Binomial Probabilities on a Multiple Choice Test

Bonnie H Litwiller and David R Duncan

Teachers are always seeking situations in which binomial probabilities can be exemplified. One such setting with which students are acquainted involves multiple choice examinations.

Suppose that Roy took a 40-question multiple choice test; each test question had five possible answers. How likely is it that his score on the test was 4 or less? To answer this question we will make repeated use of the binomial formula:

> Suppose that n independent trials are performed. For each trial the probability of success is p while the probability of failure is q, where $p + q = 1$. The probability of exactly r successes and $n - r$ failures in these n trials is then
>
> $\frac{n!}{(r!)(n-r)!} p^r q^{n-r}$, where the notation $C(n, r)$ is commonly used to symbolize the coefficient $\frac{n!}{(r!)(n-r)!}$.

Case 1: Assume that Roy guessed on every question. Using binomial probability, the probability of at most 4 correct responses can be computed as follows:

Exactly 4 correct (and 36 wrong):

For each of the 40 questions, the probability that Roy selects the correct answer is $\frac{1}{5}$, while the probability he selects the incorrect answer is $\frac{4}{5}$. The probability of exactly 4 correct and 36 incorrect responses is then

$C(40, 4)(0.2)^4(0.8)^{36} = 0.04745$

Exactly 3 correct: $C(40, 3)(0.2)^3(0.8)^{37} = 0.02052$

Exactly 2 correct: $C(40, 2)(0.2)^2(0.8)^{38} = 0.00648$

Exactly 1 correct: $C(40, 1)(0.2)^1(0.8)^{39} = 0.00133$

None correct: $C(40, 0)(0.2)^0(0.8)^{40} = 0.00013$

Since these are mutually exclusive events, the total probability for these five instances is then the sum of the five individual probabilities, or 0.0759 (to four decimal places).

Case 2: Suppose that Roy found a single question for which he knew the answer, then guessed on the remaining 39. Since one correct response is assured, we calculate the probability that he correctly answered 0, 1, 2 or 3 of the remaining 39 questions. This probability is

$C(39, 0)(0.2)^0(0.8)^{39} + C(39, 1)(0.2)^1(0.8)^{38} +$
$C(39, 2)(0.2)^2(0.8)^{37} + C(39, 3)(0.2)^3(0.8)^{36} = 0.0332$

Case 3: Roy knows the answers to exactly 2 questions. The probability that he answers at most two of the remaining 38 questions correctly is:

$C(38, 0)(0.2)^0(0.8)^{38} + C(38, 1)(0.2)^1(0.8)^{37} +$
$C(38, 2)(0.2)^2(0.8)^{36} = 0.0113$

Case 4: Roy knows the answer to exactly 3 questions. The probability that he answers at most one of the remaining 37 questions correctly is

$C(37, 0)(0.2)^0(0.8)^{37} + C(37, 1)(0.2)^1(0.8)^{36} = 0.0050$

Case 5: If Roy knows exactly 4 answers, the probability that he fails to answer a single other question correctly is

$(0.8)^{36} = 0.0003$

Challenges to the reader and students

1. Redo these types of problems, varying the maximum score, the length of test and the number of responses per question.
2. Alter the problem by supposing that Roy is able to eliminate a certain number of responses per question, but must guess among the remaining possibilities.
3. Determine how a calculator could be used to sum the probabilities in the cases described above without having to calculate each probability separately.
4. Suppose that the probability of recovery within one year after therapy is 0.7 for patients with a certain disease. Find the probability that exactly two of the four patients currently being monitored recover within one year after therapy. Find the probability that at least two patients recover. If n patients are receiving therapy, write an expression that will give the probability that exactly k of them will recover, in terms of n and k.

5. A baseball player has a batting average of 0.250. Determine the probability that this player gets
   a) exactly one hit in his next five times at bat.
   b) at least three hits in his next five times at bat.
   c) a hit each time in his next five times at bat.

   Draw a graph of $P(k)$ vs k, where k is the number of hits during the next five times at bat, $k \in \{0, 1, 2, 3, 4, 5\}$ and $P(k)$ is the probability of k hits.

Bonnie Litwiller and David Duncan are emeritus professors of mathematics at the University of Northern Iowa. They taught a number of both content and methods courses for prospective and inservice teachers. Both are past presidents of the Iowa Council of Teachers of Mathematics. Bonnie Litwiller is also a past board member of both the National Council of Teachers of Mathematics and the School Science and Mathematics Association. They have published over 950 articles for mathematics teachers of all levels.