FEATURE ARTICLES

The Teacher/Researcher and Proving in High School Mathematics

Shannon Sookochoff, Elaine Simmt and David Reid

Proof and proving is largely unsuccessful in school mathematics (Reid 1995). "This lack of success seems to be related to an incompatibility between the picture of proving portrayed in schools, and the role of deductive reasoning in professional mathematics and in students' lives" (PhD diss). Based on our reading of the high school program of studies and support materials, there is certainly a narrow view of what constitutes proof and reasoning, and little, if any, advice on how to teach and assess it. For example, the *Assessment Standards for Pure Mathematics 20* relegates reasoning to a set of seven seemingly simple outcomes[1] (Alberta Learning 2002), which some teachers choose to cover in a short take-home package.[2] Further, proof and reasoning is addressed in one of the two approved mathematics textbooks (Addison-Wesley) as formal and deductive geometric proofs that are based on a set of given and well-established geometric truths rather than based on and emergent from student meaning making. The teacher is left to develop instruction that engages students in mathematical reasoning. Interestingly, the current focus on proof as formal verification has been shown to be a poor motivator for students, since it does not address their needs when doing mathematics (Reid 1995). In this study, we explored possibilities for more meaningful and engaging experiences for high school students with mathematical proof by being attentive to their meaning making, their need for proof and their actions toward proving.

We investigated what happens when a teacher broadens her vision of proof and proving in mathematics. Reid's work points to a variety of possibilities for proving in mathematics; yet the reasoning in high school mathematics classrooms is often limited to proof as verification, reasoning as deductive, proving as mechanical and formulaic (as illustrated by algebraic proofs in trigonometry) and proofs as formal (eg, two-column proofs in Euclidian geometry). Specifically, we asked

- what is the nature of the questions/tasks offered to students that encourage proof and proving actions?
- how might a teacher recognize features of proving and proof in student conversation and in the questions students pose?
- how would valuing student need for proof and proving change current evaluation practices and rubrics for assessing student work?
- how much room is there in the curriculum for analogical reasoning, unformulated proving, and preformal proof?[3]

Using Action Research to Prompt the Growth of Pedagogical Understanding

Over the last decade, Sookochoff has found herself doing action research "by accident." She describes her pattern of professional improvement in the following passage.

> First, I have to face some truths that are uncomfortable to some: I do not know everything about the mathematics I teach and even what I do know is worth reconsideration; nor do I know everything about the students I teach; nor do I have the time to keep abreast of research literature that might inform and shape my practice. What I do have is a sincere curiosity about mathematics and about knowing it—so much so that I am compelled to engage in an intense process of action research, despite the constraints of time I face as a practising classroom teacher. Here is what I have done.

First, I identify a central theme around which I want to grow. I have used polynomials, irrational numbers, rational expressions, linear algebra and, in the case of this paper, reasoning and proving. All of these explorations started with a great dissatisfaction with the way my teaching and knowing mathematics were playing out in the classroom. My irritation was motivated sometimes by student frustration, sometimes it was by a sense of confusion, as though the curriculum were not supporting me, or as though the topic were fractured and riddled with unimportant details that I could not possibly make relevant to students. Once, as in the case of linear algebra, I just wanted to expose myself to hard ideas and feel what it was like to learn outside my area of comfort.

Then (and I only see this pattern now) I summon my math friends, people I've taken courses with or from, teachers and mathematicians I have met at conferences. Mostly, I do this through e-mails, written late at night, when my own kids are asleep, and when my mind is burning with a question from the day's activity. I fire out the e-mail and I wait. And the miracle is that I don't wait long. It is amazing to me how fellow teachers and mathematicians are eager to engage in thinking about the work we love.

Things move quickly and somewhat chaotically after that. Letters go back and forth about the research that might relate or about some mathematics that could enrich what I am working on. My teaching and mathematics evolve daily. My students become part of the process, empowered by the fact that I take their questions and utterances to my network of internet colleagues. And I end up feeling so charged with the energy of all these interactions.

There are results I can see right away. But, as years pass, at least two, I seem to enter a new phase. The excitement fades and I can more calmly select from the sea of ideas that were generated in that recursive chaos. It is in this phase that I am able to edit out the mathematics that might have fascinated me, but that obscured something for my students. It is in this phase that I see the curriculum clearly and with ownership. I know I've accomplished something important in my teaching when that feeling of irritation and confusion is gone. Instead, I am steady.

This paper is a record of some of my working through of the proving outcomes in Mathematics 20 Pure. The impulse started out of a feeling of irritation (that in this case is not quite resolved) with the way the resources, the curriculum, my background and my students were interacting. I found myself complaining about the unit. The complaints led me to my friend, Elaine Simmt, who suggested I talk to David Reid, whose research has centred on student proof and reasoning for some 10 years now. From there, the process evolved just as I described above—with one exception: after about a year of informal chat on the matter of proving, we applied for and received a research grant from the Alberta Advisory Committee for Educational Studies. This allowed us to document some of what happened in the classroom.

Sookochoff initiated an action research project in which she incorporated outcomes for formal reasoning in a number of units of instruction, rather than only in the unit typically used to teach proof and reasoning. The study involved two Pure Mathematics 20 classes and one Mathematics 20 International Baccalaureate class over the course of one semester. With her coresearchers, Sookochoff worked on her practice through conversations about proof, proving, reasoning, tasks and assessment. In the spirit of action research, she began with a question about practice and worked on it through cycles of planning, implementing, evaluating and questioning. With three classes each working on the same content and processes, she was able to try out tasks and strategies with one class, develop the tasks and strategies further and use them with another class. Because the research took place over a whole term she also was able to work through the cycle in the context of differing content.

All unit planning documents, student handouts and assessment tools, as well as student work, were collected. So, too, were e-mail messages among the research team and notes based on their face-to-face conversations. Approximately 20 per cent of the lessons were observed by one of the coresearchers. In those classes, observation notes were taken and audio tapes of selected student groups were made.

In summary, Sookochoff, in consultation with Simmt and Reid, developed lessons and units to promote proving. She taught the lessons and then reflected on them through further discussion; those discussions informed subsequent lessons. In this paper we elaborate on the action research project through the presentation and discussion of one particular lesson. We include records of e-mail correspondence, a transcript from the lesson, and an analysis of the relationship between those two things and Sookochoff's evolving understanding of teaching and mathematics.[4]

A Lesson in Definitions

The following section is intended to illustrate the action research process and a particular pedagogical concern that arose in Sookochoff's lesson (includes the planning, implementation and reflection of the lesson). The section begins with an e-mail message from Sookochoff to her collaborators prior to her planned lesson and continues with their responses. A short transcript taken from the lesson as it played out in the classroom prompts further reflection by Sookochoff.

On Monday, September 06, 2004, at 6:40 PM Shannon Sookochoff wrote:

> Hi again,
> I have asked my IB students (they are running ahead of the others) to think about the following for homework:
>
> $x + 5y = 10$
> $3x + 15y = 30$
> What values for (x, y) satisfy both equations?
>
> Then, when they are confronted with an infinite set, I will ask: *Given x/3=20, how do we solve?" Students will respond. "What mathematics do you know that makes you able to ______________ ____________? (Perhaps they will say "multiply both sides of the equation by 3.")* My idea is that we will come face to face with properties of equivalence relations that we need to build on in order to make solving by elimination work.
> How might I explain why I can multiply both sides of an equation by a constant without changing the solution? Well, I think that I would draw an analogy: *Do you accept that an equivalence relation is like a balance scale? If so, then let's put one third of an object on one side, and 20 units on the other. They balance. But we don't like only knowing what one third of the object weighs. We reason that we could triple each of these equal sides, making the unknown object whole and making the other side of the scale 20 × 3, or 60. So we see that the unknown object weighs 60. And we can verify that one third of 60 is 20.*
>
> Is this analogical reasoning sound? How might you demonstrate the reasoning behind what ends up being a property of equivalence relations?
> Shannon

Sookochoff, in planning for her lesson, is doing a thought experiment as to how the lesson might play out. She draws from her understanding of linear equations and linear systems, as well as from her understanding of equivalence. David Reid responds to her inquiry by suggesting she use the notion of an axiom—something that ties directly to proof and proving. Sookochoff is excited by this possibility and searches for meaning and ways to include the notion of *axiom* into the class discussion. This tone of inquiry is at the heart of action research.

On Tuesday, September 07, 2004, at 4:32 AM, David A Reid wrote:

> The analogy is a fine analogy, but why not try to make reference to something they might accept as an axiom/postulate?
> If $a = b$, then $ka = kb$.
>
> An equation is a statement that two things are equal. If they are equal, then the above axiom/postulate lets us multiply both sides by anything we want. I suspect IB students can cope with that.
>
> One tricky thing is the inequality. It is not true that:
> If $a <> b$, then $ka <> kb$
> because k might be 0.
>
> This is the basis for some nice proofs that 1 = 2.
> David

On Tuesday, September 07, 2004, at 6:54 AM, Shannon Sookochoff wrote:

> Excellent! So, what is it about this truth that makes it an axiom/postulate? What tells me that I cannot prove it? What are the signs to the thinker that an idea is axiomatic? I think that this will need to become explicit today when the IBs come back with their explanations. So, if you are at your computer, David, do tell! Hey. I just thought of an answer to my question. Could it be that something is axiomatic in a particular community if no one in the community can prove it but everyone agrees that it is true?
> Shannon

The conversation with Reid is important to her pedagogical understanding. Just writing to him is enough to trigger responses to her own questions.

On Tuesday, September 07, 2004 7:52 AM, David A Reid wrote:

> > Hey. I just thought of an answer to my question. Could it be that something is axiomatic in a particular community if no one in the community can prove it but everyone agrees that it is true?
>
> Exactly! I paused for a moment about what to call the thing. First I put only *axiom* but then I decided it might not be so *self-evident*. Then I thought about how to prove it. I suspect I would have to go back to the definition of multiplication. I also suspect that what I am calling an axiom might be part of the definition.

> Excellent! So, what is it about this truth that makes it an axiom/postulate? What tells me that I cannot prove it? What are the signs to the thinker that an idea is axiomatic? I think that this will need to become explicit today when the IBs come back with their explanations. So, if you are at your computer, David, do tell!

Just now I checked to see if it is part of the definition based on Peano's axioms, which are a popular starting point for number theory. This is what I found: www.cut-the-knot.org/do_you_know/mul_num.shtml

Now I THINK I can prove IF $a = b$ THEN $ka = kb$.

But I have to go now.

David

After these fast-paced exchanges Sookochoff teaches the lesson. She writes to her research collaborators immediately after teaching the lesson to her three classes.

On Tuesday, September 07, 2004 9:19 PM, Shannon Sookochoff wrote:

Zowee!

Even though my intent was to move from an inconsistent system, ie, no solution, ie, parallel lines, on to deriving elimination, instead, I was able to bring our entire conversation to bear. Here is how.

Having asked students to consider why we can multiply both sides of an equation by a constant, they came back perplexed. I sensed a tension. "This is easy but it is puzzling." Nicolai said just about that exact phrase. We noticed that in our various explanations, we ended up using the fact to explain the fact. It felt circular, yet we all agreed that the fact was true. I then defined the tension we were noticing as characteristic of an axiom. We generated one for addition and two corollaries (usage?) dealing with division and subtraction.

Then, I put four cases (A–D) on the board.

A) What is the solution to $3x + 2y = 104$ (related to but not limited by "three shirts and two sweaters cost \$104")? I have an infinite solution set of all points (x, y) where $y = (-3/2)x + 52$. I have a line if I think geometrically. Or I have a table of values with lots of possibilities. In fact I can see a pattern (the slope) in the integral points that I can generate. I use the following string of deductions: 1) Using axioms of equivalence relations to isolate y, I recognize $y = mx + b$. 2) When I graph an equation of this form, I generate a line. (Would we call this an axiom for now?) 3) Lines are made of a set of infinite points, the slope of which is consistent (a definition). 4) If the equation generates a line, then the solution is the line, which is an infinite set of points.

Other cases we considered:

B) $3x + 2y = 104$ and $2x + y = 60$.

C) $x + 3y = -10$ and $3x + 9y = -30$.

D) $x + 2y = 4$ and $2x + 4y = 3$.

With each system, the entire class worked to describe the solution set using a string of deductions.

This we got to with the IBs. (Then developed solving by elimination.)

The regulars are still thinking about cases C and D above. Their homework is to consider how they would describe the solutions sets for C and D. And your questions, Elaine [What does it mean to find a solution?], have been so powerful for me. The students really need time to see that a linear equation in x and y (especially where y is not isolated) generates a straight line and thus has infinite solutions. They will benefit from the discussion we had today in IB.

I was surprised that all this came together today and that I was able to say *axiom* without feeling too much like an impostor.

I just looked at that Peano stuff and found it really hard to read. On the other hand, the stuff I read about in Lyn English's book was more lived reasoning, I think. Am I pointing to a distinction that you two have noticed? One where Mason, English, Johnson and Lakoff are on one side of the continuum, and Peano (and others I don't know) are on the other? Is one considered more rigorous?

Shannon

As exciting and rewarding as the lesson was for Sookochoff, her desire to deeply understand proof and reasoning is growing. She takes Peano and English, a mathematician and an educational researcher respectively, and asks how they are helping her make sense of proof and proving.

On Wednesday, September 08, 2004 2:27 AM, David A Reid wrote:

All seems to be working out well.

There is certainly a tension between Peano and English et al.[5] It is a tension that has caused trouble for mathematics education from the start. There are two different starting points: mathematics and minds. Or if you like, logic and psychology. To an expert mathematician, an axiomatic system seems really easy. You start with things that everyone recognizes as true, and then you deduce everything else according to ways of thinking that everyone accepts. This was the basis for most of the New

Math movements. It turned out not to work very well. What Lakoff, English, Varela, etc, tell us is that the ways of thinking that mathematicians assume everyone accepts are in fact not all that applicable outside mathematics, and so naturally most people don't use them all that much. The trouble is that nature doesn't give babies the axioms at birth. They have to figure stuff out other ways: by analogies, metaphors, abductions, generalizations (there are a lot of words for this thinking, but none of it is well defined). And having figured out their whole world in this way, they figure out mathematics in the same way. In fact, something that is a fascinating (but hard) question for me is how those of us who have figured out how to reason mathematically when all we had to use were nonmathematical ways of reasoning.

I haven't looked hard at the Peano stuff yet, but I will now.

David

As her conversation with Reid continues, so do her mathematics classes. She keeps her collaborators informed of pedagogical moments from her class.

On Wednesday, September 8, 2004, at 3:02 PM, Shannon Sookochoff wrote:

Must write fast:

Phillip: Why is it that a pair of intersecting lines have only one solution?

Me: (*with some fluster and some panic thinking of the word axiom*) Maybe this is an axiom. (Write on the board AXIOM, having never mentioned it before.) I think it comes down to a decision by some mathematicians. We agree to consider a line to be a blah, blah, and we agree that when two lines intersect, they intersect at one point.

Phillip: Kind of a definition, then. I see.

Kelsey: So what is an axiom? A system of two intersecting lines?

Me: No, it is truth that we know to be true but are unable to explain why. Like ... I don't know. We will be talking about this more today though.

Blaine: I know an axiom. (*Holds up two fingers.*) How many fingers am I holding up ...

Gotta go get Jack. I'll send this home and pick up on it tonight.

That night Sookochoff elaborated on the events of the September 8 Math 20 Pure class on systems of equations.

Phillip: Why is it that a pair of intersecting lines have only one solution?

Me: *(with some fluster and some panic thinking of the word axiom)* Maybe this is an axiom. (Write on the board AXIOM having never mentioned it before.) I think it comes down to a decision by some mathematicians. We agree to consider a line to be an infinite array of points (each of which is a solution to an equation that we recognize to be $y = mx + b$). And we agree that when two lines intersect, they intersect at one point.

Phillip: Kind of a definition, then. I see.

Kelsey: So what is an axiom? A system of two intersecting lines?

Me: No, it is a truth that we know to be true but are unable to explain why. Like ... I don't know. We will be talking about this more today though.

Blaine: I know an axiom. (*He holds up two fingers.*) How many fingers am I holding up?

Me: Two.

Blaine: How do you know? Which one is one? Which one is two?

Me: *(smiling without anything to say)* Let's make it even simpler. I'll hold up one finger. How many fingers am I holding up?

Many: One.

Me: How do you know? (*Students are pleased.*)

Phillip: Because you have five fingers, you are holding 4 down, which leaves one standing. (*The class is happy to have proven what I suggested was unprovable.*)

Later in the class, after groups of four worked on explaining why we can multiply both sides of an equation by 3 (or any number). Their explanations ranged from a concrete example: $2 \times 3 = 6$, $(2 \times 3) \times 3 = 6 \times 3$, $18 = 18$, so it works, to "each side of the equal sign is in direct proportion to itself"; lots of mention of balance; one student related the equality to a basketball game in which, when subbing in and out, each side must always have five players on the court at one time. After sharing all of this I drew their attention to the difficulty of the task; they seemed to need to state the truth within the truth. Yet, we all understood. "That," I said, "makes this idea an axiom. No one in this room can explain it. We accept all of the examples and comparisons. We agree that we can *multiply both sides of an equation by a constant and not change the equality. So it is our axiom."*

Kelsey: So something like "cookies are sweet" is an axiom?

Me: I don't know; can anyone in here point to an explanation of cookies are sweet?

Phillip: Yes, it has something to do with taste buds and biology.

Me: So, Kelsey, Phillip thinks he could get to an explanation about that, so no, it is not an axiom.

And then I gave them notes stating the axiom in the form "if m = n, *then* km = kn*" and expanded from here to division, addition, and subtraction.*
Note: Earlier in the day (period 1) a student, I can't recall who, said that "we can multiply both sides of an equation (she was thinking about an equation in two variables) by a constant because when we do, the new line generates a new point on top of the original point in the original line." She was referring to coincident lines and I think she said it better. I'll ask tomorrow!
Shannon

The e-mails and transcript above could be analyzed in number of ways. But, because this is action research, and Sookochoff is reflecting on her own teaching, the analysis here examines how the above exchanges have transformed her thoughts and opened new possibilities for future teaching. Sookochoff writes:

> Phillip's first question asking how we are sure that there is only one point of intersection has me playing out some of the other options that I had in forming my response.
>
> 1. Probe Phillip's question more to determine whether it was grounded in the graph or the solution set for the system. If Phillip was thinking completely graphically, then I might have asked him to visualize two intersecting lines, two coincident lines and two parallel lines. But he may have been asking for my help to make the leap from two different solution sets from two different linear equations to the solution set for the system of equations. Or he may have been constructing the link between the solution set and the graphical representation of the system and its solution.
> 2. Move toward a group explanation of why two distinct and nonparallel lines in a plane intersect in exactly one point. I don't think I had thought of an explanation at the time, so I could not have led such a discussion until these last few weeks.
> 3. Call the knowledge axiomatic and explain what that means. This is what I chose to do and I am convinced that my choice, although fine, was influenced by my not having an explanation at the time and my then current struggle with the meaning of axiom. It worked well to engage students in meaning making and group discussion. Students seemed to like talking about the explainability of an idea and were intrigued with the idea that definitions and axioms stand outside the assertions that we can reason out.
> 4. Or I could have offered something that combined my response to Phillip with my response to Kelsey. I could have said, "I don't have an explanation right now. Does anyone else? Do we all accept that it is true? Can anyone think of an example? How about a counterexample? Well, IF we do not have an explanation AND we accept the idea to be true, THEN in our classroom at this moment we will call the idea an axiom. If we are able to find a convincing explanation in the next while, we will move it from the axiom board and onto the theorem board."
>
> I like #4 the best right now, because it brings all sorts of reasoning to bear. Had I used that response, I would have been asking kids to sort types of truth, the proven versus the axiomatic. In the few questions I have listed in response #4, I have referred to all of the reasoning outcomes from the curriculum.[6] The call for a specific example or counterexample builds toward an inductive approach to testing the idea, alluding to outcomes 4.1 and 4.3. By using the connecting word *and* and structuring the definition of axiom as an if–then statement, I embed outcomes 4.2 and 4.4 into a student-initiated conversation. And last, in the sorting of mathematical truths into axioms and theorems, we create a space and community-specific need for what I see as the most difficult of the reasoning outcomes: proving an assertion (outcome 4.5). Teaching in this way elevates mathematical reasoning from a discrete unit to an ongoing process and the connective syntax of the mathematical concepts we study.
>
> I think, too, that #4—let's call it "Attempt/postpone the explanation and sort the assertion"---can live in many contexts in the mathematics classroom. Students are remarkable in their ability to question why an assertion is true; they ask their teacher, "Why does the discriminant tell us how many roots we can expect for a quadratic?"; "Why do we switch the inequality sign when we multiply both sides of the inequality by a negative number?"; "Why do the roots of an equation have so much to do with the factors when the equation is set to zero?".[7] Their questions point, I think, to our students' inherent need for proof. Recognizing the students' questions as evidence of their need for proof allows the teacher to feed that need and thus brings students into the culture of proving in mathematics.
>
> This brings me to focus on the two categories of truth I mentioned above: proven and axiomatic.

I suggest that much of teaching explores the tension between these two ways of viewing assertions. I also suggest that one problem in our mathematics classrooms (and maybe in many other classrooms, as well) is our treatment of most knowledge as axiomatic—"It just is!" My own jump to label "two distinct and nonparallel lines in space intersect in exactly one point" as axiomatic is a case in point. In asking me why, Phillip challenged my mathematics ability. To answer him, I needed to honestly ask myself why. I needed to resist panic in the face of public uncertainty. I needed to make transparent a vital mathematical task, one in which I ask why something I take to be true is indeed true. And I needed to know, in both an emotional and an intellectual way, that not knowing why is legitimate. In exposing the struggle to explain why and entertaining the possibility that we cannot, teachers can underscore the nature of the mathematical assertions brought forth in the classroom. It should be noted, too, that the explainability of a given assertion can be decided in each specific classroom—what is an axiom for me and my students today may not be for my colleague down the hall. And six months from now, my students and I may find that we can indeed explain what we thought was an axiom.

Proof, Proving and Reasoning Through Action Research

With the illustration above we are able to respond to the research questions we posed when we began this study. But in the true nature of action research, these questions are not answered once and for all. Rather, we are able to identify additional questions to work on.

There are some things, Sookochoff believes, that worked to promote proving and reasoning activity among her students in those Grade 11 mathematics classes. We did well to

- integrate reasoning and proof into all content areas;
- put students into groups for proving together in discussion with one another;
- post theorems, colour-coded as to proven or accepted as true and identified as Grade 11 or pre-Grade 11 theorems, so as to have them available in the public domain;
- deal with theorems and vocabulary as needed;
- publish a collection of all that we know to be true for the class members; and
- ask the question, "We seem stuck; can anyone offer something that might get us unstuck?"

Of course, there are the things that Sookochoff will further develop the next time she teaches. Her notes to herself include the following advice:

- Keep expectations for student proving focused.[8] (To some extent, Sookochoff found that the specific standards for circle geometry,[9] which require students to prove two particular theorems from circle geometry, did not encourage this focused approach.)
- Go back and forth between specifying, proving and applying. (In Sookochoff's teaching of geometry, it was tempting to separate these proving activities, which she thinks obscured the connections between them.)
- Engage students in the issues of proof. The conversations would ideally come out of student questions and comments. However, some topics that a teacher might consider and could deliberately initiate, perhaps in a daily 10-minute group conversation, are listed below.
    - When can we name a proof as "_____ Theorem" and never again prove it? (We can make this happen by clearly titling and posting the assertions that our community accepts or proves to be true.)
    - Who decides how much is enough explanation?
    - Which truths have converses and contrapositives that are true? Which do not?
    - What does shifting from "Is this true?" to "What makes you sure this is true?" signify?
    - Is proof beautiful? (This could be a chance to share some particularly beautiful proofs from our canon—perhaps the Pythagorean Theorem, with its many proofs and unquestionable fame, could highlight the desire to explain *why* we know over *whether* we know.)
    - What is the difference between a definition, an axiom, a theorem and a postulate? (This relates nicely to the sample lesson discussed earlier.)
    - What is the nature and structure of a legal argument? Forensic evidence? Literary essay? Opinion paper? And how do they compare to a mathematical proof?

In terms of a teacher's practice we have addressed the questions that focused our study. We leave the reader with some more pointed responses to those same questions.

We asked, "What is the nature of the questions/tasks offered to students that encourage proof and proving actions?" Every high school teacher has asked some version of the question Sookochoff posed to her IB students:

$$x + 5y = 10$$
$$3x + 15y = 30$$

What values for (x, y) satisfy both equations?

There is nothing remarkable in the question. The difference lives in the context in which the questions were posed. From our work we have seen that it is essential to create space for responding to questions that arise from the desire for meaning making. (This works at the level of both teacher and student meaning making.) For the students, those spaces emerged in large group discussions that invited conjectures, challenged ideas and demanded reasoning. For the teacher, the space was created by having colleagues interested in the conversation of proof and proving. Our research suggests to us that the questions teachers ask must be accompanied by an inquiring stance, intense curiosity and a desire for things mathematical.

Also of interest to us was how a teacher might recognize features of proving and proof in student conversation and in the questions students pose. In this case, it was evident that the conversation between Sookochoff and Reid was key in Sookochoff's meaning making. However, it was close listening—that is, listening for student meaning making rather than listening for an expected particular response—that led to opportunities for Sookochoff to recognize proof and proving in student responses.

We asked how valuing student need for proof and proving would change current evaluation practices and rubrics for student work. Clearly, asking students to reason is key to any assessment. Finding ways to evaluate their responses is the challenge, and we will address it in a future paper.

Finally, we wondered how much room there is in the curriculum for analogical reasoning, unformulated proving and preformal proof. We purposely used an illustration from a nontraditional topic for addressing proof and proving in our high school mathematics curriculum. The wonderful part of this action research study was the deliberate intention to integrate proof, proving and reasoning throughout all the topics in the curriculum. Further evidence that there is plenty of room for reasoning in school mathematics will be offered in future papers.

In this paper we have illustrated how a teacher, engaged in action research in collaboration with colleagues, worked on her own understanding of mathematics, mathematics pedagogy and mathematics curriculum. We hope that classroom teachers benefit from our research in two ways: (1) as a strategy for working on their own teaching questions, and (2) for working on proof, proving and reasoning in high school mathematics. Further, we hope that university-based and school-based researchers find in our study some inspiration to work towards truly collaborative approaches to educational research that creates deeper understanding of mathematics teaching and learning.

Notes

1. Formal reasoning outcomes:
   - 4.1 Differentiate between inductive and deductive reasoning.
   - 4.2 Explain and apply connecting words, such as *and*, *or* and *not*, to solve problems.
   - 4.3 Use examples and counterexamples to analyze conjectures.
   - 4.4 Distinguish between an *if–then* proposition, its converse and its contrapositive.
   - 4.5 Prove assertions in a variety of settings, using direct reasoning.

   Circle geometry outcomes:
   - 5.2 Prove the following general properties, using established concepts and theorems:
     - The perpendicular bisector of a chord contains the centre of the circle.
     - The angle inscribed in a semicircle is a right angle.
     - The tangent segments to a circle from any external point are congruent.
   - 5.5 Verify and prove assertions in plane geometry, using coordinate geometry and trigonometric ratios as necessary.

2. During the marking of the June 2005 diploma exams, teachers talked with one another about getting better results on the exam. One strategy was to reduce the time spent on the Math 20 Pure units that had no follow-through in Math 30 Pure to give them more time to spend on items that relate directly to Math 30 Pure. They specifically talked of reducing the item on reasoning to a take-home booklet.

3. This last question requires some clarifications. The importance of analogical reasoning in mathematics has been described at length by Pólya (1968). It involves making a conjecture based on similarities between two situations. *Formulation* of proving refers to the reasoners' knowledge or awareness of their own reasoning. *Unformulated proving* refers to deductive reasoning of which the reasoner is mostly or completely unaware. *Preformal proofs* (Blum and Kirsch 1991) are a step in the direction of acceptable mathematical proofs. They might involve hidden assumptions and use informal language and notation, and might also include references to analogical or inductive evidence for a conjecture.

4. We chose not to elaborate on the evolution of the co-researchers' understanding.

5. This tension is related to another that also causes difficulty for mathematics educators. *Proof* has different meanings in different institutional contexts (Recio and Godino 2001). Most important here, *proof* has one meaning in logic and the foundations of mathematics, and another meaning in the practice of professional mathematicians. In logic and foundations of mathematics, proof is connected to deductive argumentations that take place in axiomatic and formal systems. In the practice of professional mathematicians, however, while "deductive proof is the prototypical pattern of mathematical proof … this formalist rigor decreases in practice" (p 94). Similarly, words like axiom have different meanings in these two institutional contexts. The Peano axioms discussed in the e-mails are not the formal versions, but rather the less formal ones used by professional mathematicians in their practice, and as Shannon is a teacher of mathematics, not of foundations or logic, her meanings for proof and axiom are based in the practices of mathematicians, not in the formalisms of logicians.

6. Formal reasoning outcomes:
   4.1 Differentiate between inductive and deductive reasoning.
   4.2 Explain and apply connecting words, such as *and*, *or* and *not*, to solve problems.
   4.3 Use examples and counterexamples to analyze conjectures.
   4.4 Distinguish between an if–then proposition, its converse and its contrapositive.
   4.5 Prove assertions in a variety of settings, using direct reasoning.

7. I deliberately did not list a question from geometry because students did not tend to offer their *why* questions there. Perhaps they needed no convincing when they could see the truth apparent in a visual illustration. It is ironic, then, that we often situate the task of proving within geometry, where students do not seem to *need* proof.

8. The task of writing proofs where algebra, plane geometry, coordinate geometry and trigonometry come to bear is highly complex. Students must bring together many years of their education in mathematics. And they must form a logical sequence of statements and reasons in a way that satisfies their teacher's idea of what proof looks like. Most Grade 11 students find this overwhelming. Instead of proving these particular assertions from circle geometry, perhaps the Grade 11 students would be better served by engaging in more narrowly defined proving tasks, such as discussing why they are sure of a particular property of equivalence relations. Alternatively, the theorem to be proven could have a greater cultural/historical importance (and thus be a better motivator for students) than the theorems from circle geometry. Examples here could be "interior angles in a triangle add up to 180 degrees" or the Pythagorean Theorem. However, a small canon of finely crafted and established proofs could make excellent class reading. As currently written, the curriculum seems to encourage students to memorize the two named proofs. And memorization is not a proving task.

9. Circle geometry outcomes:
   5.2 Prove the following general properties, using established concepts and theorems:
   - The perpendicular bisector of a chord contains the centre of the circle.
   - The angle inscribed in a semicircle is a right angle.
   - The tangent segments to a circle from any external point are congruent.

   5.5 Verify and prove assertions in plane geometry, using coordinate geometry and trigonometric ratios as necessary.

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