Looking at the Algorithm of Division of Fractions Differently: A Mathematics Educator Reflects on a Student's Insightful Procedure

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Introducing the Procedure

A colleague of mine, upon return from China, reported to me this procedure to divide fractions used by an 11-year-old[1]:

$$\frac{26}{20} \div \frac{2}{5} = \frac{26 \div 2}{20 \div 5} = \frac{13}{4}$$

My first reaction was to doubt the correctness of the procedure and solution, but then I realized that it was indeed mathematically correct and also that many interesting connections could be established with other operations on fractions (+, – and ×).

This article is in the spirit of, and takes its insights from, articles by Robert Davis (1973) and Stephen Brown (1981). Each had observed an interesting and nonstandard mathematics procedure carried out by a student and decided to report on it to bring forth the insights and the underpinning connections and concepts. This endeavour appears to be quite rich on many points, as Brown explained some 25 years ago:

> One incident with one child, seen in all its richness, frequently has more to convey to us than a thousand replications of an experiment conducted with hundreds of children. Our preoccupation with replicability and generalizability frequently dulls our senses to what we may see in the unique unanticipated event that has never occurred before and may never happen again. That event can, however, act as a peephole through which we get a better glimpse at a world that surrounds us but that we may never have seen in quite that way before. (Brown 1981, 11)

Now let's have a deeper look into this intriguing division procedure.

First Question: Is That Procedure Correct?

The first questions that come to mind concerning that procedure are "Is this correct? If so, how does it work?" Then, when we answer these questions, we ask "Why weren't we taught that in schools?" or "Why don't we teach that in schools?"

One first way of being convinced of its correctness is to solve it ourselves, for example, by using the invert and multiply algorithm: $\frac{26}{20} \div \frac{2}{5} = \frac{26}{20} \times \frac{5}{2} = \frac{130}{40} = \frac{13}{4}$. However, arriving at the same answer in a particular instance can leave some doubt that it would always work, even if it seems so. A more interesting question is "Why does it work?"

Looking closely at the multiplication algorithm, one realizes that it is mostly the same procedure, which is—in a very dry manner—to multiply the numerators together and multiply the denominators together. In this case, it is dividing the numerators and dividing the denominators. Hence, because division is also a multiplicative instance,[2] this procedure is indeed correct. From this, the following generalization can be deduced:

$$\frac{a}{b} \div \frac{c}{d} = \frac{a \div c}{b \div d}$$

And by playing with the multiplication algorithm, we can arrive at it directly, since

$$\frac{a}{b} \times \frac{d}{c} = \frac{a \times d}{b \times c} = \frac{a \times d}{c \times b} = \frac{a}{c} \times \frac{d}{b} = (a \div c) \times \frac{d}{b} = (a \div c) \times \frac{1}{\left(\frac{d}{b}\right)^{-1}}$$

$$= (a \div c) \times \frac{1}{\left(\frac{d^{-1}}{b^{-1}}\right)} = (a \div c) \times \frac{1}{\left(\frac{b}{d}\right)} = (a \div c) \times \frac{1}{(b \div d)} = \frac{a \div c}{b \div d}$$

Second Question: Why Don't We Teach This in Schools?

The answer to this second question lies in the fact that this procedure is only helpful in a limited number of cases. For example, if the fractions to be divided are $\frac{2}{5}$ and $\frac{3}{7}$, this procedure does not bring us very far toward the answer:

$$\frac{2}{5} \div \frac{3}{7} = \frac{2 \div 3}{5 \div 7}$$

And so, even though $\frac{2}{5} \div \frac{3}{7} = \frac{2 \div 3}{5 \div 7}$ is correct, it is simply not very helpful in finding an answer. However general in the sense that it is applicable in all cases, it cannot be considered a good algorithm since it sheds some light on the answer for only a small number of cases for which the numerators and the denominators are respectively divisible. Because this algorithm (dividing numerators together and dividing denominators together) helps in only a specific number of cases, it can be seen as a "particular" procedure.

Bringing This Procedure to Mathematics Teachers

In my research, I brought this interesting procedure to the secondary-level mathematics teachers with whom I work in professional development sessions. As predicted, they were amazed and curious about the correctness of this procedure to calculate with fractions. (Of course, I brought one that worked and produced results!)

A comment was made, however, that it could be interesting to work toward a generative way or an overall procedure of computing with all types of operations on fractions, because the teachers said that students have a hard time making sense of all four operations and their algorithms.[3] Thinking about what is normally done in addition and subtraction—that is, to write the fractions with a common denominator—one teacher wondered if we could not do this for the multiplication of fractions also, which would be

$$\frac{a}{b} \times \frac{c}{d} = \frac{ad}{bd} \times \frac{cb}{db} = \frac{adcb}{(bd)^2} = \frac{ac \cdot bd}{bd \cdot bd} = \frac{ac}{bd}.$$

As I explained that this was unnecessary and could complicate the calculations for no reason, we realized that maybe that *was* what was needed in the previous division algorithm to make it work. Indeed, transforming each fraction to have a common denominator makes the algorithm useful for any division of two fractions because both fractions' denominators would be the same. Therefore, making the new fraction obtained out of 1, creating a division by 1,

$$\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bd} \div \frac{cb}{db} = \frac{ad \div cb}{bd \div db} = \frac{ad \div cb}{1} = \frac{ad}{cb}$$

This makes the "particular" algorithm an encompassing and efficient algorithm that always brings us to the answer for dividing any two fractions. However, the person using this procedure needs to know conceptually that two numbers dividing each other can also be written in the fraction form—something that is not obvious and needs to be worked on (Davis 1975). At the secondary level, though, it can be assumed that students can make or even create that link.

The Final Question: Why Does This Work Again? Why Does the Multiplication of Fractions Algorithm Work?

Maybe this last set of questions sounds obvious, but the whole premise of accepting that we can indeed divide the numerators together and the denominators together is based on the acceptance of the multiplication algorithm. Hence, I started to wonder why this algorithm works: Why can we multiply fractions that way? What is the meaning behind this algorithm?

Of course, as I often do with pre- and inservice teachers, it is possible to illustrate it with the multiplication of fractional areas, or with folding pieces of paper (eg, Boissinotte 1998). These approaches represent very nice ways to make sense of the algorithm itself. For example, to multiply $\frac{2}{3} \times \frac{1}{4}$, I can say and show by folding areas of paper that I take a quarter of $\frac{2}{3}$ of a piece of paper. This is very nice, but am I able to make sense of it by only using the numbers themselves with no recourse to material? Can I explain why $\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$ works?

In fact, I must explain it if I want to use it as an argument to assert that the new division algorithm is indeed suitable. In order to make sense of it, I thought about the following explanation.

To stay concrete and less abstract, let's take an example: $\frac{4}{5} \times \frac{2}{3}$, which, with the algorithm, would give $\frac{8}{15}$. If I say it in words, it is "four fifths multiplied by two thirds," so I multiply by two thirds. One way to see it is that I first multiply by one third, and then I double the amount, because I wanted to multiply it by twice as much (by *two* thirds and not by *one* third). So, first, let's multiply by one third.

As I said before for one quarter, multiplying by a third means that I want a third of the amount. Wanting a third of an amount means that I want to divide the amount in three. Doing that makes each part of the amount three times smaller—they are indeed divided in three. So, in the case of $\frac{4}{5}$, each fifth becomes a fifteenth, and so I have four fifteenths now (instead of four fifths). Doing this explains why I have to multiply the denominator 5 with the denominator 3—because each part becomes three times smaller, and so becomes a fifteenth. So, the first sequence just explained could be represented like this:

$$\frac{4}{5}\times\frac{2}{3} = \frac{4}{5}\times\left(\frac{1}{3}\times 2\right) = \left(\frac{4}{5}\times\frac{1}{3}\right)\times 2 = \left(\frac{4}{5}\div 3\right)\times 2 = \left(\frac{4}{15}\right)\times 2$$

Now, I have four fifteenths ($\frac{4}{15}$). I still have to double it, because I multiplied it by $\frac{1}{3}$ and not by $\frac{2}{3}$. Because $\frac{2}{3}$ is twice as big as $\frac{1}{3}$, my answer should be twice as big. I have $\frac{4}{15}$ and I want twice that, and so my answer should be $\frac{8}{15}$. Here, because the number of parts is represented by the numerator, and I want twice that number of parts, I multiply the numerator 4 by the 2 in the algorithm. Hence

$$\left(\frac{4}{15}\right)\times 2 = \frac{4\times 2}{15} = \frac{8}{15}$$

Doing that is dismantling the algorithm into a sequence of conceptual steps, which, in an algorithm, are normally hidden (Bass 2003). And so, multiplying by a fraction means to (1) make each part a denominator number of times smaller and (2) take a numerator number of times the parts that are there. This can be summarized by the following generalization.[4]

$$\frac{a}{b}\times\frac{c}{d} = \frac{a}{b}\times\left(\frac{1}{d}\times c\right) = \left(\frac{a}{b}\times\frac{1}{d}\right)\times c$$

$$= \left(\frac{a}{b}\div d\right)\times c = \frac{a}{bd}\times c = \frac{a\times c}{bd} = \frac{ac}{bd}$$

Concluding Remarks

This new procedure for dividing fractions, which at first seemed wrong and did not feel genuine, created for me a list of connections regarding operations on fractions. Whereas I might have been the only one to be puzzled by it, I realized that my own colleagues and the secondary-level mathematics teachers with whom I worked were also very intrigued and surprised by the correctness of that procedure.

This procedure brought me to try to better understand other algorithms and operations on fractions, especially that of multiplication, which I realized, by asking these around me, is mostly taken for granted. In that sense, while the question of "why does the algorithm of multiplication work?" can sound obvious, its answer is not immediately obvious. Even now, you may not be convinced of the tentative explanation that I have elaborated above—and maybe neither am I!

I did not intend this article to show a better or a new algorithm to divide fractions, and certainly never aimed to solve the overarching, difficult problem of understanding this sort of computation. It is tempting to say that difficulties experienced in the domain of division of fractions will remain, because division of fractions is difficult to understand and conceptualize. The goal of this short article was to raise awareness of this issue, to play with numbers and, hopefully, to bring new ideas and insights about these calculations to the everyday mathematics classroom.

Notes

1. The colleague is David Pimm, whom I thank for the conversations on the issue. I also want to thank Mary Beisiegel for many discussions on this.

2. Indeed, problems requiring the operations of multiplication or division are often seen as part of the same family of problems. See, for example, the work of Vergnaud (1988) or Carpenter et al (1999).

3. Of course, it could be argued that operations on fractions need not be reduced to their algorithm (and research has shown that in many cases), but this is in another domain of discussion.

4. The explanation I have offered here mostly serves as an aid to understanding and not as a mathematical proof. It does, however, serve well its goal of bringing meaning to the algorithm of multiplication of fractions.

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