Solving First Order Linear Differential Equations by Using Variation of Parameters

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1. Introduction

One of the most significant roles of mathematics has been to address the so-called "inverse problem" in science. This concerns providing information about a quantity y on the basis of having measured some experimental trace that has been left by y. In calculus, when y is a differentiable function, one such particularly relevant trace is the derivative $y' = \frac{dy}{dx}$. In solving an ordinary differential equation (ODE), one seeks to determine all the solutions y that satisfy the given ODE. The theory of ODEs (as in, for instance, Nagle, Saff and Snider 2004) makes it useful to know the order of an ODE—namely, the highest integer n such that $y^{(n)}$, the n^{th} derivative of y, appears nontrivially in the given ODE. The cases $n = 1, 2$ are of particular importance because of applications in science and engineering (with the second derivative often playing the role of acceleration in applications of Newton's Second Law of Motion). The most complete theory in the subject has been developed for the class of linear ODEs; that is, ODEs dubbed "linear" because their analysis is often facilitated with the aid of matrix theory, which is also known as linear algebra. Occasionally, more elementary algebra becomes relevant, as in the classical solution of the second order ODE with constant coefficients, $ay'' + by' + cy = g(x)$, where the roots of the associated quadratic polynomial, $aT^2 + bT + c = 0$, play a crucial role (see Nagle, Saff and Snider 2004, chapter 4).

For several decades, the method of variation of parameters (also known as "variation of constants") has been a mainstay in the typical first course on ODEs. This method for solving n^{th} order linear ODEs is usually considered first for the case $n = 2$ (as in Nagle, Saff and Snider, 2004, section 4.6) and, in some courses and texts, later for the case $n \geq 3$ (as in Nagle, Saff and Snider, 2004, section 6.4). The treatment of variation of parameters is central to any first course on ODEs, as it is part of a (hopefully, gentle) introduction to the study of linear operators and the principle of superposition. Remarkably, this central role could be played earlier, when considering the case $n = 1$, which is instead usually treated by various ad hoc methods. The main purpose of this note is to rectify matters by showing how the above-stated principles of variation of parameters can be introduced very early in a course on ODEs to solve the general first order linear ODE. Moreover, our treatment of the case $n = 1$ has the classroom advantage of being able to focus on the differential equation aspects, as we will need none of the algebraic machinery and background (such as determinants, Wronskians and Cramer's Rule) that are needed to implement variation of parameters in the case $n > 1$.

Section 2 contains a derivation of the method promised in the title of this note. We have found that this theoretical presentation is well received in the first unit of an ODE course. Examples 3.1 and 3.2 illustrate the use of this method. Rather than depending on the abstract considerations in section 2, the presentation of examples 3.1 and 3.2 repeats some of those ideas in a concrete situation and is thus essentially self-contained. In this way, examples 3.1 and 3.2 can serve as models for a classroom presentation of our method in classes where the abstract considerations in section 2 may seem inappropriate. Remark 3.3 identifies what we see as the two most important pedagogical advantages of our method over the usual method that involves integrating factors. In closing, remark 3.4 suggests a new role for the topic of integrating factors in a first course on ODEs.

2. A Derivation Based on the Homogeneous Case

The method of variation of parameters works in general as follows. To solve a nonhomogeneous linear ODE, first obtain a formula for the general solution of the corresponding homogeneous linear ODE, and then determine how the arbitrary *constants* appearing in that formula would have to be reinterpreted as *functions* in order for the reinterpreted formula to produce a solution of the given nonhomogeneous ODE. Let us now see how this method can be applied to solve the general first order linear ODE. $y' + P(x)y = Q(x)$ (where $P(x)$ and $Q(x)$ are continuous functions defined on some open interval and, as above, y' means $\frac{dy}{dx}$).

The corresponding homogeneous linear ODE is $y' + P(x)y = 0$ or, equivalently, $\frac{dy}{y} = -P(x)dx$. This is a (variables) separable ODE, which is often the only type of ODE whose solution is typically studied before the topic of first order linear ODEs is considered in an ODE course. As usual, one can solve this separable ODE by integrating both sides, with the result that $\ln(|y|) = -\int P(x)dx + C^*$, where C^* is an arbitrary constant. Exponentiation leads to the formula $y = Ke^{-\int P(x)dx}$, where $K = \pm e^{C^*}$. Let $v := e^{-\int P(x)dx}$, ignoring the constant of integration in the exponent. (It is interesting, but not essential, to note that $v = \mu^{-1}$, where $\mu := e^{\int P(x)dx}$ is the integrating factor that is used in the typical textbook solution of first order linear ODEs.) Thus, the general solution of the corresponding homogeneous linear ODE is $y = Kv$. It follows that v is a particular solution of $y' + P(x)y = 0$. We proceed to *vary the parameter* K—that is, to determine how to interpret K as a *function*—so that $y = Kv$ is a solution of $y' + P(x)y = Q(x)$.

Since $y = Kv$, we can find y' by using the product rule: $y' = v'K + K'v$. Substituting into the given ODE leads to $v'K + K'v + P(x)Kv = Q(x)$ or, equivalently, $K'v + K(v' + P(x)v) = Q(x)$. Since $v' + P(x)v) = 0$, the above condition on K simplifies to $K'v = Q(x)$ or, equivalently, $K' = v^{-1}Q(x)$. Then, by the very meaning of indefinite integration, we have $K = \int v^{-1}Q(x)dx + C$, where C is an arbitrary constant. Therefore, the general solution of $y' + P(x)y = Q(x)$ is $y = Kv = (\int v^{-1}Q(x)dx + C)v = (\int v^{-1}Q(x)dx)v + Cv$. Since $v = \mu^{-1}$, this formula can be rewritten in the more familiar way as $y = (\int \mu Q(x)dx)\mu^{-1} + C\mu^{-1}$. Of course, it is not necessary for users of our method to remember this formula (or the formula for μ), as they need only implement the above steps.

3. Some Examples and Pedagogical Remarks

Examples 3.1 and 3.2 illustrate how to use the methodology in section 2 to find the general solution of a typical first order linear ODE. Remark 3.3 compares the details of example 3.1 with the details in the usual solution via the integrating factor method (as in Nagle, Saff and Snider 2004, section 2.3). In this way, we have a concrete example illustrating the advantages that we ascribe to the method in section 2. Of course, as we observed at the end of section 2, the two methods give the same answer. For a variety of reasons, instructors who include the method of section 2 in their curriculum for a first course on ODEs may also wish to include the integrating factor method. For such curricula, it may be advisable to identify an additional role that integrating factors can play in such a course, and remark 3.4 offers one suggestion along these lines.

Example 3.1. Use the method of section 2 to solve the following ODE: $2x^2y' + xy = 6x^2$ (for $x > 0$).

Solution. The given ODE is not in the standard form of a first order linear ODE; namely, $y' + P(x)y = Q(x)$. To find an equivalent ODE that is in this standard form, divide the given ODE by $2x^2$ (that is, by the coefficient of y'). The result is in standard form, with $P(x) = \frac{x}{2x^2} = \frac{1}{2x}$ and $Q(x) = \frac{6x^2}{2x^2} = 3$. According to the method of variation of parameters, we must first find the general solution of the corresponding homogeneous (first order) linear ODE, $y' + P(x)y = 0$ (namely, $y' + \frac{1}{2x}y = 0$). This ODE can be rewritten as $\frac{dy}{y} = -\frac{dx}{2x}$, a (variables) separable ODE whose general solution can be found in the usual way, as follows: $\int \frac{1}{y}dy = -\int \frac{1}{2x}dx + C^*$, or $\ln|y| = -\frac{1}{2}\ln|x| + C^* = -\ln(|x|^{1/2}) + C^*$. By a law of logarithms, this solution of the homogeneous ODE can be rewritten as $\ln(|y||x|^{1/2}) = C^*$ or, equivalently, as $y = Kx^{-1/2}$, where the arbitrary constant $K = \pm e^{C^*}$.

We now proceed to vary the *parameter* K that appeared in the above solution of the homogeneous ODE. As in the usual textbook treatments for the case $n \geq 2$, this amounts to asking for necessary and sufficient conditions on a *function* K so that $y = Kx^{-1/2}$ is a solution of the given (nonhomogeneous) ODE. Substituting this expression for y into the given ODE (and differentiating it using the product rule from the prerequisite differential calculus), we obtain $2x^2(-\frac{1}{2}x^{-3/2}K + K'x^{-1/2}) + xKx^{-1/2} = 6x^2$. This is algebraically equivalent to $K' = 3x^{1/2}$, whose solution (using the prerequisite integral calculus) is $K = \int 3x^{1/2}dx = 2x^{3/2} + C$. Accordingly, the general solution of the given ODE is $y = Kx^{-1/2} = (2x^{3/2} + C)x^{-1/2} = 2x + Cx^{-1/2}$.

For classes with enough time for additional applications of the method being proposed here, example 3.2 provides two more illustrations of that method. Example 3.2(a) is easier than example 3.1 in that the differential equation that one must solve to find K in example 3.2(a) is easy (namely, $K' = 1$), while example 3.2(b) is more difficult than example 3.1 because the differential equation that one must solve to find K in example 3.2(b) requires integration by parts.

Example 3.2. Use the method of section 2 to solve the following ODEs:

(a) $y' - 2y = e^{2x}$; and

(b) $y' - 2y = x$.

Solution (Sketch). The given ODEs are both in the standard form of a first order linear ODE; namely, $y' + P(x)y = Q(x)$ (with $P(x) = -2$ in both cases). The general solution of the corresponding homogeneous (first order) linear ODE, $y' + P(x)y = 0$ (namely, $y' - 2y = 0$), is found to be $y = Ke^{2x}$. Viewing K as a function and requiring $y = Ke^{2x}$ to satisfy the ODE in (a) leads, after some algebraic simplification, to $e^{2x}K' = e^{2x}$, whence $K' = 1$ and $K = \int 1dx = x + C$, where C is an arbitrary constant. Thus, the general solution for (a) is $y = (x + C)e^{2x} = xe^{2x} + Ce^{2x}$.

A similar approach in (b) leads to $K' = xe^{-2x}$, whence integration by parts gives us that $K = \int xe^{-2x}dx = x(-\frac{1}{2}e^{-2x}) - \int -\frac{1}{2}e^{-2x}dx$. Thus, $K = -\frac{xe^{-2x}}{2} - \frac{e^{-2x}}{4} + C$, and so the general solution for (b) is $y = (-\frac{xe^{-2x}}{2} - \frac{e^{-2x}}{4} + C)e^{2x}$, which simplifies to $y = -\frac{x}{2} - \frac{1}{4} + Ce^{2x}$.

Remark 3.3. In Nagle, Saff and Snider (section 2.3), the general first order linear ODE, $y' + P(x)y = Q(x)$, is solved by using the rather unmotivated introduction of the integrating factor $\mu = e^{\int P(x)dx}$ (which leads to the equivalent ODE, $\frac{d}{dx}(\mu y) = \mu Q(x)$, which can be solved by separation of variables). Thus, the usual textbook solution of the ODE given in Example 3.1 would use the integrating factor $\mu = e^{\int P(x)dx} = e^{\int \frac{1}{2x}dx} = e^{\frac{1}{2}\ln(|x|)} = |x|^{1/2} = x^{1/2}$. That solution is then $y = (\int \mu Q(x)dx)\mu^{-1} + C\mu^{-1} = (\int x^{1/2}3dx)x^{-1/2} + Cx^{-1/2} = (2x^{3/2})x^{-1/2} + Cx^{-1/2} = 2x + Cx^{-1/2}$, which agrees with the answer found in example 3.1.

A comparison of the above calculation with the details in example 3.1 shows that both involve the same mechanical skills. However, the solution in example 3.1 (and the same can be said for the solutions in example 3.2) has what we view as the two most important advantages for the method introduced in this note: (1) it does not require one to memorize the integrating factor formula $\mu = e^{\int P(x)dx}$ and (2) it introduces variation of parameters in a context ($n = 1$) that can avoid the matrix algebra that complicates the treatment in case $n \geq 2$.

Remark 3.4. In closing, we pursue the comment in the introduction that most current textbooks deal with first order linear ODEs in an ad hoc manner. (After drafting this manuscript, we came across a couple of recent textbooks that do introduce variation of parameters in case $n = 1$: see Diacu (2001, 32–33) and Logan (2006, 62–63).) Recall that the standard textbook solution of the general first order linear ODE, $y' + P(x)y = Q(x)$, is carried out with the aid of the integrating factor $\mu = e^{\int P(x)dx}$ (which leads to the equivalent ODE, $\frac{d}{dx}(\mu y) = \mu Q(x)$, which can be solved by separation of variables). Rather than appealing to separation of variables, an earlier edition of Nagle, Saff and Snider justified the integrating factor method by using the theory of exact ODEs. (The topic of exact ODEs has been moved to section 2.4 of Nagle, Saff and Snider.) This style of justification suffers the criticism of depending on Calculus III, especially on concepts involving partial derivatives and simply connected regions. On the other hand, for students with this background from Calculus III, if an instructor wishes to emphasize the integrating factor method in conjunction with a discussion of exact ODEs, then the topic of integrating factors could be made more central to the course by including the following theorem. *Any* (not necessarily linear) first order ODE, $M(x, y)dx + N(x, y)dy = 0$, has an integrating factor (that is, a function $\mu = \mu(x, y)$ such that $\mu M(x, y)dx + \mu N(x, y)dy = 0$ is an exact ODE), provided that M and N have continuous first partial derivatives defined over some open rectangle. The proof of this theorem depends on the fundamental existence and uniqueness theorem for initial value problems and can be found in Ford (1955, Theorem, 54).

References

Diacu, F. 2001. *An Introduction to Differential Equations (Order and Chaos).* San Francisco: W H Freeman.

Ford, L R. 1955. *Differential Equations.* 2nd ed. New York: McGraw-Hill.

Logan, J D. 2006. *A First Course in Differential Equations.* New York: Springer.

Nagle, R K, E B Saff and A D Snider. 2004. *Fundamentals of Differential Equations.* 6th ed. Boston: Pearson/Addison-Wesley.

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