A Refined Algorithm for Solving Polynomial Equations

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The following article describes a sequence of steps designed to reduce to a minimum the number of eligible factors when solving analytically polynomial functions with integral roots and coefficients. Our objective is to list all possible factors and then select only those factors that satisfy certain criteria. This is done by incorporating Descartes' Rule of Signs and the factors of the sum of the numerical coefficients of the given polynomial. The algorithm consists of the following steps:

1. Apply Descartes' Rule of Signs to determine possible numbers of positive or negative real numbers and/or complex roots.
2. Find the sum of the numerical coefficients or the polynomial.
3. List all the factors of this sum and then add 1 to each factor. This becomes set B.
4. List all the factors of the constant of the polynomial (Integral Factor Theorem). This becomes set A.
5. Find the intersection of sets A and B, listing only the common elements.
6. Use step 1 to focus on the number of positive or negative real roots.

Using a variety of examples, let us examine the usefulness of this algorithm.

Example 1: Solve $x^3-5x^2-8x+12=0$

Step 1: There are two variations in sign for $p(x)$: Case 1, two positive real roots and one negative real root: or Case 2, no positive real roots, one negative real root and two complex roots.

Step 2: The sum of the numerical coefficients, $1-5-8+12$, is zero. Since the sum is zero, the number 1 is a root (Remainder Theorem) and $x-1$ is a factor of $p(x)$ (Integral Factor Theorem). We go immediately to either long or synthetic division to find the other two roots. This gives a quotient of $x^2-4x-12$ or $(x-6)(x+2)$. We go no further because we have all three factors: $(x-1),(x-6)$ and $(x+2)$ and thus the roots 1, 6, and –2.

Example 2: Solve $x^3-4x^2+x+6=0$

Step 1: There are two variations in sign for $p(x)$: Case 1, two positive real roots and one negative real root; or Case 2, no positive real roots, one negative real root and two complex roots.

Step 2: a) $1-4+1+6=+4\neq 0$, so 1 is not a root.

b) $1+1=2$ and $-4+6=2$. Since these sums match (Remainder Theorem), -1 is a root and $x+1$ is a factor.

We go immediately to long or synthetic division in order to obtain the other two factors $(x-2)$ and $(x-3)$. We go no further because we have all three factors; $(x+1),(x-2)$ and $(x-3)$.

The above two examples serve to show the importance of immediately looking for 1 or -1 as a factor. However, not all polynomials contain these factors, and so we continue with the next example.

Example 3: Solve $x^3-4x^2-11x+30=0$

Step 1: There are two variations in sign for $p(x)$: Case 1, two positive real roots and one negative real root; or Case 2, no positive real roots, one negative real root, and two complex roots.

Step 2: $1-4-11+30=16\neq 0\ \therefore 1$ is not a root and $1-11\neq -4+30\ \therefore -1$ is not a root

Step 3: From step 2, 16 is the sum and its factors are $-1,-2,-4,-8,-16$ and $1,2,4,8,16$. Adding 1 to each factor gives $0,-1,-3,-7,-15$ and $2,3,5,9,17$; Set B = $\{0,-1,-3,-7,-15,2,3,5,9,17\}$.

Step 4: From the Integral Factor Theorem, the factors of 30 are placed in Set A; Set A = {–1, –2, –3 –5, –6, –10, –15, –30} {1, 2, 3, 5, 6, 10, 15, 30}

Step 5: The intersection of the two sets gives $\{-1,-3,-15,2,3,5\}$. Since -1 has already been eliminated (step 2), the list of possible factors is really {–3, –15 ,2, 3, 5}.

Step 6: From step 1, only one of the elements –3 and –15 can be a factor, while two of the three elements 2, 3 and 5 are factors.

If we compare the original number of factors of 30 (16) to the elements in $A \cap B$, we have narrowed it down to 5. Incidentally, the probability of selecting the correct negative root is ½, while the probability of selecting the correct positive root is $\frac{2}{3}$. Trying the positive roots in ascending order where the probability is higher, we have $p(2) = 20$. The other roots become 5 and –3 (both in the final list).

Quartics

When we get into the realm of even-numbered polynomials, the rule for 1 works but the rule for –1 does not. However, everything else holds. We now examine two more examples, one of which emphasizes the strength of Descartes' Rule of Signs.

Example 4: $x^4 5x^2 + 1 = 0$

Step 1: There are no variations in $p(x)$, so there are no positive real roots. The same is true in $p(-x)$, so there are no negative real roots. We conclude that there are four complex numbers, then move on!

Example 5: $x^4 + 6x^3 + x^2 - 24x - 20 = 0$

Step 1: There is one variation in sign for $p(x)$, so there we have the following possibilities: Case 1, one positive real root and three negative real roots; or Case 2, one positive real root, one negative real root and two complex roots.

Now $p(-x) = x^4 - 6x^3 + x^2 + 24x - 20$.

Since there are three variations in sign, we will look for three negative real roots.

Step 2: The sum of the numerical coefficients is $-36 \neq 0$. $\therefore$ 1 is not a root.

Step 3: The factors of the sum 36 are –1, –2, –3, –4, –6, –9, –12, –18, –36 and 1, 2, 3, 4, 6, 9, 12, 18, 36.

Adding 1 to each factor gives set B: {0, –1, –2, –3, –5, –8, –11, –17, –35} {2, 3, 4, 5, 7, 10, 13, 19, 37}

Step 4: The factors of the constant 20 gives set A: {–1, –2, –4, –5, –10, –20} {1, 2, 4, 5, 10, 20}

Step 5: $A \cap B = \{-1, -2, -5, 2, 4, 5, 10\}$

Step 6: From Step 1, there are three negative real roots, so –1, -2, and –5 qualify, and since $p(2) = 0$, we have all four roots.

Naturally, five examples do not make the case for all polynomial functions, as we have not explored rational and irrational roots. However, for the monic polynomial with integral roots, we have a method that cuts down the guessing of factors in order to solve a polynomial function. The combination of Descartes' Rule of Signs along with the intersection of two sets reduces remarkably the number of possible factors to be considered. This in turn reduces time spent on any one question and reduces the frustration of guessing, when solving analytically.

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