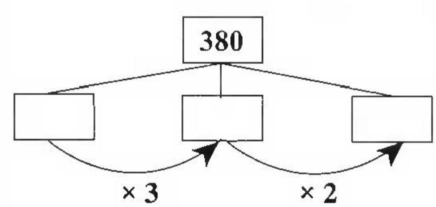
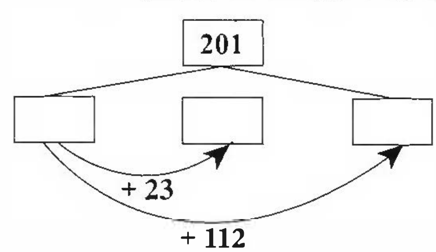
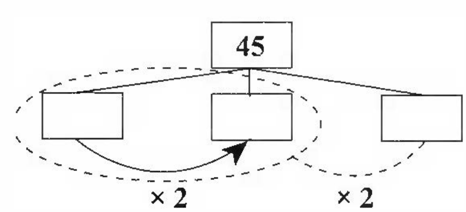
The three problems above can be solved with arithmetic skills, and algebra is not even needed to solve them. This is quite important, because students do not see how powerful algebra is. We need to show them the relevance of algebra, so the challenge here is for teachers to offer more and more difficult problems to show students the advantages of using algebra in comparison to arithmetic. In doing this, algebraic solutions gain more power and relevance because they help students succeed in solving more complicated problems.

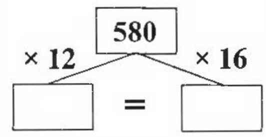
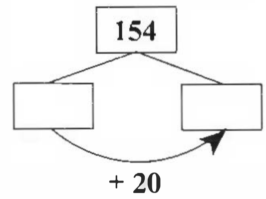
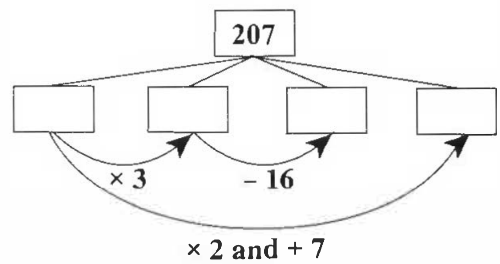
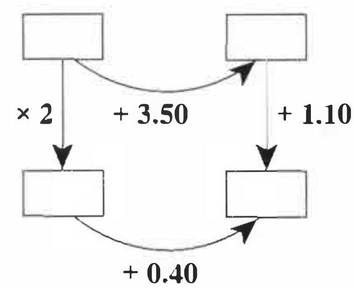
Marchand and Bednarz (1999, 40) argue:

> In effect, the choice of the situations is not haphazard, since it is determining the way in which the students will see or not see the relevance of a passage to the algebraic reasoning, and will seize the eventual power of algebra to solve a class of problems for which the arithmetic reasoning becomes insufficient. (my translation)

Showing the limits of arithmetic and the power of algebra is important because students begin to use algebra to solve problems and to opt for this algebraic reasoning. This is also important, because algebra can succeed in solving problems that other skills cannot.

Following is a list of problems (and their structures) that could extend the previous problems. The level of difficulty becomes more and more important.

	Problem	Structure
4	380 CDs are placed in three different rooms in the house. There are 76 more CDs in the living room than in the bedroom, and there are 114 more CDs in the kitchen than in the living room. How many CDs are there in each room?	
5	380 students are registered in 3 sports activities offered at school. Basketball has 3 times more students than skating, and swimming has twice as many students as basketball. How many students are registered in each activity?	
6	Three kids are playing marbles. Altogether they have 201 marbles. Claude has 23 more marbles than Andrew, and Luis has 112 more marbles than Andrew. How many marbles does each kid have?	
7	Electricians use black, red and white wires. On a construction site, they have used twice as much white wire than black wire and red wire altogether. They have used 45 metres of wire in total. They have used twice less red wire than black wire. How much wire of each colour was used?	

8	Two trains have to carry a total of 588 travellers. The first train has cars that contain 12 seats each, and the second train has cars that contain 16 seats each. If both trains have the same number of cars, how many travellers can ride in each train?	
9	Lynn and Mary have $154 altogether. Mary gets $20 more. Now they have the same amount of money. How much did they have in the beginning?	
10	207 members worldwide were present at the last sports drug-testing meeting held in Canada. There were 3 times more American representatives than Asian, and 16 fewer European representatives than Americans. There were 7 more African representatives than the double of the Asian representatives. How many representatives were there for each group?	
11	Luc has $3.50 less than Michael. Luc doubles his amount, whereas Michael adds $1.10 to his. Now Luc has $0.40 less than Michael. How much did each have in the beginning? How much do they each have now?	

As shown above, using arithmetic skills to solve complicated problems is difficult, although, it is not impossible. For example, by the seventh problem, arithmetic thinking becomes quite difficult. This list is also limited in itself; much more difficult problems could be highlighted, and the limit of arithmetic skills would become even more obvious. However, the fact that the difficulty level of the problems slowly and gradually increases is important because the idea is not to create a break but to facilitate the transition. So by gradually upgrading the difficulty level of the problems, algebra slowly obtains a greater status of relevance. In that sense, the transition from arithmetic to algebra is eased.

However, the limit of arithmetic thinking may not be the same for all students. Although some students will experience it as early as the third problem, others well rooted in and comfortable with arithmetic thinking may need more problems to find a limit to their thinking and give relevance to the algebraic approach.

Historical Account on Algebra Teaching

Historically, algebra teaching was strongly linked to what I offer here. In the beginning of the 20th century, algebra was introduced and taught in schools by creating parallels between arithmetic and algebraic solutions (Chevallard 1985). These steps were aimed at showing the power of algebra to solve a class of problems. Schmidt (1994, 71) highlights the same historical event:

> Arithmetic and algebraic ways of solving were offered, and an emphasis was placed on the power of algebra to solve other problems of the same type. In this approach, algebra was offered as a new tool that, while rooted in arithmetic traditions and knowledge, enabled the solving of problems that arithmetic was not able to solve locally. (my translation)

Conclusion

This approach offered attempts to facilitate the transition from arithmetic to algebraic thinking by clarifying the links and differences of arithmetic thinking (where students are at) to algebraic thinking (where they are expected to be). As we have seen, although these solutions are different, they do have similarities. By paralleling and comparing them, a better sense of each can emerge.

In brief, this approach is twofold: (1) to connect the arithmetic and algebraic solutions and introduce students to algebra by establishing links with their already known strategies, and (2) to create a parallel between solutions to gradually show how the algebraic solutions are more advantageous, thus creating relevance for using algebra. The fact that solutions will be placed in parallel will highlight the power of algebra quite clearly. This paralleled exposure introduces students to algebra as a new tool, and its relevance is shown and put forward because it can solve problems that can't be solved with other methods.

A key aspect here is the idea of gradually augmenting the difficulty level of the problems. This is central to easing the transition to algebraic thinking because it slowly demonstrates the limits or complexity of arithmetic solutions. Simultaneously, it shows how algebraic solutions can continue to solve more complex problems. By explicitly showing the limits of arithmetic solving, algebraic solving will gain strength and relevance for the student.

The same can be said about the introduction of solving algebraic problems using two variables. Unfortunately, in many textbooks, most of the problems offered for systems of equations are easily solved by using only one variable and even sometimes by using arithmetic procedures. In fact, the list of problems presented above often represents the type of problems offered in chapters on systems of equations. In that case, the relevance of now opting for two variables is definitely absent, and this becomes problematic and unfortunate because using two variables becomes an imposition and not a powerful strategy to opt for.

Finally, it should not be surprising to see students struggle with the idea of operating on unknowns. In effect, as I have mentioned before, it represents an important step to accept and understand, and the history of mathematics shows how it is difficult. However, it seems important to flag and explain that it is indeed possible to operate on unknowns (algebraic letters) in the same way that we operate on known quantities, precisely because the letters are not simply unknowns, but are unknown quantities. This is a nuance, but an important one: the letter x is not a thing in itself but represents a number of; x is not an unknown, it is an unknown quantity.

Again, these subtleties can appear to be of small importance for the expert solver, but for teachers they are important because they relate to students' difficulties with algebra. Being sensitive to these difficulties is key to facilitating and improving students' transition from an arithmetic mode of thought to an algebraic one.

Notes

1. As the anonymous reviewers have highlighted, students do some algebraic thinking in the elementary grades when they explore patterns and generalizations. However, this is not the same domain as solving word problems, in which the letters are used to represent unknown quantities and not to establish a rule or describe a pattern. Students will indeed come to postsecondary education with some experience using algebraic letters, but not for solving word problems. They also arrive with many previously acquired tools and concepts (for example, letters, equality sign and structured arithmetic strategies) that will possibly help them in solving algebraic word problems. This article mainly focuses on students' previously developed arithmetic strategies for solving word problems.

2. Note here that I am referring to school algebra, not abstract algebra that pure mathematicians work on. Abstract algebra does represent a discrete topic in research in pure mathematics, but school algebra does not reside in that sphere.

3. These problems, their structures and some student solutions are inspired from the work of Nadine Bednarz, from the Université du Québec à Montréal, and her colleagues (see Bednarz and Janvier 1996; Schmidt 1994; Marchand and Bednarz 1999, 2000). I am grateful to her for having introduced me to her work.

4. I italicize "adequate" because there are probably a lot of inadequate solutions that students would and could use. Here, I want to focus on efficient solutions (in the sense that the students arrive at adequately solving the problem), look at them and try to analyze them in relation to algebraic solving.

5. Here, I pay more attention to the control and structure strategies. In fact, guess-and-check is closely linked to the control strategy, whereby the difference is situated in the systematic trials present in the control one. Therefore, because it is linked to the control strategy, no specific attention will be paid to the guess-and-check one.

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A Geometrical Meaning for the Correlation Coefficient

Murray L Lauber

Having the privilege of teaching many mathematics strands has allowed me to make connections between concepts that on the surface seem unrelated. High school and undergraduate math students have this opportunity but may not have the time or the intimate knowledge of the subject matter to form such connections. A reward of repeatedly teaching the same math courses is that one's knowledge of the subject matter deepens to the point where, with some exploration, such connections can become apparent. Additional rewards are modelling such exploration for students and encouraging them to explore on their own.

This article describes the fundamental connection between the concept of the correlation coefficient from statistics and that of the angle between two vectors from linear algebra. That connection became apparent to me over a few years while teaching vectors in linear algebra and, at the same time, some elementary statistics in a precalculus course. It initially sprouted from a concept that had incubated when I was a student in a statistics course many years earlier. In the chapter of the course textbook pertaining to the correlation coefficient, Ferguson (1981, 132) describes how the correlation coefficient is related to the angular separation between two regression lines. The ensuing discussion is general enough to leave room for questions and to invite exploration. In fact, Ferguson's observations seemed inaccurate because a full mathematical explanation was not given. At the least, they lodged in the back of my mind as a kind of healthy dissonance. They were not completely resolved until I taught a linear algebra course where the concept of the angle between two n-dimensional vectors was fully developed as the generalization of the geometrical angle between two 2- or 3-dimensional vectors. The angle between a pair of 2- or 3-dimensional vectors can be visualized intuitively and is easily calculated using simple trigonometry. The angle between two n-dimensional vectors is then defined as a generalization of the intuitive notions applying to 2- or 3-dimensional vectors.

What follows is the full development of a geometrical meaning for the correlation coefficient based on the notions of the previous paragraph. It is related to Ferguson's observations but, given an understanding of some basic concepts of vectors, seems more elegant in its simplicity.

The Correlation Coefficient—A Brief Review

The peripheral correlation coefficient is a precise comparison of two sets of scores that measures the degree to which corresponding scores deviate from their respective means. Do the sizes and directions of the deviations of corresponding data elements from their respective means tend to correspond? If so, the correlation coefficient will be high (close to 1). Does there appear to be little relationship between how corresponding data elements deviate from their respective means in the two sets of scores? If so, the correlation coefficient will be low (close to 0). Do the deviations from their respective means for corresponding elements tend to be in opposite directions (scores above the mean for the one data set correspond to scores below the mean for the other set, and vise versa)? If so, the correlation coefficient will be negative (perhaps as negative as −1). Consider the following simple example for two sets of scores, x and y.

x	y
1	2
2	4
3	6
4	8

[1]

Intuitively, these two sets of data are as closely related as any two distinct sets of data can be; therefore, the correlation coefficient should be 1. This will be demonstrated shortly.

The correlation coefficient may be defined as the ratio of the average of the sum of products of the deviations of corresponding elements from their respective means to the product of the standard deviations of the two sets of scores. Consider the following two sets of scores[1].

$x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n)$

The correlation coefficient r between these two sets of scores is defined formulaically as follows:

$$r = \frac{(x_1 - \bar{x})(y_1 - \bar{y}) + (x_2 - \bar{x})(y_2 - \bar{y}) + \ldots(x_n - \bar{x})(y_n - \bar{y})}{ns_x s_y}$$

$$\text{or } r = \frac{\sum(x - \bar{x})(y - \bar{y})}{ns_x s_y} \quad [2]$$

In this formulation:

$\bar{x}$ is the mean for the set x
$\bar{y}$ is the mean for the set y
s_x is the standard deviation for the set x [3]
s_y is the standard deviation for the set y
n is the number of scores in each data set

Definition [2] readily shows that the numerator will be large and positive if corresponding scores from x and y deviate proportionately in the same direction from their respective means. On the other hand, it will be large and negative if corresponding scores from x and y deviate proportionately in opposite directions from their respective means. And it will be small if there is little connection between how corresponding scores from x and y deviate from their respective means. The numerator alone, though, would not adequately define any measure of comparison between two sets of scores. We would be left with the questions, "How large is large?" and "How small is small?" But definition [2] taken altogether is ingenious in that dividing by $ns_x s_y$ ensures that the value of r is between -1 and 1 for any two sets of scores with 1 representing the highest possible positive correlation and -1 representing the lowest possible negative correlation. The proof is not included here but can be formed using the definitions of s_x, s_y and r.

By way of illustration, Table A shows the calculations used in determining r for example [1]. Recall that we had already anticipated that the value of r for this case should be 1. Note here that $\bar{x} = 2.5$, $\bar{y} = 5$ and $n = 4$. From Table A, we have

$$s_x = \sqrt{\frac{\sum(x - \bar{x})^2}{n}} = \frac{\sqrt{5}}{2}, \; s_y = \sqrt{\frac{\sum(y - \bar{y})^2}{n}} = \sqrt{5}.$$

$$\text{Then } r = \frac{\sum(x - \bar{x})(y - \bar{y})}{ns_x s_y} = \frac{10}{4\left(\frac{\sqrt{5}}{2}\right)(\sqrt{5})} = 1.$$

This is as we expected.

Correlation Coefficients from Standard Scores

When comparing two data sets, it often helps to first convert the raw scores into standard scores or z-scores. The z-score of a particular score in a set of raw scores is the measure of how many standard deviations the raw score is above or below the mean. Suppose, for example, that for a set of scores x, the mean and standard deviation are $\bar{x} = 10$ and $s_x = 2$, respectively. Then a raw score of 12 would have a z-score of 1 because it is exactly one standard deviation above the mean. In general, the z-score, z_x, of a particular raw score x from the set of scores x where the mean is $\bar{x}$ and the standard deviation is s_x is defined as

$$z_x = \frac{x - \bar{x}}{s_x}$$

The formulaic representation for the correlation coefficient r is simpler when standard scores are used. Recall that

$$r = \frac{\sum(x - \bar{x})(y - \bar{y})}{ns_x s_y} \quad [3]$$

Since $z_x = \frac{x - \bar{x}}{s_x}$ and $z_y = \frac{y - \bar{y}}{s_y}$, we have

$$r = \frac{\sum z_x z_y}{n} \quad [4]$$

Table A

x	y	$x - \bar{x}$	$y - \bar{y}$	$(x - \bar{x})^2$	$(y - \bar{y})^2$	$(x - \bar{x})(y - \bar{y})$
1	2	−1.5	−3	2.25	9	4.5
2	4	−0.5	−1	.25	1	.5
3	6	0.5	1	.25	1	.5
4	8	1.5	3	2.25	9	4.5
				$\sum(x - \bar{x})^2 = 5$	$\sum(y - \bar{y})^2 = 20$	$\sum(x - \bar{x})(y - \bar{y}) = 10$

This formula for r will be revisited after the concept of the angle between two vectors has been fully developed. It is most useful as a theoretical tool for developing other relationships. However, by way of illustration, it is applied to example [1] in Table B below. Recall that in this example, $\bar{x} = 2.5$, $\bar{y} = 5$, $s_x = \frac{\sqrt{5}}{2}$, and $s_y = \sqrt{5}$. By way of illustration, the values of z_x, z_y, and $z_x z_y$ in the first row were computed as follows.

$$z_x = \frac{-1.5}{\sqrt{5}/2} = \frac{-3}{\sqrt{5}},\ z_y = \frac{-3}{\sqrt{5}},\ \text{and } z_x z_y = \left(\frac{-3}{\sqrt{5}}\right)\left(\frac{-3}{\sqrt{5}}\right) = \frac{9}{5}$$

Table B

x	y	$x-\bar{x}$	$y-\bar{y}$	z_x	z_y	$z_x z_y$
1	2	−1.5	−3	$-3\sqrt{5}$	$-3\sqrt{5}$	9/5
2	4	−0.5	−1	$-1\sqrt{5}$	$-1\sqrt{5}$	1/5
3	6	0.5	1	$1\sqrt{5}$	$1\sqrt{5}$	1/5
4	8	1.5	3	$3\sqrt{5}$	$3\sqrt{5}$	9/5
						$\sum z_x z_y = 4$

Using the results from the table, $r = \frac{\sum z_x z_y}{n} = \frac{4}{4} = 1$.

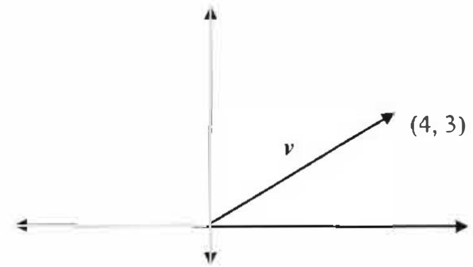
One other concept pertaining to z-scores will be needed to show the relationship between the correlation coefficient and the angle between two vectors. It is that of the magnitude of the vector formed by the z-scores of a data set. This notion will be easy to formulate but must await some basic concepts pertaining to vectors.

Vectors and Their Relevant Properties

What Is a Vector?

A vector is a directed line segment. A vector in the Cartesian plane is called a 2-dimensional geometric vector; a vector in Cartesian 3-space is called a 3-dimensional geometric vector. If the vector's initial point is at the origin of the Cartesian coordinate system, then the vector is in standard form. A vector not in standard form with initial point A and terminal point B is denoted $\mathbf{AB}$ (in bold case). For convenience a vector may also be denoted as a single letter in bold case; for example, $\mathbf{v}$. If a 2- or 3-dimensional vector is in standard form, then it is determined by its terminal point. This leads to the following algebraic definitions for these vectors: a 2-dimensional vector is an ordered pair of real numbers (a, b); a 3-dimensional vector is an ordered triple of real numbers (a, b, c). Two- and 3-dimensional vectors can be represented geometrically. For example, the vector $\mathbf{v} = (4, 3)$ is illustrated in Figure 1.

Figure 1



Although we cannot picture more than three dimensions, the notions pertaining to algebraic vectors can be extended to any number of dimensions. An n-dimensional vector is defined as an ordered n-tuple $(x_1, x_2, \ldots, x_n)$ of real numbers. An n-dimensional vector is said to have n components. The i^{th} component is x_i.

The Length or Magnitude of a Vector

The length of a 2- or 3-dimensional vector can be determined easily using the formulas for the distance between a pair of points in 2- or 3-space, respectively. For example, the length of the vector $\mathbf{v} = (4, 3)$ in Figure 1 is $\|\mathbf{v}\| = \sqrt{4^2 + 3^2} = 5$. The magnitude of an algebraic vector is defined as being equal to the length of its corresponding geometric vector. So the terms *length* and *magnitude* are interchangeable. If $\mathbf{v} = (a, b)$ $a, b \in \mathbf{R}$, where $\mathbf{R}$ is the set of real numbers, then the magnitude of $\mathbf{v}$, $\|\mathbf{v}\|$, is defined by $\|\mathbf{v}\| = \sqrt{a^2 + b^2}$; if $\mathbf{v} = (a, b, c)$, $a, b, c \in \mathbf{R}$, then $\|\mathbf{v}\| = \sqrt{a^2 + b^2 + c^2}$. These notions can be extended to n-dimensional vectors: if $\mathbf{v} = (x_1, x_2, \ldots, x_n)$, then

$$\|\mathbf{v}\| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2} \quad [5]$$

Of course, one cannot picture (at least in a sober state) the length of an n-dimensional vector if $n > 3$, but this definition is a reasonable abstraction consistent with our intuitive understanding of the lengths of 2- and 3-dimensional vectors.

The Inner (Dot Product) of Two Vectors

A number of operations are defined on vectors. Among them are two important products that involve pairs of vectors: the inner product and the cross product. Both have important applications as well as theoretical value. The one of relevance here is the inner product because it is useful in defining the angle between two vectors. If $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$, then the inner product $x \cdot y$ of x and y is

$$x \cdot y = x_1 y_1 + x_2 y_2 + \dots + x_n y_n = \sum xy \qquad [6]$$

For example, if $x = (4, 3)$ and $y = (5, 6)$, then $x \cdot y = 4 \cdot 5 + 3 \cdot 6 = 38$.

The Angle Between Two Vectors

Consider the vectors $x = (4, 3)$ and $y = (5, 6)$ as illustrated in Figure 2. One can use the law of cosines to determine the angle θ between x and y:

$$||AB||^2 = ||x||^2 + ||y||^2 - 2||x|| \, ||y|| \cos\theta$$

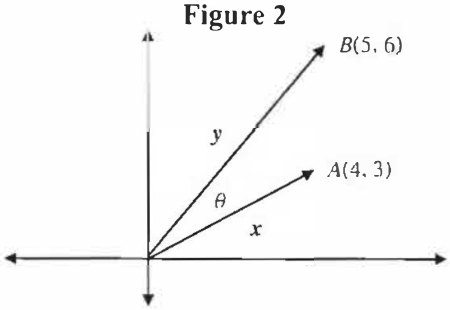
$$\Rightarrow (5-4)^2 + (6-3)^2 = (4^2 + 3^2) + (5^2 + 6^2) - 2\sqrt{4^2+3^2}\sqrt{5^2+6^2}\cos\theta$$

$$\Rightarrow 10 = 86 - 2 \cdot 5\sqrt{61}\cos\theta$$

$$\Rightarrow \cos\theta = \frac{76}{10\sqrt{61}}$$

$$\Rightarrow \theta \cong 13.32^\circ$$

Figure 2



Consider the general case for the angle between a pair of 2-dimensional vectors $x = (a, b)$ and $y = (c, d)$ in standard position as illustrated in Figure 3. Then, as in the previous example, $||AB||^2 = ||x||^2 + ||y||^2 - 2||x|| \, ||y|| \cos\theta$

$$\Rightarrow \cos\theta = (||x||^2 + ||y||^2 - ||AB||^2) / (2||x|| \, ||y||)$$

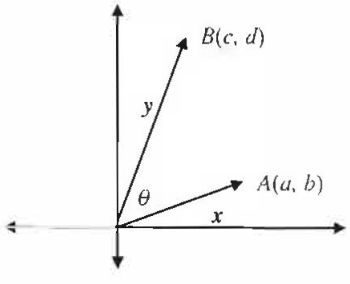
$$\Rightarrow \cos\theta = (a^2 + b^2 + c^2 + d^2 - ((c-a)^2 + (d-b)^2)) / (2||x|| \, ||y||)$$

$$\Rightarrow \cos\theta = (a^2 + b^2 + c^2 + d^2 - c^2 + 2ac - a^2 - d^2 + 2bd - b^2) / (2||x|| \, ||y||)$$

$$\Rightarrow \cos\theta = (ac + bd) / (||x|| \, ||y||)$$

$$\Rightarrow \cos\theta = (x \cdot y) / (||x|| \, ||y||) \qquad [7]$$

Figure 3



The result [7] provides a simple way of thinking about the angle θ between a pair of 2-dimensional vectors: the cosine of θ is just the inner product of the two vectors divided by the product of their magnitudes. Consider again the two vectors $x = (4, 3)$ and $y = (5, 6)$. Using [7] the angle θ between the two vectors is given by

$$\cos\theta = (x \cdot y)/(||x|| \, ||y||) = (4 \cdot 5 + 3 \cdot 6)/(\sqrt{4^2+3^2}\sqrt{5^2+6^2})$$

$$= 38 / (5\sqrt{61}).$$

This is the same value as that obtained earlier by more laborious methods.

The result [7] applies to 3-dimensional vectors as well. This can be seen by applying the law of cosines to a pair of 3-dimensional vectors $x = (a, b, c)$ and $y = (d, e, f)$. The steps are the same as those used above for 2-dimensional vectors. Verification of this result is left to the reader.

Although one cannot visualize the angle between two n-dimensional vectors for $n > 3$, it is reasonable to think of the angle between such a pair of vectors as a generalization of the angle between 2- or 3-dimensional vectors. This leads to the following definition. If $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$ are any pair of n-dimensional vectors, then the angle θ between them is defined by

$$\cos\theta = (x \cdot y) / (||x|| \, ||y||) \qquad [8]$$

Consider a simple example of a pair of 5-dimensional vectors $x = (1, 2, 3, 4, 5)$ and $y = (2, 4, 6, 8, 10)$. The vectors x and y have an obvious intuitive relationship to each other. In the precise language of vector algebra, y is said to be a scalar multiple of x. In general, a vector y is said to be a scalar multiple of vector x if each component of y is obtained from the corresponding component of x by multiplying by the same constant or scalar. In this case the constant is 2 and we write $y = 2x$. It is easy to appreciate why two n-dimensional vectors that are positive scalar multiples of each other are defined to have the same direction. Thus, in the above example, the vectors x and y should have the same direction and the angle between them should be 0°. Using definition [8] as follows yields a result that is consistent with this.

$$\cos\theta = (x \cdot y) / (||x|| \, ||y||)$$

$$\Rightarrow \cos\theta = (1 \cdot 2 + 2 \cdot 4 + 3 \cdot 6 + 4 \cdot 8 + 5 \cdot 10) / (\sqrt{1^2+2^2+\dots+5^2}\sqrt{2^2+4^2+\dots+10^2})$$

$$\Rightarrow \cos\theta = = 110 / (\sqrt{55}\sqrt{220}) = 110 / (\sqrt{110^2}) = 1$$

$$\Rightarrow \theta = 0^\circ$$

Data Sets as Vectors

With this framework, it is easy to see that a set of data can be represented as a vector. Consider the two data sets $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ represented as vectors. Then it should be possible to formulate the correlation coefficient in terms of the vector concepts outlined in section 4 above. It turns out that the formulation is simpler if each set of scores is first converted to standard form; that is, each x_i and y_i is first converted to a z-score. We will refer to these vectors as the standard score vectors of x and y and denote them $z_x = (z_{x_1}, z_{x_2}, \ldots, z_{x_n})$ and $z_y = (z_{y_1}, z_{y_2}, \ldots, z_{y_n})$, respectively. Using [8] the angle θ between z_x and z_y is given by

$$\cos\theta = (z_{x_1} z_{y_1} + z_{x_2} z_{y_2} + \ldots + z_{x_n} z_{y_n}) / (\|z_x\| \, \|z_y\|)$$

$$\Rightarrow \cos\theta = z_x \cdot z_y / (\|z_x\| \, \|z_y\|)$$

$$\Rightarrow \cos\theta = \sum z_x z_y / (\|z_x\| \, \|z_y\|) \quad [9]$$

We encountered the numerator of the right side of [9] earlier: it is also the numerator of the correlation coefficient in [4]. Let us examine the denominator $\|z_x\| \, \|z_y\|$. It can be shown that for the standard score vector z_x of any n-dimensional vector x, $\|z_x\| = \sqrt{n}$ as follows. Note that

$$\|z_x\| = \sqrt{z_{x_1}^2 + z_{x_2}^2 + \ldots + z_{x_n}^2} = \sqrt{\sum z_{x_i}^2}$$

$$\text{But } z_{x_i} = \frac{x_i - \bar{x}}{s_x} \text{ and } s_x = \sqrt{\frac{\sum (x - \bar{x})^2}{n}}$$

$$\text{So } z_{x_i} = \frac{x_i - \bar{x}}{\sqrt{\frac{\sum (x - \bar{x})^2}{n}}} = \frac{(x_i - \bar{x})\sqrt{n}}{\sqrt{\sum (x - \bar{x})^2}}$$

Then from [5], $\|z_x\| = \sqrt{\frac{(x_1 - \bar{x})^2 n}{\sum (x - \bar{x})^2} + \frac{(x_2 - \bar{x})^2 n}{\sum (x - \bar{x})^2} + \ldots + \frac{(x_n - \bar{x})^2 n}{\sum (x - \bar{x})^2}}$

$$\Rightarrow \|z_x\| = \sqrt{\frac{\sum (x - \bar{x})^2 n}{\sum (x - \bar{x})^2}} = \sqrt{n} \quad [10]$$

Since z_x and z_y are both standard score vectors, $\|z_x\| = \sqrt{n}$ and $\|z_y\| = \sqrt{n}$. Thus [9] becomes $\cos\theta$

$$= \frac{\sum z_x z_y}{n} \quad [11]$$

The right side of [11] is the correlation coefficient between the set of scores x and y shown in formula [4]. Thus we have the result that has been the object of this article: the correlation coefficient between two sets of scores is just the cosine of the angle between their standard form vectors.

Applying formula [11] to the special cases where $\theta = 0°$, $90°$ and $180°$ and noting that $\cos 0° = 1$, $\cos 90° = 0$ and $\cos 180° = -1$ yields the following intriguing results about the value of the correlation coefficient r between the standard score vectors of two sets of scores:

* $r = 1$ if and only if the standard score vectors have the same direction.
* $r = -1$ if and only if the standard score vectors are in opposite directions.
* $r = 0$ if and only if the standard score vectors are perpendicular.
* r has a value between 0 and 1 if and only if the standard score vectors are somewhere between perpendicular and in the same direction.
* r has a value between 0 and −1 if and only if the standard score vectors are somewhere between perpendicular and in opposite directions.

Conclusion

This article demonstrates the relationship between correlation coefficient and the angle between two vectors. The beauty of this relationship is that it provides a simple geometrical meaning for the correlation coefficient that appeal to the intuition. There is also beauty and satisfaction in the processes underlying the discovery and development of this relationship. The exploratory and deductive methods used illustrate how mathematical connections are discovered and verified. Mathematics teachers who look for connections are in a good position to uncover such connections by virtue of the intimate knowledge of the subject matter that accompanies teaching. Further, they can model both the excitement and the discipline that is involved in carrying the discovery process to its conclusion. Teachers who are captivated by the exploration process will find ways to allow students to be captivated as well.

Notes

1. The two sets of data are presented here in vector notation; that is, as ordered n-tuples. This is a convenient notation and appropriate for the purposes of this article.

2. The formulations for standard deviation and the correlation coefficient used in this article are those pertaining to a whole population rather than a sample. Using n rather than the usual n-1 makes the demonstration of the relationship between the correlation coefficient and the angle between two vectors more transparent. But it is possible to demonstrate the relationship using n-1 as well.

3. The reader will recall that the standard deviation s_x for the data set $x = (x_1, x_2, \ldots, x_n)$ is a measure of how the data is distributed about the mean $\bar{x}$. It is defined as follows.

$s_x = \sqrt{\frac{\sum (x - \bar{x})^2}{n}}$ and can be described as the root of the mean of the squares of the deviations of the individual scores from the mean.

Reference

Ferguson, G. A. 1981. *Statistical Analysis in Psychology and Education*. 5th ed. New York: McGraw-Hill.

Note: The vector and statistics concepts underlying and related to this article can be found in any introductory linear algebra textbook or introductory statistics textbook.

Murray L Lauber is an associate professor of mathematics in the Augustana Faculty of the University of Alberta in Camrose, Alberta. He has taught a variety of courses including precalculus, introductory calculus, linear algebra, discrete mathematics, history of mathematics and higher arithmetic. He believes that mathematics is a potent tool for expanding the intellectual capacities of all students. He shares his passion for mathematical exploration and discovery with his students and his colleagues and peers, particularly through his writing. Many of his articles, in which he shares intriguing mathematical relationships that he has uncovered during the course of his teaching, have been published in delta-K *and in the* Mathematics Teacher, *a publication of the National Council of Teachers of Mathematics.*

When Technology Integration Goes to Math Class

Brenda Dyck

This article previously appeared in Meridian, *a Middle School Computer Technologies Journal, Vol 6, No 2, Summer 2003. Minor changes have been made to fit ATA style.*

> The principal goal of education is to create men who are capable of doing new things, not simply of repeating what other generations have done—men who are creative, inventive and discoverers. The second goal of education is to form minds that can be critical, can verify, and not accept everything they are offered; we need pupils who are active, who learn early to find out by themselves, partly by their own spontaneous activity and partly through materials we set up for them; we learn early to tell what is verifiable and what is simply the first idea to come to them.
>
> —*Jean Piaget*

> I think you should learn, of course, and some days you must learn a great deal. But you should also have days when you allow what is already in you to swell up inside of you until it touches everything. If you never take time for that to happen, then you just accumulate facts, and they begin to rattle around inside you. You can make noise with them, but never really feel anything with them. It's hollow.
>
> —*From* The Mixed-Up Files of Mrs. Basil E. Frankweiler *by E L Konigsburg, 1967*

Math classes from my learning past had a definite cookie-cutter appearance: rows of desks, lined scribblers, pencils, textbooks and the teacher at the front. These were housed within the most predictable of all, a quiet classroom. Math instruction is well suited to a traditional teaching format. Because of the logical and sequential nature of math, it often attracts teachers whose thinking and learning styles match the subject. Therefore while other teaching disciplines are branching out to encompass a constructivist style of instruction that is full of collaboration and technology integration, many middle and high school math teachers continue to use a more teacher-centred approach.

As a math teacher, I believe change is on the horizon. Language arts and social studies colleagues are embracing the power of the Web to push their students' creative and critical thinking skills, and many math teachers are looking for ways to enhance the curriculum using digital media. Realizing that digital media can facilitate critical thinking and higher-order learning, teachers are looking for math-related online projects and resources that will help students and challenge their thinking skills.

Telecollaborative Projects

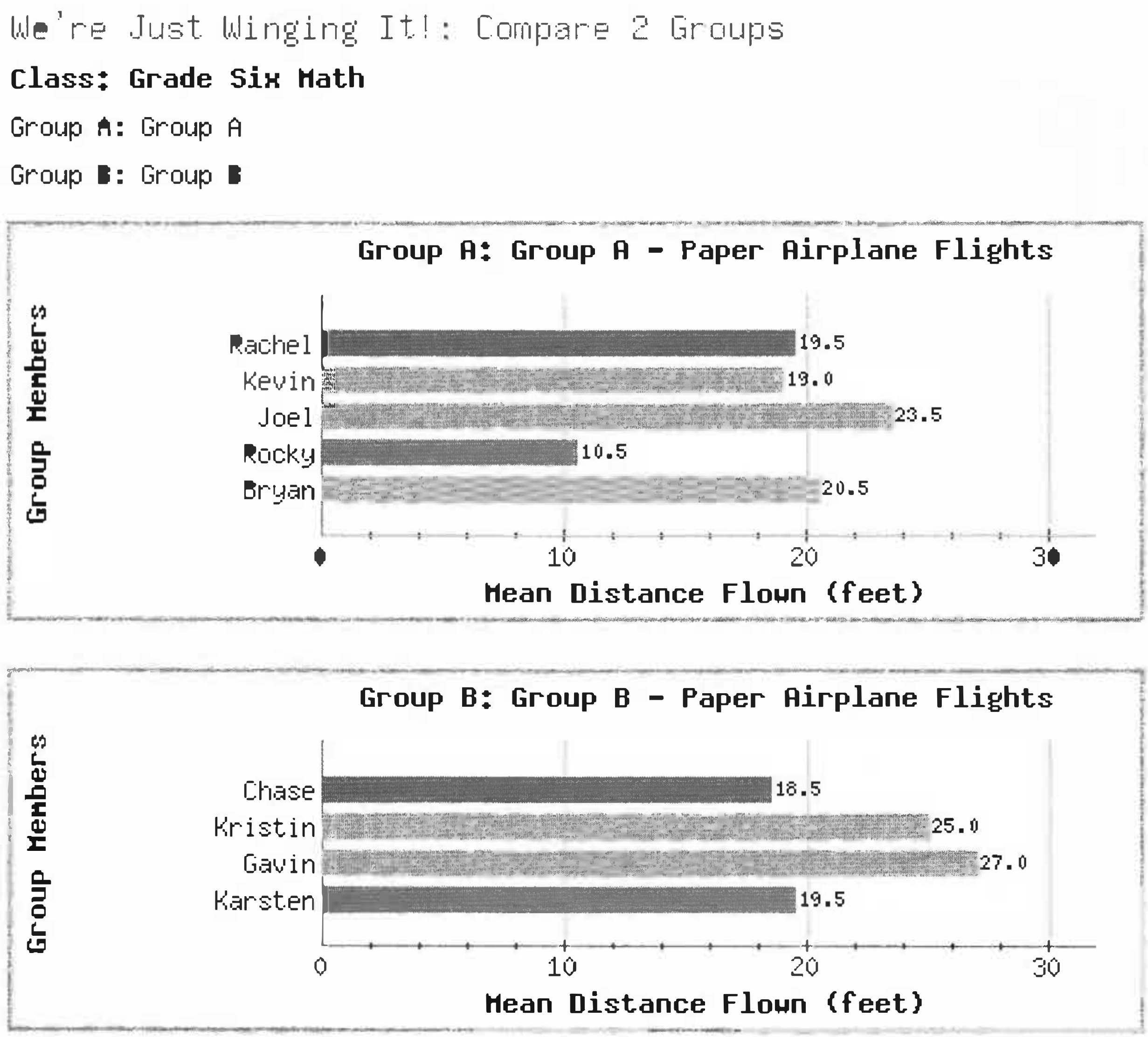
Statistics: A Curiosity Factor (www.masters.ab.ca/bdyck/Staff/Olson2) was my first attempt at integrating telecollaborative project work into math class. I had developed many language arts and social studies-based telecollaborative projects to connect learners in other countries. Shared learning projects could challenge students' critical thinking skills, engage their interest and expand their global perspective while covering curriculum requirements. The question was, how could I use this style of instruction in math class? Using the unit on Collecting and Analyzing Data as a jumping-off point, I began looking for Internet resources that would add pizzazz to a unit that had, in my class, been traditionally textbook driven. I uncovered an abundance of exciting statistical resources that would grab student interest:

- Articles that shed light on how numbers can inform or misinform readers
- Online surveys that explored hot topics, such as spam and property rights in cyberspace

- The Gallup Poll's webpage containing information on how the Gallup Organization uses polls to predict trends and inform the public. This site is loaded with videos examining everything from cloning to those sticky ethical questions that students love to debate.
- An online site that turned student data into a variety of colourful graphs with just a click of the mouse
- An array of sites that provided up-to-date information on topics that interest all kinds of learners

Using these resources, students developed a deeper understanding of how numbers can lead or mislead the usefulness of unbiased data, the art of creating a good survey question and how to analyze data and present the results effectively. For examples, see the Student Work section of the website, Statistics: A Curiosity Factor. Without question, using technology engaged them in a way that textbook graphs and data charts never did. Knowing that their learning would be online encouraged the students to put more effort into their work and to increase their global perspective as schools from Ohio, Florida, Pennsylvania, Texas and Canada joined in to share survey results with each other.

Several years ago, Houghton Mifflin's Education Place website, which no longer exists, contained a resource called the Data Place. After registering, teachers had access to grade-appropriate collaborative projects whereby students collected and worked with real data by analyzing and drawing conclusions. Not only did students compare data from their own classrooms, they had access to a data bank containing project results from classrooms worldwide. The



activities were interactive, imaginative and thought provoking.

We encountered the Data Place through a project called, We're Just Winging It! In this project students made paper airplanes, gathered data about how far the airplanes flew, compared results with the class and then, using the graphs created on the Data Place website, compared their results with their peers and other Data Place users from other countries, such as Thailand and Australia.

According to the students, the best part of this project was throwing the paper airplanes down the hall. Surprisingly, these enthusiastic data collectors were totally on task and meticulous about measuring the distance the planes flew. Students took their results and, using an online metric converting tool, changed their metric measurements into the American Imperial Measurement system. From here they calculated the mean distances (individually and as a class) and entered their data into the Data Place website. Everyone was delighted with the colourful graphs that appeared within seconds.

During the following class, I hooked up an LCD projector. The students and I analyzed the graphs and discussed the variables that would have made some airplanes fly farther than others. Student thinking was evident as they suggested that flying distances could have been affected by differences in size, how airplanes were folded, weight of the paper, unexpected breezes in the hall, direction of breezes, styles of throwing or the humidity in the air.

To extend the learning gleaned during the We're Just Winging It! activities, I created a telecollaborative project called Come Fly With Me! (www.masters.ab.ca/bdyck/Fly). This hands-on, technology-supported project merged data collection and analysis skills with a science unit on flight. Throwing airplanes down the hall, using technology to analyze the learning, competing with students across the world and having their math class results on the Web made for one of the best math classes of the year!

Brenda Dyck is a sessional instructor in the Faculty of Elementary Education at the University of Alberta. Her Hot Links column is a regular feature in Middle Ground Magazine, *a publication of the National Middle School Association (NMSA). Brenda's book,* The Rebooting of a Teacher's Mind, *was recently published by NMSA. Brenda is also a teacher-editor for* MidLink Magazine *(www.cs.ucf.edu/~MidLink) and serves on the provincial executive for the Middle Years Council of the Alberta Teachers' Association (MYCATA).*

Ready, Set, Decorate!

Abbey Alexander

Mathematical Concept

Surface area and volume of prisms, cylinders and cones

Grade

Mathematics 9

Purpose

Culminating activity for the shape and space unit

Objective

Students will apply their knowledge and understanding of the area of two-dimensional shapes and the surface area and volume of prisms, cylinders and cones to a room design or set-up activity.

Concepts Addressed

(From Alberta Education Program of Studies)
Shape and Space (Measurement)
Describe and compare everyday phenomena, using either direct or indirect measurement

General Outcome

Describe the effects of dimension changes in related two-dimensional shapes and three-dimensional objects in solving problems involving area, perimeter, surface area and volume

Specific Outcomes

- Relate expressions for volumes of pyramids to volumes of prisms, and volumes of cones to volumes of cylinders
- Calculate and apply the rate of volume to surface area to solve design problems in three dimensions
- Calculate and apply the rate of area to perimeter to solve design problems in two dimensions

Materials

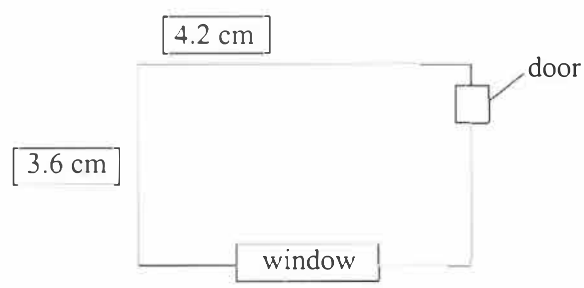
- Copy of activity (see Description)
- Blank pieces of paper (two per student)
- Overhead transparency of activity
- Pencil and eraser
- Calculator if needed
- Two worksheets of room to decorate and set up (one for rough copy, one for good copy)

Vocabulary

Surface area—the amount of material required to cover an object
Volume—the amount of space that an object occupies
Feature wall—a wall in a room that stands out from the rest

Description

Scenario: Your parents and/or guardians are renovating their home, and you must decide how to renovate your bedroom. Your bedroom is 4.2 m long, 3.6 m wide and 3 m high. The door (2.5 cm × 0.5 cm) and window (1 cm × 1.5 cm) are marked in the diagram below (scale: 1 cm = 1 m).



Decorate and set up your bedroom by doing the following:

1. Decide on the paint colour and quantity. Choose any colour you want; paint all walls the same colour, each a different colour or have a feature wall if you like. You must justify the amount of paint you need (surface area).
2. Hang a minimum of three pictures of any dimension or shape in your room. For each picture, justify why you chose the particular dimensions and shape, and why you hung it in a certain place (area in 2-D).
3. In your bedroom set-up, you must have a minimum of one window, one bed, one dresser, one garbage

can, one lamp and lamp shade, one hanging light fixture and four trinkets.

4. First figure out how much space you have to work with (volume). Then decide what the items in your bedroom will look like and include dimensions. Use at least one of each 3-D shape listed below:
   - cube
   - square pyramid
   - triangular pyramid
   - rectangular prism
   - rectangular pyramid
   - cone
   - triangular prism
   - cylinder

   Note: For some students, building nets may be a useful strategy to employ.
5. Indicate how much space each item in your bedroom takes up.
6. When you are done setting up your bedroom, indicate the total space your set-up occupies.
7. When you have completed the activity, we will showcase the bedrooms for all to see.

For the Student

Complete a rough copy of your room. Then do a good copy to hand in, and write your name, date and class on the back. Accompanying your room design on a separate piece of paper, provide the following:

Paint

- The paint colour you chose (provide a sample if possible) and, if more than one colour was used, indicate the colour of each wall
- How much paint you needed and your justification for the amount of paint needed; that is, surface area

Pictures

- The dimensions and shapes of pictures you hung in your room and justification for your decisions; that is, how much area the picture took up on the wall versus how much area you had to work with
- Explain why pictures were hung in certain places

Set-Up

- The amount of space you had to work with in your room; that is, volume
- The dimensions you used for each item and how much total space each item occupied
- The total amount of space all your items used and how much space was left over

Grading Rubric

Level	Descriptors
A	The work is exceptional and exceeds minimum expectations of the project. Justifications for choices are clear and logical. The student demonstrates initiative, creativity, insight and ability to solve problems.
B	The work is generally of high quality. It is accurate and meets minimal requirements. Most justifications are clearly stated. However, the project is not creative or insightful in the judgment of the teacher and problem-solving skills are not exceptional.
C	The work is adequate but unexceptional. Significant errors in understanding, ability to problem solve, superficial justification or poorly described ideas are evident.
D	The work is inadequate or nonexistent. No requirements are met.

Abbey Alexander is in her final practicum in the University of Lethbridge bachelor of education program. She is interested in how we can make mathematics fun and intriguing for students and teachers. She became interested in education because of her high school math teacher and her experience coaching a girls' rep soccer team.

Factoring Trinomials: A Student's Perspective

Duncan E McDougall and Pega Alerasool

It is when we least expect it that a student will make an observation about a conventional method that we virtually take for granted. In this case, it was factoring the trinomial form $ax^2 + bx + c$, where the factors of 'ac' add up to 'b' by decomposition. In particular, I presented $3x^2 - 2x - 8$ for consideration. The question posed was, "How can we factor this if the 3 is not a perfect square?" Pega Alerasool, an industrious student with a different view of this procedure, did not understand the conventional algorithm as it had been presented to her. It was clear that my perception of this method and hers were very different. So after explaining that the 3 or any other coefficient of the x^2 term didn't have to be a perfect square, she asked, "Why can't it be a perfect square?" That is, why couldn't the coefficient of x^2 always be a perfect square? Now I was curious and asked her to explain her approach. In essence, let $A = 3x^2 - 2x - 8$ and then multiply both sides by 3 in order to make the coefficient of x^2 a perfect square. The advantage here is that when we apply decomposition, we do not have to worry about the positioning of the constant terms. The illustrated format looks like this:

Factor	$3x^2 - 2x - 8$	Given
Let	$A = 3x^2 - 2x - 8$	Substitution
then	$3A = 9x^2 - 2(3)x - 24$	Multiplication
	$3A = (3x)^2 - 2(3x) - 24$	
	$3A = (3x - 6)(3x + 4)$	$9x^2 = 3x \bullet 3x$ and the factors of -24 whose sum is -2 are - 6 and 4

As with previous methods of factoring, the position of the variable and constant terms is crucial. Here, however, we no longer have this problem. Since the numerical coefficient of the x term is the same in both sets of parentheses, it does not matter where we place the constant terms, factors, in this case -6 and 4. Continuing this process we have:

$3A = (3x - 6)(3x + 4)$	from above
$3A = 3(x - 2)(3x + 4)$	factor out the GCF
$A = (x - 2)(3x + 4)$	division

In general, the sequence looks like this:

Factor	$ax^2 + bx + c$	where the factors of ac add up to be factorable
Let	$A = ax^2 + bx + c$	by substitution
Then	$Aa = a^2x^2 + abx + ac$	by multiplication to make the coefficient of x^2 a perfect square
Now	$Aa = (ax + a)(ax + c)$	factors of ac are a and c regardless of order
	$Aa = a(x + 1)(ax + c)$	factor out the GCF
	$A = (x + 1)(ax + c)$	division by a

The only real question remaining is what to do when the coefficient of x^2 in a given question is already a perfect square. As tempting as it is to factor "as is," the procedure doesn't work as shown below (Example 1). We must still multiply the coefficient of x^2 by itself so that we can get the constant factors to work properly, as in (Example 2).

Example 1

$4x^2 - 5x + 1$

does not work as there are no values a and b for which the form $(2x-a)(2x-b)$ works.

Example 2

Let	$A = 4x^2 - 3x - 1$	substitution
	$4A = 16x^2 - 3(4)x - 4$	multiplication by 4
	$4A = (4x - 4)(4x + 1)$	$16x^2 = 4x \bullet 4x$ and $-4 = -4 - 1$ and $-3 = -4 + 1$
	$4A = 4(x - 1)(4x + 1)$	factor out the GCF
	$A = (x - 1)(4x + 1)$	division

The above was Pega's take on factoring by decomposition. This is how it made sense to her. If it appeals to other students like Pega, then it may become an alternative method of factoring. Try it, you'll like it!

Duncan McDougall has been teaching for 27 years, including 13 years of teaching in the public school systems of Quebec, Alberta and British Columbia. During the past 15 years, he has taught mathematics to high school and university students and to elementary school teachers. He owns and operates TutorFind Learning Centre in Victoria, British Columbia. Pega Alerasool is in her third year as a coop student in mechanical engineering. The work presented in this article occurred when she was a member of a high-energy Grade 11 honours math class that savoured all the material Duncan McDougall presented to them. She was a keen student who liked to tinker with methods and techniques demonstrated in math and science classrooms.

Travelling to South Africa: An Application Task

Kim Runnalls

Topic: Time zones

Objectives

- Identify different time zones.
- Determine the relationship between various time zones.
- Research and plan how to travel between time zones in an allotted period of time and budget.
- Convert values of time into different time zones.

Grade Level: 7

Materials

- World map with marked time zones
- Computers with Internet connections (www.timeanddate.com)
- Problem sheet (one copy per student)

Task Description

1. Students will be given the problem sheet and must decide without teacher instruction how to tackle the problem.
2. Using the flight listings sheet and the Internet, students must choose a satisfactory flight itinerary.
3. Students write a rationale for why they chose their itinerary by using descriptions and justifications of the time zone changes, layover and in-flight times, and price.
4. The rationale with a printout of the flight itinerary must be submitted to the teacher for assessment.

Mathematical Concepts: time zones, time zone adjustments, time zone determination and unevenness

Vocabulary: itinerary, time zone, layover/stops, EST, GMT, International Date Line

Adaptations: If students are struggling, they can practise converting between time zones by using a time zone converter online (www.timeanddate.com/worldclock/converter.html). Having Erin fly from Calgary to Capetown rather than going across the International Date Line can make the task easier. Prices can be altered so that exchange rates must be used to increase the difficulty of budgeting. Also, task details can be changed so that Erin's starting location matches wherever the students completing the task live. This makes the task relevant to any student.

Rationale: By applying the concept of time zones to a highly authentic alternative context, students see the value and need for this mathematical knowledge. This task is highly significant in that it takes on the form of vicarious relevance. This full-class task requires students to integrate and synthesize their knowledge of time zones, number sense and budgeting. Drill and practice text questions (which typically are used for teaching these concepts) cannot attain this level of relevance or require this higher taxonomic level of thinking; whereas a task like this can. Finally, a problem task such as this provides students with a feeling of success and achievement because their work is purposeful and their solution is actually useful for something other than just a mark in the grade book.

Teacher Reflection: This task can be quite engaging as long as the students have a clear idea as to the research/decision-making process and what is expected of their paragraph rationales. This requires the teacher to take time to fully review the new vocabulary terms, such as *itinerary*, and go through exactly how flights are booked and how to read flight itineraries (that is, what do stops mean, what times and time zones are usually given, boarding times and so on). If this is covered well, then students get the idea of how to understand flight plans, how to match them with their flight needs and then how to realistically see how time zones apply to life.

Problem Sheet

Erin Runnalls is doing a law internship this summer in Cape Town, South Africa. She will be working in Cape Town starting July 10, 2006, and living there until the end of August 2006. Before she starts work, she would like to fly from Calgary, Alberta, to Auckland, New Zealand, around June 15 so she can do some travelling before she heads to South Africa for the rest of the summer. She wants to fly from Auckland to Cape Town to be at her job for July 10.

Important things to take note of:

- Erin prefers to fly on well-known airlines with good reputations.
- The maximum total budget for flight costs (including the flight to Auckland plus the flight to South Africa) is C$6,500.
- Although her job doesn't start until July 10, it would be nice if she had some time to settle in her new house before work starts.
- Erin would like to spend as much time in Auckland as possible.
- Erin doesn't like to be rushed to catch flights in layover destinations.
- More direct flights with fewer layovers are better for Erin.

Through your Internet researching, you must take note of flight times, destination time zones and budget to find an itinerary that satisfies Erin's travelling needs.

After you have found and printed an itinerary, you must write a paragraph explaining why your combination of flights will mathematically work and be appropriate for Erin, while also answering the following questions:

1. Erin's mother wants her to phone home whenever she arrives in a destination. List each of Erin's flight destinations (including layover destinations) and what time it would be at home when she calls her mom in each of these locations.
2. Once Erin is living in Cape Town, what would be an appropriate time for Erin to call her mother so that it's a convenient time for both of them?

Rubric

Score / Criteria	In-Province Traveller (1–3 marks)	The Tourist Destination Traveller (4–6 marks)	Far Reaches of the World Traveller (7–9 marks)
Time and Date Conversions	Most time and date conversions are incorrect.	Most time and date conversions are correct.	All time and date conversions are correct.
Itinerary	The itinerary chosen satisfies few to none of the parameters.	The itinerary chosen satisfies most of the parameters.	The itinerary chosen satisfies all of the parameters.
Rationale	The rationale is not well thought out and fails to support most of the decisions made.	The rationale is thoughtful and supports most of the decisions made.	The rationale is thoughtful and supports all of the decisions made.

The total score for the application task is out of a total 27 marks.

Kim Runnalls is in her final year at the University of Lethbridge, completing a mathematics education combined degree. She is passionate about mathematics and tries to stimulate an interest in math in every child. She strives to grow and develop her understanding and skills by teaching mathematics in a relevant experiential manner rather than through traditional rote textbook work.

Reducing Fractions and Its Application to Rational Expressions

Duncan E McDougall

There are various reasons a given student doesn't learn or master a presented method or technique. Teachers are aware of diverse learning styles and conditions in the classroom. But sometimes despite hard work and willingness to learn on the part of the student, the objective is not met. What do we do? Alternative approaches to a problem are often sought, and this is where thinking outside the box may come in handy, especially when a novel idea works and appeals to others.

Imagine the plight of high school math students whose factoring skills are less than adequate. They may find factoring a chore because of a lack of success with a conventional method. Students might need another approach to help them succeed.

Consider reducing the fraction 39/65 to lowest terms. If we know that 13 is a factor of both 39 and 65, then we can write it as $\frac{39}{65}=\frac{13\times 3}{13\times 5}=\frac{3}{5}$. The educator knows that 13 is the greatest common factor, but the student may not. Similarly, how would it be apparent to a student in an expression like $\frac{x^2-2x-3}{x^2-7x+12}$?

Previously, I have demonstrated the premise that the only possible factors available to reduce a fraction to lowest terms come from the difference between the numerator and the denominator (McDougall 1990). Numerically, it looks like this:

Reduce $\frac{39}{65}$

Steps:

(1) 65 – 39 = 26 demonstrates the difference between the numerator and the denominator

(2) Factors of 26 1, 2, 13, 26

(3) Disregard 1. Disregard 2 and 26 because they are even

(4) Try 13. If 13 doesn't work, then nothing else will.

(5) $\frac{39}{65}\div\frac{13}{13}=\frac{3}{5}$

When transferring this concept to rational expressions, examine the following two examples:

Example 1

Reduce $\frac{x^2-2x-3}{x^2-7x+12}$

Steps:

(1) $(x^2-2x-3)-(x^2-7x+12)$

$=x^2-2x-3-x^2+7x-12$

$-5x-15$

$=5(x-3)$

(2) Disregard 5 and consider (x–3) because 5 doesn't divide evenly into the numerator or the denominator but (x–3) might.

(3) $x^2-2x-3=(x-3)(x+1)$ and $x^2-7x+12=(x-3)(x-4)$

At this stage, we can factor more easily because we know that (x–3) is one of the desired factors; failing that, use long division

(4) $\frac{x^2-2x-3}{x^2-7x+12}=\frac{x+1}{x-4}$

Example 2

Reduce $\frac{x^3-1}{x-1}$

Steps:

(1) $(x^3-1)-(x-1)$

$=x^3-1-x+1$

$=x^3-x$

$=x(x^2-1)$

$=x(x-1)(x+1)$

(2) Disregard x and (x+1) but consider (x–1) because neither x nor (x+1) divide evenly into neither the numerator nor the denominator; the only remaining factor to consider is (x–1).

(3) $x^3 - 1 = (x-1)(x^2+x+1)$

(4) $\frac{x^3-1}{x-1} = \frac{(x-1)(x^2+x+1)}{x-1}$
$= x^2 + x + 1$

What I also like about this method is that we can discover which factors will not work in a given situation:

Consider $\frac{x^2-x-12}{x^2-x-6}$ **Steps:**

(1) $(x^2-x-12)-(x^2-x-6)$
$= x^2 - x - 12 - x^2 + x + 6$
$= -6$

Immediately we can see that there cannot be a common factor of the form (x+a) (assuming that the algebra is done correctly of course).

As we can see, there is no point in factoring and looking for a common term if indeed none exists to begin with. Now it's no secret that a method for finding the GCF of two polynomials does exist, but it does involve long division, and therefore, it would look something like this:

Find the GCF for x^2-2x-3 and $x^2-7x+12$, or GCF $(x^2-2x-3, x^2-7x+12)$.

Divide one into the other, and keep track of the remainder. Now divide the remainder into the previous divisor, and again, keep track of the remainder. Continue this last step until the remainder is zero. The divisor, which gives zero as a remainder, is our GCF. This means we would have:

$$\begin{array}{r} 1 \\ x^2-2x-3 \overline{) x^2-7x+12} \\ \underline{x^2-2x-3} \\ -5x+15 = -5(x-3) \end{array}$$

Take only (x–3) because –5 is not a factor of the form (x+a) and because –5 doesn't divide evenly into either the numerator or the denominator.

$$\begin{array}{r} x+1 \\ x-3 \overline{) x^2-2x-3} \\ \underline{x^2-3x} \\ x-3 \\ \underline{x-3} \\ 0 \end{array}$$

Actually, finding the GCF this way is part of the reason the above method of subtraction works. The teacher now has more than one way of presenting this material to various types of learners and can provide alternatives for the reluctant student. A welcome application of this approach is the calculation of limits for the calculus student. In general, we have:

$$\lim_{x \to -a} \frac{x^2 + x(a+b) + ab}{x+a}$$

Instead of evaluating directly, and giving the indeterminate form $0/0$, we can subtract the two polynomials, factor this difference, and then try to reduce it to its lowest terms. This would create the following:

$$\begin{aligned}(x^2 + x(a+b) + ab) - (x+a) &= x^2 + ax + bx + ab - x - a \\ &= x(x+a) + b(x+a) - 1(x+a) \\ &= (x+a)(x+b-1)\end{aligned}$$

This reveals that (1) the expression can be reduced, and (2) (x+a) is the common factor. A numerical example would look like:

$$\lim_{x \to -2} \frac{x^2+3x+2}{x+2}$$

$$\begin{aligned}&(x^2+3x+2)-(x+2) \\ &= x^2+3x+2-x-2 \\ &= x^2+2x \\ &= x(x+2)\end{aligned}$$

$$= \lim_{x \to -2} \frac{(x+1)(x+2)}{x+2}$$

Disregard x and consider (x+2) because x doesn't divide evenly into the numerator or the denominator.

$$= \lim_{x \to -2} x+1$$
$$= -2 + 1$$
$$= -1$$

In summary, some students see math as a necessary evil. However, now they can get into it a bit more because someone has found a method that makes sense to them.

Reference

McDougall, D E. 1991. "Reducing Fractions." *International Journal of Mathematics Education and Science Technology* 22, no 4: 683–93.

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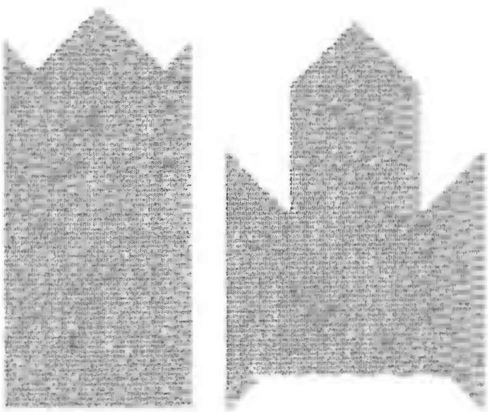
A Page of Problems

A Craig Loewen

Elementary

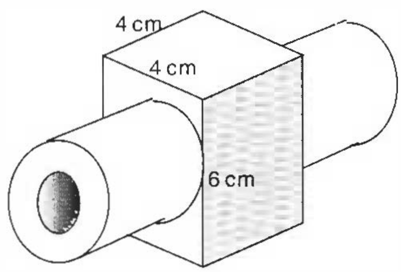
Build the silhouettes below using one seven-piece set of tangrams for each.



Launch Pad Tower and Space Shuttle

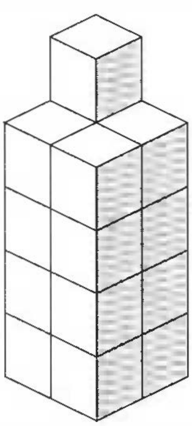
High School

If the hole passes completely through the cylinder, and the tube extends 4 cm on each side of the rectangular centre, calculate both the surface area and volume for the object below.



Outer Diameter = 4 cm
Inner Diameter = 2 cm

Middle School

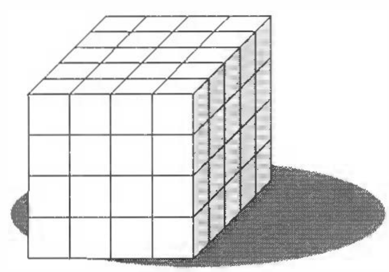


Build the shape shown on the left using exactly 16 linking cubes.

How many solutions can you find if you may use 16 or fewer cubes? What is the smallest number of cubes you could use to build this object?

Elementary

Several sugar cubes are stacked four wide, four high and five deep. The outside of the stack is then painted red.



How many of the sugar cubes are red on one side? How many are red on two sides? Three sides? Four or more sides? How many receive no paint at all?

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