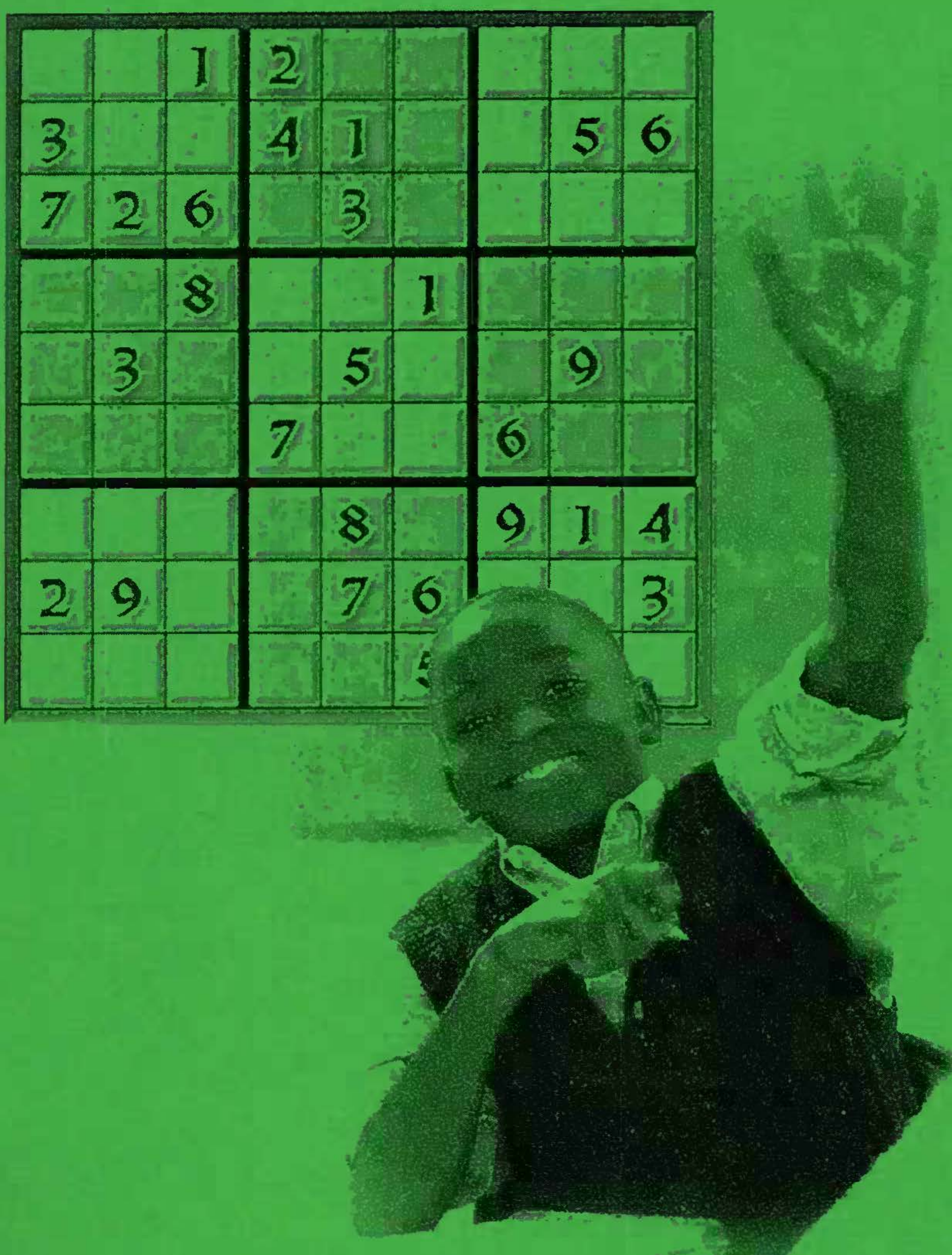
delta-k

JOURNAL OF THE
MATHEMATICS COUNCIL
OF THE ALBERTA
TEACHERS' ASSOCIATION

6789 123456789 123456789 123456789 123456789 123456789 123456789 123456789 123456789 123456789 123456789 123456789 123456789 123456789 123456789 123456789 1234567

Volume 44, Number 2 June 2007

6789 123456789 123456789 123456789 123456789 123456789 123456789 123456789 123456789 123456789 123456789 123456789 123456789 123456789 123456789 123456789 1234567



GUIDELINES FOR MANUSCRIPTS

delta-K is a professional journal for mathematics teachers in Alberta. It is published twice a year to

- promote the professional development of mathematics educators, and
- stimulate thinking, explore new ideas and offer various viewpoints.

Submissions are requested that have a classroom as well as a scholarly focus. They may include

- personal explorations of significant classroom experiences;
- descriptions of innovative classroom and school practices;
- reviews or evaluations of instructional and curricular methods, programs or materials;
- discussions of trends, issues or policies;
- a specific focus on technology in the classroom; or
- a focus on the curriculum, professional and assessment standards of the NCTM.

Suggestions for Writers

1. *delta-K* is a refereed journal. Manuscripts submitted to *delta-K* should be original material. Articles currently under consideration by other journals will not be reviewed.
2. All manuscripts should be typewritten, double-spaced and properly referenced. All pages should be numbered.
3. The author's name and full address should be provided on a separate page. If an article has more than one author, the contact author must be clearly identified. Authors should avoid all other references that may reveal their identities to the reviewers.
4. All manuscripts should be submitted electronically, using Microsoft Word format.
5. Pictures or illustrations should be clearly labelled and placed where you want them to appear in the article. Please also include all graphics as separate files (JPEG, GIF, TIF). A caption and photo credit should accompany each photograph.
6. References should be formatted using *The Chicago Manual of Style*'s author-date system.
7. If any student work is included, please provide a release letter from the student's parent/guardian allowing publication in the journal.
8. Limit your manuscript to no more than eight pages double-spaced.
9. Letters to the editor and reviews of curriculum materials are welcome.
10. Send manuscripts and inquiries to the editor: Gladys Sterenberg, 195 Sheep River Cove, Okotoks, AB T1S 2L4; e-mail gladys.sterenberg@uleth.ca.

MCATA Mission Statement

Providing leadership to encourage the continuing enhancement of teaching, learning and understanding mathematics.

delta-K

Volume 44, Number 2 June 2007

CONTENTS

 delta-K is published by the ATA for the Mathematics Council (MCATA). EDITOR: Gladys Sterenberg, 195 Sheep River Cove, Okotoks, AB T1S 2L4; e-mail gladys.sterenberg@uleth.ca. EDITORIAL AND PRODUCTION SERVICES: Document Production staff, ATA. Opinions expressed herein are not necessarily those of MCATA or of the ATA. Address correspondence regarding this publication to the editor. *delta-K* is indexed in the Canadian Education Index. ISSN 0319-8367

Individual copies of this journal can be ordered at the following prices: 1 to 4 copies, $7.50 each; 5 to 10 copies, $5.00 each; over 10 copies, $3.50 each. Please add 5 per cent shipping and handling and 6 per cent GST. Please contact Distribution at Barnett House to place your order. In Edmonton, dial (780) 447-9400, ext 321; toll free in Alberta, dial 1-800-232-7208, ext 321.

EDITORIAL

As the school year draws to a close, I am looking forward to some deserved relaxation. However, I know that many of us use the summer break as an opportunity to reflect on our teaching practice and to search for new ideas for engaging students in mathematical inquiry. This issue presents such opportunities.

Our annual convention of the Mathematics Council of the Alberta Teachers' Association (MCATA) in October was very successful. I'm continually impressed by our teachers' level of commitment to professional development in mathematics education. The photographs, Martina's report and this invitation to the next conference remind us that these events are important professional and social occasions. Please consider joining us for the 2007 MCATA convention, "Mathematical Tapestries: Weaving the Connections," at the Fantasyland Hotel in Edmonton, October 18 to 20, 2007.

I am thankful to the authors and reviewers who work diligently to provide us with thought-provoking ideas and suggestions for teaching. I encourage those of you who have created learning opportunities for your students to share these ideas with the community. Lesson plans, teaching ideas, and research papers related to teaching are all welcome. The summer presents an opportunity for you to write an article for *delta-K*. Please contact me if you want more information or help with this process.

In past issues I have invited your responses to articles in *delta-K*, but they have not been forthcoming. This publication offers an excellent opportunity for you to respond to ideas being presented here. If you have experimented with a teaching idea or have reflected on the content of an article, please write an editorial or reader's response and forward this to me for inclusion in the next issue. Also, if you are encountering challenges or joys in the planning or implementation of the new curriculum, please share them. Your responses can provoke additional thought and can initiate a professional conversation in our community.

Gladys Sterenberg

From the President's Pen

Algebra has become the new Latin. In the past, the ability to master the classic languages of Latin and Greek was seen as an indication that a person had the intellectual ability to deal with higher education. Of course, knowledge of Latin and Greek also meant that a person had money and had been educated in the finest fashion.

In the new age of technology, Calculus and Geek have replaced the old classics. Postsecondary institutions, particularly those in Alberta, are using a certain kind of mathematics as their filter to decide whom to admit to their institutions. It is a little known fact that you can be admitted to university in Alberta with a second language (perhaps Latin, but more likely French) and without mathematics. This of course limits your options. If, however, you wish to be able to select from a wider range of possibilities then only algebra and calculus will do.

Reality says that in any situation where there is a need to limit access to a scarce resource (in this case, postsecondary education) a mechanism will be found to decide who does and does not fit. I have no desire to argue whether or not mathematics should be that limiting factor. In some cases the mathematics itself is essential; in others the ability to think logically and analytically is what is needed. In some cases it is difficult to understand why mathematics is the chosen filter.

As a teacher and principal of students in elementary school, I see that the effects of "mathematics as gatekeeper" are reaching downward. The proposed new curriculum for K–9 mathematics in Alberta has a much more explicit emphasis on representing algebraic expressions.

Recently I have been engaged in discussions about the development of algebraic thinking in elementary students. There are those who think that if students have difficulty with algebra in junior and senior high school, then starting algebra earlier is the solution. They seem to think that substituting letters for unknowns in simple equations prepares children for quadratics.

I would argue that extensive experience with algebraic thinking rather than algebra would better serve our elementary students. When children extend patterns beyond what can easily be listed and when they argue the reasons behind their conjectures, they are engaging in algebraic thinking. When students state pattern rules in words, they are thinking algebraically. Moving flexibly between pattern representations, stating the relationships that underlie those patterns and detecting regularities in the world all prepare children for later algebra.

Young children are context dependent. A relevant problem that is set in a meaningful context is easily solved. A problem in an unfamiliar or unspecific context is much more difficult for them. By providing extensive and deep experience with patterns in context we provide a strong foundation for the later work of abstract algebra. Those who confuse algebraic thinking with the use of algebraic notation and who focus on the latter will confuse students and cause them to avoid algebra.

This is my last "From the President's Pen" for *delta-K*. It has been a challenging two-year term and I have appreciated the opportunity to express my ideas in writing. I look forward to reading many more issues of *delta-K*.

Janis Kristjansson

MCATA Conference 2006

Message from the Conference Chair

Jasper Park Lodge provided a beautiful setting for Conference 2006: Pathways to Understanding. It was a long trip for most who attended, but we received very positive feedback about the venue.

Dr Anne Watson (Oxford University, UK) led Thursday's leadership symposium. Participants engaged in many mathematical activities that both challenged conventional ways of thinking and allowed them to experience how their own examples can be used to open up new levels of understanding and new possibilities for investigation. Anne invented a new verb when she announced that "I can see you've all been Tom Kierened," a nice tribute to Alberta's mathematical community.

Professor John Mason from the Open University in the United Kingdom officially kicked off the conference with a thought-provoking talk about how we might direct attention in the mathematics classroom. Rumour has it he was up a good part of the night incorporating new ideas into his talk to address common concerns that he became aware of through many conversations during the week prior to the conference. Even with a crowd of over 300, he was able to stimulate thought-provoking conversations about mathematics.

John and Anne have generously provided copies of all of their talks, and they are available on the MCATA website at www.mathteachers.ab.ca.

We hosted 63 sessions on Friday and Saturday, and we received a great deal of positive feedback on their quality. We offer a huge thank you to the speakers who shared so much of their passion, time, and energy in presenting their ideas and engaging in the meaningful conversation that emerged from them.

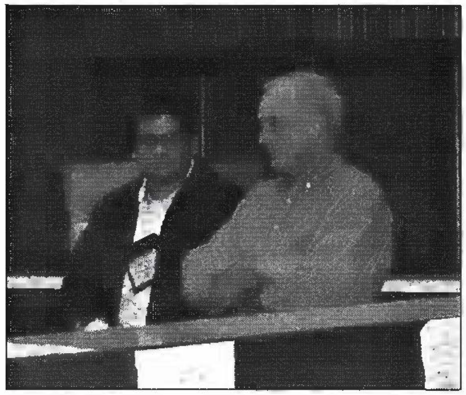
Dr Edward Burger's after-dinner talk was both inspiring and entertaining. I have heard his reminder to ask, "What will they remember 20 years from now?" repeated several times since the conference. For those who are interested, his "Top Ten Life Lessons" are also posted on the MCATA website.

On behalf of MCATA, I would like to extend congratulations to Nicole Patrie, this year's Dr Arthur Jorgensen Chair Award winner, and to Gerald Krabbe, the 2006 Mathematics Educator of the Year. Also, a big thank you to this year's Friends of MCATA, Geri Lorway and Len Bonifacio.

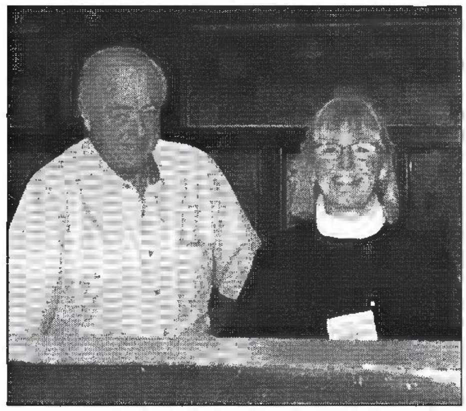
On behalf of the conference committee, I would like once again to thank the speakers, displayers, and participants who came together to share their common interest in mathematics education at the 2006 conference.

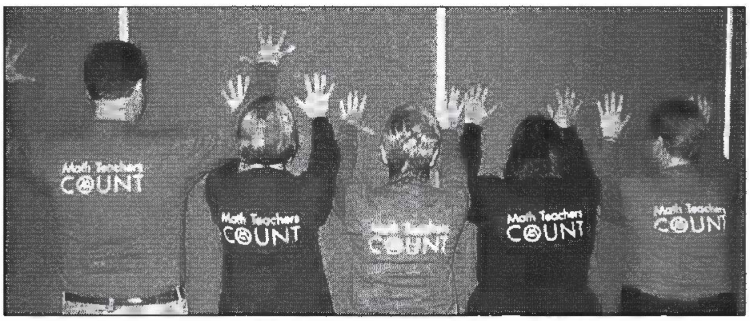
Martina Metz

Convention Committee—Susan Ludwig, Daryl Chichak, Martina Metz, Sharon Gach, Donna Chanasyk (missing: Chenoa Marcotte, Rebecca Steel, Pat Chichak, Geri Lorway, Lisa Hauk-Meeker



Registration Desk—
(above) Indy Lagu, Dave Jeary;
(below) Dave Jeary, Anne MacQuarrie

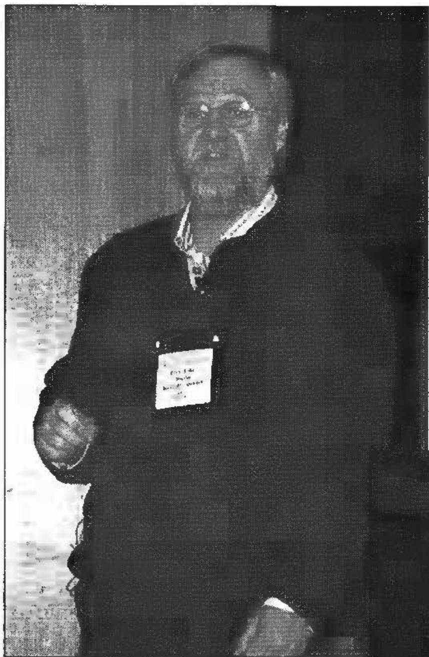




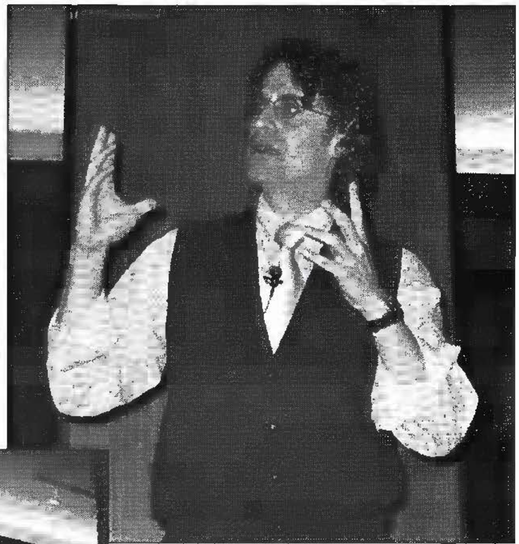
Displays

Door prizes—
Janis Kristjansson,
Martina Metz

Conference Chair Martina Metz



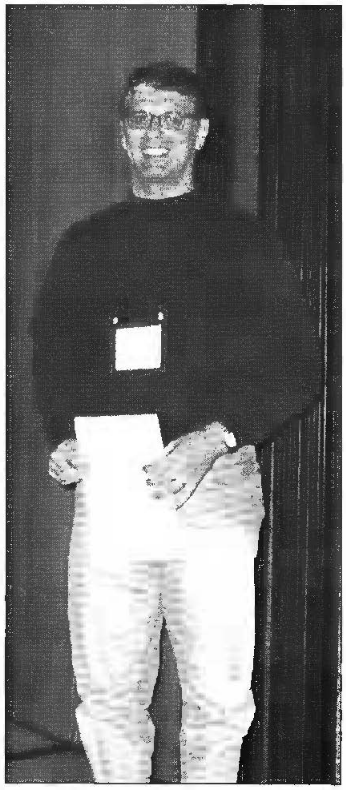
Keynote Speaker
John Mason



Keynote Speaker
Edward Burger

Dr Arthur Jorgensen Chair Award for 2006
Nicole Patrie
(presented by Sharon Gach)

2006 Alberta Mathematics Educator Award Recipient Gerald Krabbe (right) receives the award from Len Bonifacio

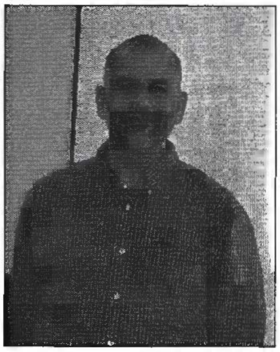


Friends of MCATA 2006
Leonard Bonifacio
(missing: Geri Lorway)


Anne Watson


Rae-Ann McMullen


Dana Newby

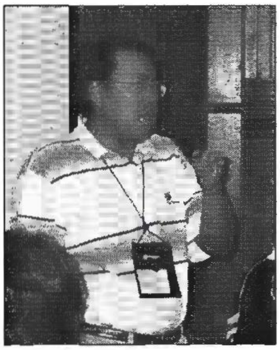

Glen Reesor


Rosalind Carson

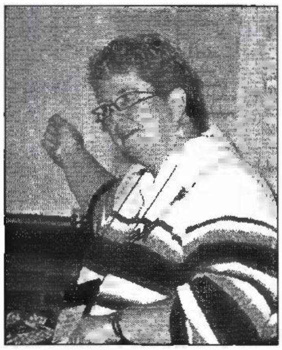

Chris Zarski


Katherine Willson


Marie Hauk


Wayne Mar

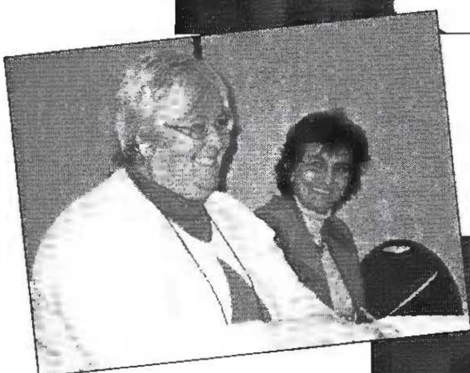

Karen Smith & Kip Rodgers


Greta Millenaar


Susan Ludwig & Carolyn Martin


Irene Meglis

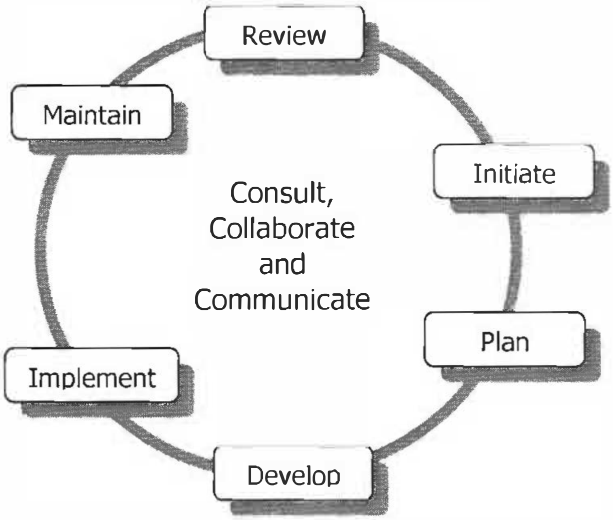

Florence Glanfield & Deanna Shostak



The Right Angle: Report from Alberta Education

Jennifer Dolecki

In the last instalment of "The Right Angle" we looked at the curriculum development and implementation cycle.



In this instalment we will take a closer look at where the mathematics programs of study fit in this cycle. The math curriculum has undergone revisions approximately every 10 to 15 years over the past 25 to 35 years.

Currently two courses are in the maintain phase: Mathematics 31 and Mathematics Preparation 10. The remainder of the K–12 mathematics program is in either the development or the implementation phase.

For K–12 mathematics, Alberta Education works with the Western and Northern Canadian Protocol (WNCP) to develop a common curriculum framework (CCF) that can be used by all of the WNCP jurisdictions to develop their own programs of studies. The WNCP CCF for K–9 Mathematics was completed in May 2006.

Since the CCF was completed, Alberta Education has worked to modify it to ensure that the specific needs of Alberta students are met. The Alberta Program of Studies for K–9 Mathematics will be completed by January 2007 and will be posted on the Alberta Education website.

The implementation schedule of the new program is shown below.

	2007	2008	2009	2010
Optional Implementation	K, 1, 4, 7	2, 5, 8	3, 6, 9	
Provincial Implementation		K, 1, 4, 7	2, 5, 8	3, 6, 9, 10

Planning for the implementation of kindergarten and Grades 1, 4 and 7 is already under way. Regional consortia offered junior high workshops during the 2005/06 school year, and in the 2006/07 school year they are offering Division I and II workshops. These workshops will continue into the 2007/08 school year. Work has started on an online guide to implementation, which will be ready for the optional implementation year. The online guide will use the same platform as the social studies online guide so that teachers will be familiar with how to navigate in that environment. Plans are also underway for a mathematics institute in July 2007 to focus on kindergarten and Grades 1, 4 and 7.

The Teacher/Researcher and Proving in High School Mathematics

Shannon Sookochoff, Elaine Simmt and David Reid

Proof and proving is largely unsuccessful in school mathematics (Reid 1995). "This lack of success seems to be related to an incompatibility between the picture of proving portrayed in schools, and the role of deductive reasoning in professional mathematics and in students' lives" (PhD diss). Based on our reading of the high school program of studies and support materials, there is certainly a narrow view of what constitutes proof and reasoning, and little, if any, advice on how to teach and assess it. For example, the *Assessment Standards for Pure Mathematics 20* relegates reasoning to a set of seven seemingly simple outcomes[1] (Alberta Learning 2002), which some teachers choose to cover in a short take-home package.[2] Further, proof and reasoning is addressed in one of the two approved mathematics textbooks (Addison-Wesley) as formal and deductive geometric proofs that are based on a set of given and well-established geometric truths rather than based on and emergent from student meaning making. The teacher is left to develop instruction that engages students in mathematical reasoning. Interestingly, the current focus on proof as formal verification has been shown to be a poor motivator for students, since it does not address their needs when doing mathematics (Reid 1995). In this study, we explored possibilities for more meaningful and engaging experiences for high school students with mathematical proof by being attentive to their meaning making, their need for proof and their actions toward proving.

We investigated what happens when a teacher broadens her vision of proof and proving in mathematics. Reid's work points to a variety of possibilities for proving in mathematics; yet the reasoning in high school mathematics classrooms is often limited to proof as verification, reasoning as deductive, proving as mechanical and formulaic (as illustrated by algebraic proofs in trigonometry) and proofs as formal (eg, two-column proofs in Euclidian geometry). Specifically, we asked

- what is the nature of the questions/tasks offered to students that encourage proof and proving actions?
- how might a teacher recognize features of proving and proof in student conversation and in the questions students pose?
- how would valuing student need for proof and proving change current evaluation practices and rubrics for assessing student work?
- how much room is there in the curriculum for analogical reasoning, unformulated proving, and preformal proof?[3]

Using Action Research to Prompt the Growth of Pedagogical Understanding

Over the last decade, Sookochoff has found herself doing action research "by accident." She describes her pattern of professional improvement in the following passage.

> First, I have to face some truths that are uncomfortable to some: I do not know everything about the mathematics I teach and even what I do know is worth reconsideration; nor do I know everything about the students I teach; nor do I have the time to keep abreast of research literature that might inform and shape my practice. What I do have is a sincere curiosity about mathematics and about knowing it—so much so that I am compelled to engage in an intense process of action research, despite the constraints of time I face as a practising classroom teacher. Here is what I have done.

First, I identify a central theme around which I want to grow. I have used polynomials, irrational numbers, rational expressions, linear algebra and, in the case of this paper, reasoning and proving. All of these explorations started with a great dissatisfaction with the way my teaching and knowing mathematics were playing out in the classroom. My irritation was motivated sometimes by student frustration, sometimes it was by a sense of confusion, as though the curriculum were not supporting me, or as though the topic were fractured and riddled with unimportant details that I could not possibly make relevant to students. Once, as in the case of linear algebra, I just wanted to expose myself to hard ideas and feel what it was like to learn outside my area of comfort.

Then (and I only see this pattern now) I summon my math friends, people I've taken courses with or from, teachers and mathematicians I have met at conferences. Mostly, I do this through e-mails, written late at night, when my own kids are asleep, and when my mind is burning with a question from the day's activity. I fire out the e-mail and I wait. And the miracle is that I don't wait long. It is amazing to me how fellow teachers and mathematicians are eager to engage in thinking about the work we love.

Things move quickly and somewhat chaotically after that. Letters go back and forth about the research that might relate or about some mathematics that could enrich what I am working on. My teaching and mathematics evolve daily. My students become part of the process, empowered by the fact that I take their questions and utterances to my network of internet colleagues. And I end up feeling so charged with the energy of all these interactions.

There are results I can see right away. But, as years pass, at least two, I seem to enter a new phase. The excitement fades and I can more calmly select from the sea of ideas that were generated in that recursive chaos. It is in this phase that I am able to edit out the mathematics that might have fascinated me, but that obscured something for my students. It is in this phase that I see the curriculum clearly and with ownership. I know I've accomplished something important in my teaching when that feeling of irritation and confusion is gone. Instead, I am steady.

This paper is a record of some of my working through of the proving outcomes in Mathematics 20 Pure. The impulse started out of a feeling of irritation (that in this case is not quite resolved) with the way the resources, the curriculum, my background and my students were interacting. I found myself complaining about the unit. The complaints led me to my friend, Elaine Simmt, who suggested I talk to David Reid, whose research has centred on student proof and reasoning for some 10 years now. From there, the process evolved just as I described above—with one exception: after about a year of informal chat on the matter of proving, we applied for and received a research grant from the Alberta Advisory Committee for Educational Studies. This allowed us to document some of what happened in the classroom.

Sookochoff initiated an action research project in which she incorporated outcomes for formal reasoning in a number of units of instruction, rather than only in the unit typically used to teach proof and reasoning. The study involved two Pure Mathematics 20 classes and one Mathematics 20 International Baccalaureate class over the course of one semester. With her coresearchers, Sookochoff worked on her practice through conversations about proof, proving, reasoning, tasks and assessment. In the spirit of action research, she began with a question about practice and worked on it through cycles of planning, implementing, evaluating and questioning. With three classes each working on the same content and processes, she was able to try out tasks and strategies with one class, develop the tasks and strategies further and use them with another class. Because the research took place over a whole term she also was able to work through the cycle in the context of differing content.

All unit planning documents, student handouts and assessment tools, as well as student work, were collected. So, too, were e-mail messages among the research team and notes based on their face-to-face conversations. Approximately 20 per cent of the lessons were observed by one of the coresearchers. In those classes, observation notes were taken and audio tapes of selected student groups were made.

In summary, Sookochoff, in consultation with Simmt and Reid, developed lessons and units to promote proving. She taught the lessons and then reflected on them through further discussion; those discussions informed subsequent lessons. In this paper we elaborate on the action research project through the presentation and discussion of one particular lesson. We include records of e-mail correspondence, a transcript from the lesson, and an analysis of the relationship between those two things and Sookochoff's evolving understanding of teaching and mathematics.[4]

A Lesson in Definitions

The following section is intended to illustrate the action research process and a particular pedagogical concern that arose in Sookochoff's lesson (includes the planning, implementation and reflection of the lesson). The section begins with an e-mail message from Sookochoff to her collaborators prior to her planned lesson and continues with their responses. A short transcript taken from the lesson as it played out in the classroom prompts further reflection by Sookochoff.

On Monday, September 06, 2004, at 6:40 PM Shannon Sookochoff wrote:

> Hi again,
> I have asked my IB students (they are running ahead of the others) to think about the following for homework:
>
> $x + 5y = 10$
> $3x + 15y = 30$
> What values for (x, y) satisfy both equations?
>
> Then, when they are confronted with an infinite set, I will ask: *Given x/3=20, how do we solve?" Students will respond. "What mathematics do you know that makes you able to _________________________? (Perhaps they will say "multiply both sides of the equation by 3.")* My idea is that we will come face to face with properties of equivalence relations that we need to build on in order to make solving by elimination work.
> How might I explain why I can multiply both sides of an equation by a constant without changing the solution? Well, I think that I would draw an analogy: *Do you accept that an equivalence relation is like a balance scale? If so, then let's put one third of an object on one side, and 20 units on the other. They balance. But we don't like only knowing what one third of the object weighs. We reason that we could triple each of these equal sides, making the unknown object whole and making the other side of the scale 20 × 3, or 60. So we see that the unknown object weighs 60. And we can verify that one third of 60 is 20.*
>
> Is this analogical reasoning sound? How might you demonstrate the reasoning behind what ends up being a property of equivalence relations?
> Shannon

Sookochoff, in planning for her lesson, is doing a thought experiment as to how the lesson might play out. She draws from her understanding of linear equations and linear systems, as well as from her understanding of equivalence. David Reid responds to her inquiry by suggesting she use the notion of an axiom—something that ties directly to proof and proving. Sookochoff is excited by this possibility and searches for meaning and ways to include the notion of *axiom* into the class discussion. This tone of inquiry is at the heart of action research.

On Tuesday, September 07, 2004, at 4:32 AM, David A Reid wrote:

> The analogy is a fine analogy, but why not try to make reference to something they might accept as an axiom/postulate?
> If $a = b$, then $ka = kb$.
>
> An equation is a statement that two things are equal. If they are equal, then the above axiom/postulate lets us multiply both sides by anything we want. I suspect IB students can cope with that.
>
> One tricky thing is the inequality. It is not true that:
> If $a <> b$, then $ka <> kb$
> because k might be 0.
>
> This is the basis for some nice proofs that $1 = 2$.
> David

On Tuesday, September 07, 2004, at 6:54 AM, Shannon Sookochoff wrote:

> Excellent! So, what is it about this truth that makes it an axiom/postulate? What tells me that I cannot prove it? What are the signs to the thinker that an idea is axiomatic? I think that this will need to become explicit today when the IBs come back with their explanations. So, if you are at your computer, David, do tell! Hey. I just thought of an answer to my question. Could it be that something is axiomatic in a particular community if no one in the community can prove it but everyone agrees that it is true?
> Shannon

The conversation with Reid is important to her pedagogical understanding. Just writing to him is enough to trigger responses to her own questions.

On Tuesday, September 07, 2004 7:52 AM, David A Reid wrote:

> > Hey. I just thought of an answer to my question. Could it be that something is axiomatic in a particular community if no one in the community can prove it but everyone agrees that it is true?
>
> Exactly! I paused for a moment about what to call the thing. First I put only *axiom* but then I decided it might not be so *self-evident*. Then I thought about how to prove it. I suspect I would have to go back to the definition of multiplication. I also suspect that what I am calling an axiom might be part of the definition.

> Excellent! So, what is it about this truth that makes it an axiom/postulate? What tells me that I cannot prove it? What are the signs to the thinker that an idea is axiomatic? I think that this will need to become explicit today when the IBs come back with their explanations. So, if you are at your computer, David, do tell!

Just now I checked to see if it is part of the definition based on Peano's axioms, which are a popular starting point for number theory. This is what I found: www.cut-the-knot.org/do_you_know/mul_num.shtml

Now I THINK I can prove IF $a = b$ THEN $ka = kb$.
But I have to go now.
David

After these fast-paced exchanges Sookochoff teaches the lesson. She writes to her research collaborators immediately after teaching the lesson to her three classes.

On Tuesday, September 07, 2004 9:19 PM, Shannon Sookochoff wrote:
Zowee!
Even though my intent was to move from an inconsistent system, ie, no solution, ie, parallel lines, on to deriving elimination, instead, I was able to bring our entire conversation to bear. Here is how.
Having asked students to consider why we can multiply both sides of an equation by a constant, they came back perplexed. I sensed a tension. "This is easy but it is puzzling." Nicolai said just about that exact phrase. We noticed that in our various explanations, we ended up using the fact to explain the fact. It felt circular, yet we all agreed that the fact was true. I then defined the tension we were noticing as characteristic of an axiom. We generated one for addition and two corollaries (usage?) dealing with division and subtraction.
Then, I put four cases (A–D) on the board.

A) What is the solution to $3x + 2y = 104$ (related to but not limited by "three shirts and two sweaters cost \$104")? I have an infinite solution set of all points (x, y) where $y = (-3/2)x + 52$. I have a line if I think geometrically. Or I have a table of values with lots of possibilities. In fact I can see a pattern (the slope) in the integral points that I can generate. I use the following string of deductions: 1) Using axioms of equivalence relations to isolate y, I recognize $y = mx + b$. 2) When I graph an equation of this form, I generate a line. (Would we call this an axiom for now?) 3) Lines are made of a set of infinite points, the slope of which is consistent (a definition). 4) If the equation generates a line, then the solution is the line, which is an infinite set of points.

Other cases we considered:
B) $3x + 2y = 104$ and $2x + y = 60$.
C) $x + 3y = -10$ and $3x + 9y = -30$.
D) $x + 2y = 4$ and $2x + 4y = 3$.

With each system, the entire class worked to describe the solution set using a string of deductions.
This we got to with the IBs. (Then developed solving by elimination.)
The regulars are still thinking about cases C and D above. Their homework is to consider how they would describe the solutions sets for C and D. And your questions, Elaine [What does it mean to find a solution?], have been so powerful for me. The students really need time to see that a linear equation in x and y (especially where y is not isolated) generates a straight line and thus has infinite solutions. They will benefit from the discussion we had today in IB.
I was surprised that all this came together today and that I was able to say *axiom* without feeling too much like an impostor.
I just looked at that Peano stuff and found it really hard to read. On the other hand, the stuff I read about in Lyn English's book was more lived reasoning, I think. Am I pointing to a distinction that you two have noticed? One where Mason, English, Johnson and Lakoff are on one side of the continuum, and Peano (and others I don't know) are on the other? Is one considered more rigorous?
Shannon

As exciting and rewarding as the lesson was for Sookochoff, her desire to deeply understand proof and reasoning is growing. She takes Peano and English, a mathematician and an educational researcher respectively, and asks how they are helping her make sense of proof and proving.

On Wednesday, September 08, 2004 2:27 AM, David A Reid wrote:

All seems to be working out well.
There is certainly a tension between Peano and English et al.[5] It is a tension that has caused trouble for mathematics education from the start. There are two different starting points: mathematics and minds. Or if you like, logic and psychology. To an expert mathematician, an axiomatic system seems really easy. You start with things that everyone recognizes as true, and then you deduce everything else according to ways of thinking that everyone accepts. This was the basis for most of the New

Math movements. It turned out not to work very well. What Lakoff, English, Varela, etc, tell us is that the ways of thinking that mathematicians assume everyone accepts are in fact not all that applicable outside mathematics, and so naturally most people don't use them all that much. The trouble is that nature doesn't give babies the axioms at birth. They have to figure stuff out other ways: by analogies, metaphors, abductions, generalizations (there are a lot of words for this thinking, but none of it is well defined). And having figured out their whole world in this way, they figure out mathematics in the same way. In fact, something that is a fascinating (but hard) question for me is how those of us who have figured out how to reason mathematically when all we had to use were nonmathematical ways of reasoning.

I haven't looked hard at the Peano stuff yet, but I will now.

David

As her conversation with Reid continues, so do her mathematics classes. She keeps her collaborators informed of pedagogical moments from her class.

On Wednesday, September 8, 2004, at 3:02 PM, Shannon Sookochoff wrote:

Must write fast:

Phillip: Why is it that a pair of intersecting lines have only one solution?

Me: (*with some fluster and some panic thinking of the word axiom*) Maybe this is an axiom. (Write on the board AXIOM, having never mentioned it before.) I think it comes down to a decision by some mathematicians. We agree to consider a line to be a blah, blah, and we agree that when two lines intersect, they intersect at one point.

Phillip: Kind of a definition, then. I see.

Kelsey: So what is an axiom? A system of two intersecting lines?

Me: No, it is truth that we know to be true but are unable to explain why. Like … I don't know. We will be talking about this more today though.

Blaine: I know an axiom. (*Holds up two fingers.*) How many fingers am I holding up ...

Gotta go get Jack. I'll send this home and pick up on it tonight.

That night Sookochoff elaborated on the events of the September 8 Math 20 Pure class on systems of equations.

Phillip: Why is it that a pair of intersecting lines have only one solution?

Me: *(with some fluster and some panic thinking of the word axiom)* Maybe this is an axiom. (Write on the board AXIOM having never mentioned it before.) I think it comes down to a decision by some mathematicians. We agree to consider a line to be an infinite array of points (each of which is a solution to an equation that we recognize to be $y = mx + b$). And we agree that when two lines intersect, they intersect at one point.

Phillip: Kind of a definition, then. I see.

Kelsey: So what is an axiom? A system of two intersecting lines?

Me: No, it is a truth that we know to be true but are unable to explain why. Like … I don't know. We will be talking about this more today though.

Blaine: I know an axiom. (*He holds up two fingers.*) How many fingers am I holding up?

Me: Two.

Blaine: How do you know? Which one is one? Which one is two?

Me: *(smiling without anything to say)* Let's make it even simpler. I'll hold up one finger. How many fingers am I holding up?

Many: One.

Me: How do you know? (*Students are pleased.*)

Phillip: Because you have five fingers, you are holding 4 down, which leaves one standing. (*The class is happy to have proven what I suggested was unprovable.*)

Later in the class, after groups of four worked on explaining why we can multiply both sides of an equation by 3 (or any number). Their explanations ranged from a concrete example: $2 \times 3 = 6$, $(2 \times 3) \times 3 = 6 \times 3$, $18 = 18$, so it works, to "each side of the equal sign is in direct proportion to itself"; lots of mention of balance; one student related the equality to a basketball game in which, when subbing in and out, each side must always have five players on the court at one time. After sharing all of this I drew their attention to the difficulty of the task; they seemed to need to state the truth within the truth. Yet, we all understood. "That," I said, "makes this idea an axiom. No one in this room can explain it. We accept all of the examples and comparisons. We agree that we can *multiply both sides of an equation by a constant and not change the equality. So it is our axiom."*

Kelsey: So something like "cookies are sweet" is an axiom?

Me: I don't know; can anyone in here point to an explanation of cookies are sweet?

Phillip: Yes, it has something to do with taste buds and biology.

Me: So, Kelsey, Phillip thinks he could get to an explanation about that, so no, it is not an axiom.

And then I gave them notes stating the axiom in the form "if m = n, *then* km = kn*" and expanded from here to division, addition, and subtraction.*
Note: Earlier in the day (period 1) a student, I can't recall who, said that "we can multiply both sides of an equation (she was thinking about an equation in two variables) by a constant because when we do, the new line generates a new point on top of the original point in the original line." She was referring to coincident lines and I think she said it better. I'll ask tomorrow!
Shannon

The e-mails and transcript above could be analyzed in number of ways. But, because this is action research, and Sookochoff is reflecting on her own teaching, the analysis here examines how the above exchanges have transformed her thoughts and opened new possibilities for future teaching. Sookochoff writes:

Phillip's first question asking how we are sure that there is only one point of intersection has me playing out some of the other options that I had in forming my response.

1. Probe Phillip's question more to determine whether it was grounded in the graph or the solution set for the system. If Phillip was thinking completely graphically, then I might have asked him to visualize two intersecting lines, two coincident lines and two parallel lines. But he may have been asking for my help to make the leap from two different solution sets from two different linear equations to the solution set for the system of equations. Or he may have been constructing the link between the solution set and the graphical representation of the system and its solution.
2. Move toward a group explanation of why two distinct and nonparallel lines in a plane intersect in exactly one point. I don't think I had thought of an explanation at the time, so I could not have led such a discussion until these last few weeks.
3. Call the knowledge axiomatic and explain what that means. This is what I chose to do and I am convinced that my choice, although fine, was influenced by my not having an explanation at the time and my then current struggle with the meaning of axiom. It worked well to engage students in meaning making and group discussion. Students seemed to like talking about the explainability of an idea and were intrigued with the idea that definitions and axioms stand outside the assertions that we can reason out.
4. Or I could have offered something that combined my response to Phillip with my response to Kelsey. I could have said, "I don't have an explanation right now. Does anyone else? Do we all accept that it is true? Can anyone think of an example? How about a counterexample? Well, IF we do not have an explanation AND we accept the idea to be true, THEN in our classroom at this moment we will call the idea an axiom. If we are able to find a convincing explanation in the next while, we will move it from the axiom board and onto the theorem board."

I like #4 the best right now, because it brings all sorts of reasoning to bear. Had I used that response, I would have been asking kids to sort types of truth, the proven versus the axiomatic. In the few questions I have listed in response #4, I have referred to all of the reasoning outcomes from the curriculum.[6] The call for a specific example or counterexample builds toward an inductive approach to testing the idea, alluding to outcomes 4.1 and 4.3. By using the connecting word *and* and structuring the definition of axiom as an if–then statement, I embed outcomes 4.2 and 4.4 into a student-initiated conversation. And last, in the sorting of mathematical truths into axioms and theorems, we create a space and community-specific need for what I see as the most difficult of the reasoning outcomes: proving an assertion (outcome 4.5). Teaching in this way elevates mathematical reasoning from a discrete unit to an ongoing process and the connective syntax of the mathematical concepts we study.

I think, too, that #4—let's call it "Attempt/postpone the explanation and sort the assertion"---can live in many contexts in the mathematics classroom. Students are remarkable in their ability to question why an assertion is true; they ask their teacher, "Why does the discriminant tell us how many roots we can expect for a quadratic?"; "Why do we switch the inequality sign when we multiply both sides of the inequality by a negative number?"; "Why do the roots of an equation have so much to do with the factors when the equation is set to zero?".[7] Their questions point, I think, to our students' inherent need for proof. Recognizing the students' questions as evidence of their need for proof allows the teacher to feed that need and thus brings students into the culture of proving in mathematics.

This brings me to focus on the two categories of truth I mentioned above: proven and axiomatic.

I suggest that much of teaching explores the tension between these two ways of viewing assertions. I also suggest that one problem in our mathematics classrooms (and maybe in many other classrooms, as well) is our treatment of most knowledge as axiomatic—"It just is!" My own jump to label "two distinct and nonparallel lines in space intersect in exactly one point" as axiomatic is a case in point. In asking me why, Phillip challenged my mathematics ability. To answer him, I needed to honestly ask myself why. I needed to resist panic in the face of public uncertainty. I needed to make transparent a vital mathematical task, one in which I ask why something I take to be true is indeed true. And I needed to know, in both an emotional and an intellectual way, that not knowing why is legitimate. In exposing the struggle to explain why and entertaining the possibility that we cannot, teachers can underscore the nature of the mathematical assertions brought forth in the classroom. It should be noted, too, that the explainability of a given assertion can be decided in each specific classroom—what is an axiom for me and my students today may not be for my colleague down the hall. And six months from now, my students and I may find that we can indeed explain what we thought was an axiom.

Proof, Proving and Reasoning Through Action Research

With the illustration above we are able to respond to the research questions we posed when we began this study. But in the true nature of action research, these questions are not answered once and for all. Rather, we are able to identify additional questions to work on.

There are some things, Sookochoff believes, that worked to promote proving and reasoning activity among her students in those Grade 11 mathematics classes. We did well to

- integrate reasoning and proof into all content areas;
- put students into groups for proving together in discussion with one another;
- post theorems, colour-coded as to proven or accepted as true and identified as Grade 11 or pre-Grade 11 theorems, so as to have them available in the public domain;
- deal with theorems and vocabulary as needed;
- publish a collection of all that we know to be true for the class members; and
- ask the question, "We seem stuck; can anyone offer something that might get us unstuck?"

Of course, there are the things that Sookochoff will further develop the next time she teaches. Her notes to herself include the following advice:

- Keep expectations for student proving focused.[8] (To some extent, Sookochoff found that the specific standards for circle geometry,[9] which require students to prove two particular theorems from circle geometry, did not encourage this focused approach.)
- Go back and forth between specifying, proving and applying. (In Sookochoff's teaching of geometry, it was tempting to separate these proving activities, which she thinks obscured the connections between them.)
- Engage students in the issues of proof. The conversations would ideally come out of student questions and comments. However, some topics that a teacher might consider and could deliberately initiate, perhaps in a daily 10-minute group conversation, are listed below.
  - When can we name a proof as "_____ Theorem" and never again prove it? (We can make this happen by clearly titling and posting the assertions that our community accepts or proves to be true.)
  - Who decides how much is enough explanation?
  - Which truths have converses and contrapositives that are true? Which do not?
  - What does shifting from "Is this true?" to "What makes you sure this is true?" signify?
  - Is proof beautiful? (This could be a chance to share some particularly beautiful proofs from our canon—perhaps the Pythagorean Theorem, with its many proofs and unquestionable fame, could highlight the desire to explain *why* we know over *whether* we know.)
  - What is the difference between a definition, an axiom, a theorem and a postulate? (This relates nicely to the sample lesson discussed earlier.)
  - What is the nature and structure of a legal argument? Forensic evidence? Literary essay? Opinion paper? And how do they compare to a mathematical proof?

In terms of a teacher's practice we have addressed the questions that focused our study. We leave the reader with some more pointed responses to those same questions.

We asked, "What is the nature of the questions/tasks offered to students that encourage proof and proving actions?" Every high school teacher has asked some version of the question Sookochoff posed to her IB students:

$$x + 5y = 10$$
$$3x + 15y = 30$$

What values for (x, y) satisfy both equations?

There is nothing remarkable in the question. The difference lives in the context in which the questions were posed. From our work we have seen that it is essential to create space for responding to questions that arise from the desire for meaning making. (This works at the level of both teacher and student meaning making.) For the students, those spaces emerged in large group discussions that invited conjectures, challenged ideas and demanded reasoning. For the teacher, the space was created by having colleagues interested in the conversation of proof and proving. Our research suggests to us that the questions teachers ask must be accompanied by an inquiring stance, intense curiosity and a desire for things mathematical.

Also of interest to us was how a teacher might recognize features of proving and proof in student conversation and in the questions students pose. In this case, it was evident that the conversation between Sookochoff and Reid was key in Sookochoff's meaning making. However, it was close listening—that is, listening for student meaning making rather than listening for an expected particular response—that led to opportunities for Sookochoff to recognize proof and proving in student responses.

We asked how valuing student need for proof and proving would change current evaluation practices and rubrics for student work. Clearly, asking students to reason is key to any assessment. Finding ways to evaluate their responses is the challenge, and we will address it in a future paper.

Finally, we wondered how much room there is in the curriculum for analogical reasoning, unformulated proving and preformal proof. We purposely used an illustration from a nontraditional topic for addressing proof and proving in our high school mathematics curriculum. The wonderful part of this action research study was the deliberate intention to integrate proof, proving and reasoning throughout all the topics in the curriculum. Further evidence that there is plenty of room for reasoning in school mathematics will be offered in future papers.

In this paper we have illustrated how a teacher, engaged in action research in collaboration with colleagues, worked on her own understanding of mathematics, mathematics pedagogy and mathematics curriculum. We hope that classroom teachers benefit from our research in two ways: (1) as a strategy for working on their own teaching questions, and (2) for working on proof, proving and reasoning in high school mathematics. Further, we hope that university-based and school-based researchers find in our study some inspiration to work towards truly collaborative approaches to educational research that creates deeper understanding of mathematics teaching and learning.

Notes

1. Formal reasoning outcomes:
   - 4.1 Differentiate between inductive and deductive reasoning.
   - 4.2 Explain and apply connecting words, such as *and*, *or* and *not*, to solve problems.
   - 4.3 Use examples and counterexamples to analyze conjectures.
   - 4.4 Distinguish between an *if–then* proposition, its converse and its contrapositive.
   - 4.5 Prove assertions in a variety of settings, using direct reasoning.

   Circle geometry outcomes:
   - 5.2 Prove the following general properties, using established concepts and theorems:
     - The perpendicular bisector of a chord contains the centre of the circle.
     - The angle inscribed in a semicircle is a right angle.
     - The tangent segments to a circle from any external point are congruent.
   - 5.5 Verify and prove assertions in plane geometry, using coordinate geometry and trigonometric ratios as necessary.

2. During the marking of the June 2005 diploma exams, teachers talked with one another about getting better results on the exam. One strategy was to reduce the time spent on the Math 20 Pure units that had no follow-through in Math 30 Pure to give them more time to spend on items that relate directly to Math 30 Pure. They specifically talked of reducing the item on reasoning to a take-home booklet.

3. This last question requires some clarifications. The importance of analogical reasoning in mathematics has been described at length by Pólya (1968). It involves making a conjecture based on similarities between two situations. *Formulation* of proving refers to the reasoners' knowledge or awareness of their own reasoning. *Unformulated proving* refers to deductive reasoning of which the reasoner is mostly or completely unaware. *Preformal proofs* (Blum and Kirsch 1991) are a step in the direction of acceptable mathematical proofs. They might involve hidden assumptions and use informal language and notation, and might also include references to analogical or inductive evidence for a conjecture.

4. We chose not to elaborate on the evolution of the co-researchers' understanding.

5. This tension is related to another that also causes difficulty for mathematics educators. *Proof* has different meanings in different institutional contexts (Recio and Godino 2001). Most important here, *proof* has one meaning in logic and the foundations of mathematics, and another meaning in the practice of professional mathematicians. In logic and foundations of mathematics, proof is connected to deductive argumentations that take place in axiomatic and formal systems. In the practice of professional mathematicians, however, while "deductive proof is the prototypical pattern of mathematical proof … this formalist rigor decreases in practice" (p 94). Similarly, words like axiom have different meanings in these two institutional contexts. The Peano axioms discussed in the e-mails are not the formal versions, but rather the less formal ones used by professional mathematicians in their practice, and as Shannon is a teacher of mathematics, not of foundations or logic, her meanings for proof and axiom are based in the practices of mathematicians, not in the formalisms of logicians.

6. Formal reasoning outcomes:
   - 4.1 Differentiate between inductive and deductive reasoning.
   - 4.2 Explain and apply connecting words, such as *and*, *or* and *not*, to solve problems.
   - 4.3 Use examples and counterexamples to analyze conjectures.
   - 4.4 Distinguish between an if–then proposition, its converse and its contrapositive.
   - 4.5 Prove assertions in a variety of settings, using direct reasoning.

7. I deliberately did not list a question from geometry because students did not tend to offer their *why* questions there. Perhaps they needed no convincing when they could see the truth apparent in a visual illustration. It is ironic, then, that we often situate the task of proving within geometry, where students do not seem to *need* proof.

8. The task of writing proofs where algebra, plane geometry, coordinate geometry and trigonometry come to bear is highly complex. Students must bring together many years of their education in mathematics. And they must form a logical sequence of statements and reasons in a way that satisfies their teacher's idea of what proof looks like. Most Grade 11 students find this overwhelming. Instead of proving these particular assertions from circle geometry, perhaps the Grade 11 students would be better served by engaging in more narrowly defined proving tasks, such as discussing why they are sure of a particular property of equivalence relations. Alternatively, the theorem to be proven could have a greater cultural/historical importance (and thus be a better motivator for students) than the theorems from circle geometry. Examples here could be "interior angles in a triangle add up to 180 degrees" or the Pythagorean Theorem. However, a small canon of finely crafted and established proofs could make excellent class reading. As currently written, the curriculum seems to encourage students to memorize the two named proofs. And memorization is not a proving task.

9. Circle geometry outcomes:
   - 5.2 Prove the following general properties, using established concepts and theorems:
     - The perpendicular bisector of a chord contains the centre of the circle.
     - The angle inscribed in a semicircle is a right angle.
     - The tangent segments to a circle from any external point are congruent.
   - 5.5 Verify and prove assertions in plane geometry, using coordinate geometry and trigonometric ratios as necessary.

References

Alberta Learning. 2002. *Outcome for Assessment Standards for Pure Mathematics 20*. Edmonton, Alta: Alberta Learning. www.education.gov.ab.ca/k_12/curriculum/bySubject/math/pure20stand.pdf

Alexander, R, and B Kelly. 1998. *Mathematics 11 (Western Canadian Edition)*. Don Mills, Ont: Addison-Wesley Longman.

Blum, W, and A Kirsch. 1991. "Preformal Proving: Examples and Reflections." *Educational Studies in Mathematics* 22: 183–203.

Pólya, G. 1968. *Mathematics and Plausible Reasoning*. (2nd ed). Princeton, NJ: Princeton University Press.

Recio, A M, and J D Godino. 2001. "Institutional and Personal Meanings of Mathematical Proof." *Educational Studies in Mathematics* 48: 83–99.

Reid, D. 1995. "The Need to Prove." PhD dissertation, University of Alberta.

Shannon Sookochoff teaches mathematics at Jasper Place High School, in Edmonton. Her ongoing research explores various facets of her classroom practice in an effort to offer meaningful mathematics experiences to all students. Shannon employs small group interaction, peer explanation, guided investigations, ongoing projects, daily reasoning experiences and family math among her teaching strategies.

Elaine Simmt is an associate professor of secondary mathematics education at the University of Alberta. Her ongoing research program includes studies of mathematics knowing-in-action, in secondary school classrooms and in the context of teachers' knowledge. She has a particular interest in understanding classrooms as complex learning systems.

David A Reid is an associate professor in mathematics education at Acadia University. His research focuses on mathematical proof, reasoning and emotion. He teaches methods and issues related to mathematics teaching to prospective elementary school teachers. For more information, see http://ace.acadiau.ca/~dreid/.

Looking at the Algorithm of Division of Fractions Differently: A Mathematics Educator Reflects on a Student's Insightful Procedure

Jérôme Proulx

Introducing the Procedure

A colleague of mine, upon return from China, reported to me this procedure to divide fractions used by an 11-year-old[1]:

$$\frac{26}{20} \div \frac{2}{5} = \frac{26 \div 2}{20 \div 5} = \frac{13}{4}$$

My first reaction was to doubt the correctness of the procedure and solution, but then I realized that it was indeed mathematically correct and also that many interesting connections could be established with other operations on fractions (+, – and ×).

This article is in the spirit of, and takes its insights from, articles by Robert Davis (1973) and Stephen Brown (1981). Each had observed an interesting and nonstandard mathematics procedure carried out by a student and decided to report on it to bring forth the insights and the underpinning connections and concepts. This endeavour appears to be quite rich on many points, as Brown explained some 25 years ago:

> One incident with one child, seen in all its richness, frequently has more to convey to us than a thousand replications of an experiment conducted with hundreds of children. Our preoccupation with replicability and generalizability frequently dulls our senses to what we may see in the unique unanticipated event that has never occurred before and may never happen again. That event can, however, act as a peephole through which we get a better glimpse at a world that surrounds us but that we may never have seen in quite that way before. (Brown 1981, 11)

Now let's have a deeper look into this intriguing division procedure.

First Question: Is That Procedure Correct?

The first questions that come to mind concerning that procedure are "Is this correct? If so, how does it work?" Then, when we answer these questions, we ask "Why weren't we taught that in schools?" or "Why don't we teach that in schools?"

One first way of being convinced of its correctness is to solve it ourselves, for example, by using the invert and multiply algorithm: $\frac{26}{20} \div \frac{2}{5} = \frac{26}{20} \times \frac{5}{2} = \frac{130}{40} = \frac{13}{4}$. However, arriving at the same answer in a particular instance can leave some doubt that it would always work, even if it seems so. A more interesting question is "Why does it work?"

Looking closely at the multiplication algorithm, one realizes that it is mostly the same procedure, which is—in a very dry manner—to multiply the numerators together and multiply the denominators together. In this case, it is dividing the numerators and dividing the denominators. Hence, because division is also a multiplicative instance,[2] this procedure is indeed correct. From this, the following generalization can be deduced:

$$\frac{a}{b} \div \frac{c}{d} = \frac{a \div c}{b \div d}$$

And by playing with the multiplication algorithm, we can arrive at it directly, since

$$\frac{a}{b} \times \frac{d}{c} = \frac{a \times d}{b \times c} = \frac{a \times d}{c \times b} = \frac{a}{c} \times \frac{d}{b} = (a \div c) \times \frac{d}{b} = (a \div c) \times \frac{1}{\left(\frac{d}{b}\right)^{-1}}$$

$$= (a \div c) \times \frac{1}{\left(\frac{d^{-1}}{b^{-1}}\right)} = (a \div c) \times \frac{1}{\left(\frac{b}{d}\right)} = (a \div c) \times \frac{1}{(b \div d)} = \frac{a \div c}{b \div d}$$

Second Question: Why Don't We Teach This in Schools?

The answer to this second question lies in the fact that this procedure is only helpful in a limited number of cases. For example, if the fractions to be divided are $\frac{2}{5}$ and $\frac{3}{7}$, this procedure does not bring us very far toward the answer:

$$\frac{2}{5}\div\frac{3}{7}=\frac{2\div 3}{5\div 7}$$

And so, even though $\frac{2}{5}\div\frac{3}{7}=\frac{2\div 3}{5\div 7}$ is correct, it is simply not very helpful in finding an answer. However general in the sense that it is applicable in all cases, it cannot be considered a good algorithm since it sheds some light on the answer for only a small number of cases for which the numerators and the denominators are respectively divisible. Because this algorithm (dividing numerators together and dividing denominators together) helps in only a specific number of cases, it can be seen as a "particular" procedure.

Bringing This Procedure to Mathematics Teachers

In my research, I brought this interesting procedure to the secondary-level mathematics teachers with whom I work in professional development sessions. As predicted, they were amazed and curious about the correctness of this procedure to calculate with fractions. (Of course, I brought one that worked and produced results!)

A comment was made, however, that it could be interesting to work toward a generative way or an overall procedure of computing with all types of operations on fractions, because the teachers said that students have a hard time making sense of all four operations and their algorithms.[3] Thinking about what is normally done in addition and subtraction—that is, to write the fractions with a common denominator—one teacher wondered if we could not do this for the multiplication of fractions also, which would be

$$\frac{a}{b}\times\frac{c}{d}=\frac{ad}{bd}\times\frac{cb}{db}=\frac{adcb}{(bd)^2}=\frac{ac\cdot bd}{bd\cdot bd}=\frac{ac}{bd}.$$

As I explained that this was unnecessary and could complicate the calculations for no reason, we realized that maybe that *was* what was needed in the previous division algorithm to make it work. Indeed, transforming each fraction to have a common denominator makes the algorithm useful for any division of two fractions because both fractions' denominators would be the same. Therefore, making the new fraction obtained out of 1, creating a division by 1,

$$\frac{a}{b}\div\frac{c}{d}=\frac{ad}{bd}\div\frac{cb}{db}=\frac{ad\div cb}{bd\div db}=\frac{ad\div cb}{1}=\frac{ad}{cb}$$

This makes the "particular" algorithm an encompassing and efficient algorithm that always brings us to the answer for dividing any two fractions. However, the person using this procedure needs to know conceptually that two numbers dividing each other can also be written in the fraction form—something that is not obvious and needs to be worked on (Davis 1975). At the secondary level, though, it can be assumed that students can make or even create that link.

The Final Question: Why Does This Work Again? Why Does the Multiplication of Fractions Algorithm Work?

Maybe this last set of questions sounds obvious, but the whole premise of accepting that we can indeed divide the numerators together and the denominators together is based on the acceptance of the multiplication algorithm. Hence, I started to wonder why this algorithm works: Why can we multiply fractions that way? What is the meaning behind this algorithm?

Of course, as I often do with pre- and inservice teachers, it is possible to illustrate it with the multiplication of fractional areas, or with folding pieces of paper (eg, Boissinotte 1998). These approaches represent very nice ways to make sense of the algorithm itself. For example, to multiply $\frac{2}{3}\times\frac{1}{4}$, I can say and show by folding areas of paper that I take a quarter of $\frac{2}{3}$ of a piece of paper. This is very nice, but am I able to make sense of it by only using the numbers themselves with no recourse to material? Can I explain why $\frac{a}{b}\times\frac{c}{d}=\frac{a\times c}{b\times d}$ works?

In fact, I must explain it if I want to use it as an argument to assert that the new division algorithm is indeed suitable. In order to make sense of it, I thought about the following explanation.

To stay concrete and less abstract, let's take an example: $\frac{4}{5}\times\frac{2}{3}$, which, with the algorithm, would give $\frac{8}{15}$. If I say it in words, it is "four fifths multiplied by two thirds," so I multiply by two thirds. One way to see it is that I first multiply by one third, and then I double the amount, because I wanted to multiply it by twice as much (by *two* thirds and not by *one* third). So, first, let's multiply by one third.

As I said before for one quarter, multiplying by a third means that I want a third of the amount. Wanting a third of an amount means that I want to divide the amount in three. Doing that makes each part of the amount three times smaller—they are indeed divided in three. So, in the case of $\frac{4}{5}$, each fifth becomes a fifteenth, and so I have four fifteenths now (instead of four fifths). Doing this explains why I have to multiply the denominator 5 with the denominator 3—because each part becomes three times smaller, and so becomes a fifteenth. So, the first sequence just explained could be represented like this:

$$\frac{4}{5}\times\frac{2}{3}=\frac{4}{5}\times\left(\frac{1}{3}\times 2\right)=\left(\frac{4}{5}\times\frac{1}{3}\right)\times 2=\left(\frac{4}{5}\div 3\right)\times 2=\left(\frac{4}{15}\right)\times 2$$

Now, I have four fifteenths ($\frac{4}{15}$). I still have to double it, because I multiplied it by $\frac{1}{3}$ and not by $\frac{2}{3}$. Because $\frac{2}{3}$ is twice as big as $\frac{1}{3}$, my answer should be twice as big. I have $\frac{4}{15}$ and I want twice that, and so my answer should be $\frac{8}{15}$. Here, because the number of parts is represented by the numerator, and I want twice that number of parts, I multiply the numerator 4 by the 2 in the algorithm. Hence

$$\left(\frac{4}{15}\right)\times 2=\frac{4\times 2}{15}=\frac{8}{15}$$

Doing that is dismantling the algorithm into a sequence of conceptual steps, which, in an algorithm, are normally hidden (Bass 2003). And so, multiplying by a fraction means to (1) make each part a denominator number of times smaller and (2) take a numerator number of times the parts that are there. This can be summarized by the following generalization.[4]

$$\frac{a}{b}\times\frac{c}{d}=\frac{a}{b}\times\left(\frac{1}{d}\times c\right)=\left(\frac{a}{b}\times\frac{1}{d}\right)\times c$$

$$=\left(\frac{a}{b}\div d\right)\times c=\frac{a}{bd}\times c=\frac{a\times c}{bd}=\frac{ac}{bd}$$

Concluding Remarks

This new procedure for dividing fractions, which at first seemed wrong and did not feel genuine, created for me a list of connections regarding operations on fractions. Whereas I might have been the only one to be puzzled by it, I realized that my own colleagues and the secondary-level mathematics teachers with whom I worked were also very intrigued and surprised by the correctness of that procedure.

This procedure brought me to try to better understand other algorithms and operations on fractions, especially that of multiplication, which I realized, by asking these around me, is mostly taken for granted. In that sense, while the question of "why does the algorithm of multiplication work?" can sound obvious, its answer is not immediately obvious. Even now, you may not be convinced of the tentative explanation that I have elaborated above—and maybe neither am I!

I did not intend this article to show a better or a new algorithm to divide fractions, and certainly never aimed to solve the overarching, difficult problem of understanding this sort of computation. It is tempting to say that difficulties experienced in the domain of division of fractions will remain, because division of fractions is difficult to understand and conceptualize. The goal of this short article was to raise awareness of this issue, to play with numbers and, hopefully, to bring new ideas and insights about these calculations to the everyday mathematics classroom.

Notes

1. The colleague is David Pimm, whom I thank for the conversations on the issue. I also want to thank Mary Beisiegel for many discussions on this.

2. Indeed, problems requiring the operations of multiplication or division are often seen as part of the same family of problems. See, for example, the work of Vergnaud (1988) or Carpenter et al (1999).

3. Of course, it could be argued that operations on fractions need not be reduced to their algorithm (and research has shown that in many cases), but this is in another domain of discussion.

4. The explanation I have offered here mostly serves as an aid to understanding and not as a mathematical proof. It does, however, serve well its goal of bringing meaning to the algorithm of multiplication of fractions.

References

Bass, H. 2003. "Computational Fluency, Algorithms, and Mathematics Proficiency: One Mathematician's Perspective." *Teaching Children Mathematics* 9, no 6: 322–327.

Boissinotte, C. 1998. "Des fractions de papier." *Envol* 102: 45–50.

Brown, S I. 1981. "Sharon's 'Kye'." *Mathematics Teaching* 94: 11–17.

Carpenter, T P, E Fennema, M L Franke, L Levi, and S B Empson. 1999. *Children's Mathematics: Cognitively Guided Instruction*. Portsmouth, NH: National Council of Teachers of Mathematics.

Davis, R B. 1973. "The Misuse of Educational Objectives." *Educational Technology* 13, no 11: 34–36.

———. 1975. "Cognitive Processes Involved in Solving Simple Arithmetic Equations." *Journal of Children's Mathematical Behavior* 1, no 3: 7–35.

Vergnaud, G. 1988. "Multiplicative Structures." In *Number Concepts and Operations in the Middle Grades*. Ed J Hiebert and B Behr. Reston, Va: National Council of Teachers of Mathematics.

Jérôme Proulx wrote this article as a doctoral candidate in the Department of Secondary Education at the University of Alberta. He is currently a professor in the Faculty of Education at the University of Ottawa.

Binomial Probabilities on a Multiple Choice Test

Bonnie H Litwiller and David R Duncan

Teachers are always seeking situations in which binomial probabilities can be exemplified. One such setting with which students are acquainted involves multiple choice examinations.

Suppose that Roy took a 40-question multiple choice test; each test question had five possible answers. How likely is it that his score on the test was 4 or less? To answer this question we will make repeated use of the binomial formula:

> Suppose that n independent trials are performed. For each trial the probability of success is p while the probability of failure is q, where $p + q = 1$. The probability of exactly r successes and $n - r$ failures in these n trials is then
> $\frac{n!}{(r!)(n-r)!} p^r q^{n-r}$, where the notation $C(n, r)$ is commonly used to symbolize the coefficient $\frac{n!}{(r!)(n-r)!}$.

Case 1: Assume that Roy guessed on every question. Using binomial probability, the probability of at most 4 correct responses can be computed as follows:

Exactly 4 correct (and 36 wrong):

For each of the 40 questions, the probability that Roy selects the correct answer is $\frac{1}{5}$, while the probability he selects the incorrect answer is $\frac{4}{5}$. The probability of exactly 4 correct and 36 incorrect responses is then

$C(40, 4)(0.2)^4(0.8)^{36} = 0.04745$

Exactly 3 correct: $C(40, 3)(0.2)^3(0.8)^{37} = 0.02052$

Exactly 2 correct: $C(40, 2)(0.2)^2(0.8)^{38} = 0.00648$

Exactly 1 correct: $C(40, 1)(0.2)^1(0.8)^{39} = 0.00133$

None correct: $C(40, 0)(0.2)^0(0.8)^{40} = 0.00013$

Since these are mutually exclusive events, the total probability for these five instances is then the sum of the five individual probabilities, or 0.0759 (to four decimal places).

Case 2: Suppose that Roy found a single question for which he knew the answer, then guessed on the remaining 39. Since one correct response is assured, we calculate the probability that he correctly answered 0, 1, 2 or 3 of the remaining 39 questions. This probability is

$C(39, 0)(0.2)^0(0.8)^{39} + C(39, 1)(0.2)^1(0.8)^{38} +$
$C(39, 2)(0.2)^2(0.8)^{37} + C(39, 3)(0.2)^3(0.8)^{36} = 0.0332$

Case 3: Roy knows the answers to exactly 2 questions. The probability that he answers at most two of the remaining 38 questions correctly is:

$C(38, 0)(0.2)^0(0.8)^{38} + C(38, 1)(0.2)^1(0.8)^{37} +$
$C(38, 2)(0.2)^2(0.8)^{36} = 0.0113$

Case 4: Roy knows the answer to exactly 3 questions. The probability that he answers at most one of the remaining 37 questions correctly is

$C(37, 0)(0.2)^0(0.8)^{37} + C(37, 1)(0.2)^1(0.8)^{36} = 0.0050$

Case 5: If Roy knows exactly 4 answers, the probability that he fails to answer a single other question correctly is

$(0.8)^{36} = 0.0003$

Challenges to the reader and students

1. Redo these types of problems, varying the maximum score, the length of test and the number of responses per question.
2. Alter the problem by supposing that Roy is able to eliminate a certain number of responses per question, but must guess among the remaining possibilities.
3. Determine how a calculator could be used to sum the probabilities in the cases described above without having to calculate each probability separately.
4. Suppose that the probability of recovery within one year after therapy is 0.7 for patients with a certain disease. Find the probability that exactly two of the four patients currently being monitored recover within one year after therapy. Find the probability that at least two patients recover. If n patients are receiving therapy, write an expression that will give the probability that exactly k of them will recover, in terms of n and k.

5. A baseball player has a batting average of 0.250. Determine the probability that this player gets
   a) exactly one hit in his next five times at bat.
   b) at least three hits in his next five times at bat.
   c) a hit each time in his next five times at bat.

   Draw a graph of $P(k)$ vs k, where k is the number of hits during the next five times at bat, $k \in \{0, 1, 2, 3, 4, 5\}$ and $P(k)$ is the probability of k hits.

Bonnie Litwiller and David Duncan are emeritus professors of mathematics at the University of Northern Iowa. They taught a number of both content and methods courses for prospective and inservice teachers. Both are past presidents of the Iowa Council of Teachers of Mathematics. Bonnie Litwiller is also a past board member of both the National Council of Teachers of Mathematics and the School Science and Mathematics Association. They have published over 950 articles for mathematics teachers of all levels.

Solving First Order Linear Differential Equations by Using Variation of Parameters

David E Dobbs

1. Introduction

One of the most significant roles of mathematics has been to address the so-called "inverse problem" in science. This concerns providing information about a quantity y on the basis of having measured some experimental trace that has been left by y. In calculus, when y is a differentiable function, one such particularly relevant trace is the derivative $y' = \frac{dy}{dx}$. In solving an ordinary differential equation (ODE), one seeks to determine all the solutions y that satisfy the given ODE. The theory of ODEs (as in, for instance, Nagle, Saff and Snider 2004) makes it useful to know the order of an ODE—namely, the highest integer n such that $y^{(n)}$, the n^{th} derivative of y, appears nontrivially in the given ODE. The cases $n = 1, 2$ are of particular importance because of applications in science and engineering (with the second derivative often playing the role of acceleration in applications of Newton's Second Law of Motion). The most complete theory in the subject has been developed for the class of linear ODEs; that is, ODEs dubbed "linear" because their analysis is often facilitated with the aid of matrix theory, which is also known as linear algebra. Occasionally, more elementary algebra becomes relevant, as in the classical solution of the second order ODE with constant coefficients, $ay'' + by' + cy = g(x)$, where the roots of the associated quadratic polynomial, $aT^2 + bT + c = 0$, play a crucial role (see Nagle, Saff and Snider 2004, chapter 4).

For several decades, the method of variation of parameters (also known as "variation of constants") has been a mainstay in the typical first course on ODEs. This method for solving n^{th} order linear ODEs is usually considered first for the case $n = 2$ (as in Nagle, Saff and Snider, 2004, section 4.6) and, in some courses and texts, later for the case $n \geq 3$ (as in Nagle, Saff and Snider, 2004, section 6.4). The treatment of variation of parameters is central to any first course on ODEs, as it is part of a (hopefully, gentle) introduction to the study of linear operators and the principle of superposition. Remarkably, this central role could be played earlier, when considering the case $n = 1$, which is instead usually treated by various ad hoc methods. The main purpose of this note is to rectify matters by showing how the above-stated principles of variation of parameters can be introduced very early in a course on ODEs to solve the general first order linear ODE. Moreover, our treatment of the case $n = 1$ has the classroom advantage of being able to focus on the differential equation aspects, as we will need none of the algebraic machinery and background (such as determinants, Wronskians and Cramer's Rule) that are needed to implement variation of parameters in the case $n > 1$.

Section 2 contains a derivation of the method promised in the title of this note. We have found that this theoretical presentation is well received in the first unit of an ODE course. Examples 3.1 and 3.2 illustrate the use of this method. Rather than depending on the abstract considerations in section 2, the presentation of examples 3.1 and 3.2 repeats some of those ideas in a concrete situation and is thus essentially self-contained. In this way, examples 3.1 and 3.2 can serve as models for a classroom presentation of our method in classes where the abstract considerations in section 2 may seem inappropriate. Remark 3.3 identifies what we see as the two most important pedagogical advantages of our method over the usual method that involves integrating factors. In closing, remark 3.4 suggests a new role for the topic of integrating factors in a first course on ODEs.

2. A Derivation Based on the Homogeneous Case

The method of variation of parameters works in general as follows. To solve a nonhomogeneous linear ODE, first obtain a formula for the general solution of the corresponding homogeneous linear ODE, and then determine how the arbitrary *constants* appearing in that formula would have to be reinterpreted as *functions* in order for the reinterpreted formula to produce a solution of the given nonhomogeneous ODE. Let us now see how this method can be applied to solve the general first order linear ODE. $y' + P(x)y = Q(x)$ (where $P(x)$ and $Q(x)$ are continuous functions defined on some open interval and, as above, y' means $\frac{dy}{dx}$).

The corresponding homogeneous linear ODE is $y' + P(x)y = 0$ or, equivalently, $\frac{dy}{y} = -P(x)dx$. This is a (variables) separable ODE, which is often the only type of ODE whose solution is typically studied before the topic of first order linear ODEs is considered in an ODE course. As usual, one can solve this separable ODE by integrating both sides, with the result that $\ln(|y|) = -\int P(x)dx + C^*$, where C^* is an arbitrary constant. Exponentiation leads to the formula $y = Ke^{-\int P(x)dx}$, where $K = \pm e^{C^*}$. Let $v := e^{-\int P(x)dx}$, ignoring the constant of integration in the exponent. (It is interesting, but not essential, to note that $v = \mu^{-1}$, where $\mu := e^{\int P(x)dx}$ is the integrating factor that is used in the typical textbook solution of first order linear ODEs.) Thus, the general solution of the corresponding homogeneous linear ODE is $y = Kv$. It follows that v is a particular solution of $y' + P(x)y = 0$. We proceed to *vary the parameter* K—that is, to determine how to interpret K as a *function*—so that $y = Kv$ is a solution of $y' + P(x)y = Q(x)$.

Since $y = Kv$, we can find y' by using the product rule: $y' = v'K + K'v$. Substituting into the given ODE leads to $v'K + K'v + P(x)Kv = Q(x)$ or, equivalently, $K'v + K(v' + P(x)v) = Q(x)$. Since $v' + P(x)v) = 0$, the above condition on K simplifies to $K'v = Q(x)$ or, equivalently, $K' = v^{-1}Q(x)$. Then, by the very meaning of indefinite integration, we have $K = \int v^{-1}Q(x)dx + C$, where C is an arbitrary constant. Therefore, the general solution of $y' + P(x)y = Q(x)$ is $y = Kv = (\int v^{-1}Q(x)dx + C)v = (\int v^{-1}Q(x)dx)v + Cv$. Since $v = \mu^{-1}$, this formula can be rewritten in the more familiar way as $y = (\int \mu Q(x)dx)\mu^{-1} + C\mu^{-1}$. Of course, it is not necessary for users of our method to remember this formula (or the formula for μ), as they need only implement the above steps.

3. Some Examples and Pedagogical Remarks

Examples 3.1 and 3.2 illustrate how to use the methodology in section 2 to find the general solution of a typical first order linear ODE. Remark 3.3 compares the details of example 3.1 with the details in the usual solution via the integrating factor method (as in Nagle, Saff and Snider 2004, section 2.3). In this way, we have a concrete example illustrating the advantages that we ascribe to the method in section 2. Of course, as we observed at the end of section 2, the two methods give the same answer. For a variety of reasons, instructors who include the method of section 2 in their curriculum for a first course on ODEs may also wish to include the integrating factor method. For such curricula, it may be advisable to identify an additional role that integrating factors can play in such a course, and remark 3.4 offers one suggestion along these lines.

Example 3.1. Use the method of section 2 to solve the following ODE: $2x^2y' + xy = 6x^2$ (for $x > 0$).

Solution. The given ODE is not in the standard form of a first order linear ODE; namely, $y' + P(x)y = Q(x)$. To find an equivalent ODE that is in this standard form, divide the given ODE by $2x^2$ (that is, by the coefficient of y'). The result is in standard form, with $P(x) = \frac{x}{2x^2} = \frac{1}{2x}$ and $Q(x) = \frac{6x^2}{2x^2} = 3$. According to the method of variation of parameters, we must first find the general solution of the corresponding homogeneous (first order) linear ODE, $y' + P(x)y = 0$ (namely, $y' + \frac{1}{2x}y = 0$). This ODE can be rewritten as $\frac{dy}{y} = -\frac{dx}{2x}$, a (variables) separable ODE whose general solution can be found in the usual way, as follows: $\int \frac{1}{y}dy = -\int \frac{1}{2x}dx + C^*$, or $\ln|y| = -\frac{1}{2}\ln|x| + C^* = -\ln(|x|^{1/2}) + C^*$. By a law of logarithms, this solution of the homogeneous ODE can be rewritten as $\ln(|y||x|^{1/2}) = C^*$ or, equivalently, as $y = Kx^{-1/2}$, where the arbitrary constant $K = \pm e^{C^*}$.

We now proceed to vary the *parameter* K that appeared in the above solution of the homogeneous ODE. As in the usual textbook treatments for the case $n \geq 2$, this amounts to asking for necessary and sufficient conditions on a *function* K so that $y = Kx^{-1/2}$ is a solution of the given (nonhomogeneous) ODE. Substituting this expression for y into the given ODE (and differentiating it using the product rule from the prerequisite differential calculus), we obtain $2x^2(-\frac{1}{2}x^{-3/2}K + K\,'\,x^{-1/2}) + xKx^{-1/2} = 6x^2$. This is algebraically equivalent to $K\,' = 3x^{1/2}$, whose solution (using the prerequisite integral calculus) is $K = \int 3x^{1/2}dx = 2x^{3/2} + C$. Accordingly, the general solution of the given ODE is $y = Kx^{-1/2}$ $= (2x^{3/2} + C)x^{-1/2} = 2x + Cx^{-1/2}$.

For classes with enough time for additional applications of the method being proposed here, example 3.2 provides two more illustrations of that method. Example 3.2(a) is easier than example 3.1 in that the differential equation that one must solve to find K in example 3.2(a) is easy (namely, $K\,' = 1$), while example 3.2(b) is more difficult than example 3.1 because the differential equation that one must solve to find K in example 3.2(b) requires integration by parts.

Example 3.2. Use the method of section 2 to solve the following ODEs:

(a) $y\,' - 2y = e^{2x}$; and

(b) $y\,' - 2y = x$.

Solution (Sketch). The given ODEs are both in the standard form of a first order linear ODE; namely, $y\,' + P(x)y = Q(x)$ (with $P(x) = -2$ in both cases). The general solution of the corresponding homogeneous (first order) linear ODE, $y\,' + P(x)y = 0$ (namely, $y\,' - 2y = 0$), is found to be $y = Ke^{2x}$. Viewing K as a function and requiring $y = Ke^{2x}$ to satisfy the ODE in (a) leads, after some algebraic simplification, to $e^{2x}K\,' = e^{2x}$, whence $K\,' = 1$ and $K = \int 1dx = x + C$, where C is an arbitrary constant. Thus, the general solution for (a) is $y = (x + C)e^{2x} = xe^{2x} + Ce^{2x}$.

A similar approach in (b) leads to $K\,' = xe^{-2x}$, whence integration by parts gives us that $K = \int xe^{-2x}dx = x(-\frac{1}{2}e^{-2x}) - \int -\frac{1}{2}e^{-2x}dx$. Thus, $K = -\frac{xe^{-2x}}{2} - \frac{e^{-2x}}{4} + C$, and so the general solution for (b) is $y = (-\frac{xe^{-2x}}{2} - \frac{e^{-2x}}{4} + C)e^{2x}$, which simplifies to $y = -\frac{x}{2} - \frac{1}{4} + Ce^{2x}$.

Remark 3.3. In Nagle, Saff and Snider (section 2.3), the general first order linear ODE, $y\,' + P(x)y = Q(x)$, is solved by using the rather unmotivated introduction of the integrating factor $\mu = e^{\int P(x)dx}$ (which leads to the equivalent ODE, $\frac{d}{dx}(\mu y) = \mu Q(x)$, which can be solved by separation of variables). Thus, the usual textbook solution of the ODE given in Example 3.1 would use the integrating factor $\mu = e^{\int P(x)dx} = e^{\int \frac{1}{2x}dx} = e^{\frac{1}{2}\ln(|x|)} = |x|^{1/2} = x^{1/2}$. That solution is then $y = (\int \mu Q(x)dx)\mu^{-1} + C\mu^{-1} = (\int x^{1/2}3dx)x^{-1/2} + Cx^{-1/2}$ $= (2x^{3/2})x^{-1/2} + Cx^{-1/2} = 2x + Cx^{-1/2}$, which agrees with the answer found in example 3.1.

A comparison of the above calculation with the details in example 3.1 shows that both involve the same mechanical skills. However, the solution in example 3.1 (and the same can be said for the solutions in example 3.2) has what we view as the two most important advantages for the method introduced in this note: (1) it does not require one to memorize the integrating factor formula $\mu = e^{\int P(x)dx}$ and (2) it introduces variation of parameters in a context ($n = 1$) that can avoid the matrix algebra that complicates the treatment in case $n \geq 2$.

Remark 3.4. In closing, we pursue the comment in the introduction that most current textbooks deal with first order linear ODEs in an ad hoc manner. (After drafting this manuscript, we came across a couple of recent textbooks that do introduce variation of parameters in case $n = 1$: see Diacu (2001, 32–33) and Logan (2006, 62–63).) Recall that the standard textbook solution of the general first order linear ODE, $y\,' + P(x)y = Q(x)$, is carried out with the aid of the integrating factor $\mu = e^{\int P(x)dx}$ (which leads to the equivalent ODE, $\frac{d}{dx}(\mu y) = \mu Q(x)$, which can be solved by separation of variables). Rather than appealing to separation of variables, an earlier edition of Nagle, Saff and Snider justified the integrating factor method by using the theory of exact ODEs. (The topic of exact ODEs has been moved to section 2.4 of Nagle, Saff and Snider.) This style of justification suffers the criticism of depending on Calculus III, especially on concepts involving partial derivatives and simply connected regions. On the other hand, for students with this background from Calculus III, if an instructor wishes to emphasize the integrating factor method in conjunction with a discussion of exact ODEs, then the topic of integrating factors could be made more central to the course by including the following theorem. *Any* (not necessarily linear) first order ODE, $M(x,y)dx + N(x,y)dy = 0$, has an integrating factor (that is, a function $\mu = \mu(x,y)$ such that $\mu M(x,y)dx + \mu N(x,y)dy = 0$ is an exact ODE), provided that M and N have continuous first partial derivatives defined over some open rectangle. The proof of this theorem depends on the fundamental existence and uniqueness theorem for initial value problems and can be found in Ford (1955, Theorem, 54).

References

Diacu, F. 2001. *An Introduction to Differential Equations (Order and Chaos)*. San Francisco: W H Freeman.

Ford, L R. 1955. *Differential Equations*. 2nd ed. New York: McGraw-Hill.

Logan, J D. 2006. *A First Course in Differential Equations*. New York: Springer.

Nagle, R K, E B Saff and A D Snider. 2004. *Fundamentals of Differential Equations*. 6th ed. Boston: Pearson/Addison-Wesley.

David E Dobbs received his BA (majors: mathematics, statistics, and philosophy) and his MA (major: geometry, minor: logic) from the University of Manitoba in his native Winnipeg. He received his doctorate in algebraic geometry from Cornell University. He has been on staff at UCLA, Rutgers University and the University of Tennessee, Knoxville, where he has held the position of professor since 1980. He has taught mathematics education courses at each of the five universities named above and has written nearly 90 teaching notes at all levels of mathematics education. He enjoys teaching at all levels, and has directed seven PhD dissertations and several dozen senior and master's theses. His main research interest is in commutative algebra, on which he has written eight books and more than 200 papers.

A Refined Algorithm for Solving Polynomial Equations

Duncan E McDougall and Ryan Willoughby

The following article describes a sequence of steps designed to reduce to a minimum the number of eligible factors when solving analytically polynomial functions with integral roots and coefficients. Our objective is to list all possible factors and then select only those factors that satisfy certain criteria. This is done by incorporating Descartes' Rule of Signs and the factors of the sum of the numerical coefficients of the given polynomial. The algorithm consists of the following steps:

1. Apply Descartes' Rule of Signs to determine possible numbers of positive or negative real numbers and/or complex roots.
2. Find the sum of the numerical coefficients or the polynomial.
3. List all the factors of this sum and then add 1 to each factor. This becomes set B.
4. List all the factors of the constant of the polynomial (Integral Factor Theorem). This becomes set A.
5. Find the intersection of sets A and B, listing only the common elements.
6. Use step 1 to focus on the number of positive or negative real roots.

Using a variety of examples, let us examine the usefulness of this algorithm.

Example 1: Solve $x^3-5x^2-8x+12=0$

Step 1: There are two variations in sign for $p(x)$: Case 1, two positive real roots and one negative real root: or Case 2, no positive real roots, one negative real root and two complex roots.

Step 2: The sum of the numerical coefficients, $1-5-8+12$, is zero. Since the sum is zero, the number 1 is a root (Remainder Theorem) and $x-1$ is a factor of $p(x)$ (Integral Factor Theorem). We go immediately to either long or synthetic division to find the other two roots. This gives a quotient of $x^2-4x-12$ or $(x-6)(x+2)$. We go no further because we have all three factors: $(x-1),(x-6)$ and $(x+2)$ and thus the roots 1, 6, and –2.

Example 2: Solve $x^3-4x^2+x+6=0$

Step 1: There are two variations in sign for $p(x)$: Case 1, two positive real roots and one negative real root; or Case 2, no positive real roots, one negative real root and two complex roots.

Step 2: a) $1-4+1+6=+4\neq 0$, so 1 is not a root.

b) $1+1=2$ and $-4+6=2$. Since these sums match (Remainder Theorem), -1 is a root and $x+1$ is a factor.

We go immediately to long or synthetic division in order to obtain the other two factors $(x-2)$ and $(x-3)$. We go no further because we have all three factors; $(x+1),(x-2)$ and $(x-3)$.

The above two examples serve to show the importance of immediately looking for 1 or -1 as a factor. However, not all polynomials contain these factors, and so we continue with the next example.

Example 3: Solve $x^3-4x^2-11x+30=0$

Step 1: There are two variations in sign for $p(x)$: Case 1, two positive real roots and one negative real root; or Case 2, no positive real roots, one negative real root, and two complex roots.

Step 2: $1-4-11+30=16\neq 0$ $\therefore$ 1 is not a root and $1-11\neq -4+30$ $\therefore -1$ is not a root

Step 3: From step 2, 16 is the sum and its factors are $-1,-2,-4,-8,-16$ and $1,2,4,8,16$. Adding 1 to each factor gives $0,-1,-3,-7,-15$ and $2,3,5,9,17$; Set B = $\{0,-1,-3,-7,-15,2,3,5,9,17\}$.

Step 4: From the Integral Factor Theorem, the factors of 30 are placed in Set A; Set A = {–1, –2, –3 –5, –6, –10, –15, –30} {1, 2, 3, 5, 6, 10, 15, 30}

Step 5: The intersection of the two sets gives $\{-1,-3,-15,2,3,5\}$. Since -1 has already been eliminated (step 2), the list of possible factors is really {–3, –15 ,2, 3, 5}.

Step 6: From step 1, only one of the elements –3 and –15 can be a factor, while two of the three elements 2, 3 and 5 are factors.

If we compare the original number of factors of 30 (16) to the elements in $A \cap B$, we have narrowed it down to 5. Incidentally, the probability of selecting the correct negative root is ½, while the probability of selecting the correct positive root is $\frac{2}{3}$. Trying the positive roots in ascending order where the probability is higher, we have $p(2) = 20$. The other roots become 5 and –3 (both in the final list).

Quartics

When we get into the realm of even-numbered polynomials, the rule for 1 works but the rule for –1 does not. However, everything else holds. We now examine two more examples, one of which emphasizes the strength of Descartes' Rule of Signs.

Example 4: $x^4 5x^2 + 1 = 0$

Step 1: There are no variations in $p(x)$, so there are no positive real roots. The same is true in $p(-x)$, so there are no negative real roots. We conclude that there are four complex numbers, then move on!

Example 5: $x^4 + 6x^3 + x^2 - 24x - 20 = 0$

Step 1: There is one variation in sign for $p(x)$, so there we have the following possibilities: Case 1, one positive real root and three negative real roots; or Case 2, one positive real root, one negative real root and two complex roots.

Now $p(-x) = x^4 - 6x^3 + x^2 + 24x - 20$.

Since there are three variations in sign, we will look for three negative real roots.

Step 2: The sum of the numerical coefficients is $-36 \neq 0$. ∴ 1 is not a root.

Step 3: The factors of the sum 36 are –1, –2, –3, –4, –6, –9, –12, –18, –36 and 1, 2, 3, 4, 6, 9, 12, 18, 36.

Adding 1 to each factor gives set B: {0,–1, –2, –3, –5, –8, –11, –17, –35} {2, 3, 4, 5, 7, 10, 13, 19, 37}

Step 4: The factors of the constant 20 gives set A: {–1, –2, –4, –5, –10, –20} {1, 2, 4, 5, 10, 20}

Step 5: $A \cap B = \{-1, -2, -5, 2, 4, 5, 10\}$

Step 6: From Step 1, there are three negative real roots, so –1, -2, and –5 qualify, and since $p(2) = 0$, we have all four roots.

Naturally, five examples do not make the case for all polynomial functions, as we have not explored rational and irrational roots. However, for the monic polynomial with integral roots, we have a method that cuts down the guessing of factors in order to solve a polynomial function. The combination of Descartes' Rule of Signs along with the intersection of two sets reduces remarkably the number of possible factors to be considered. This in turn reduces time spent on any one question and reduces the frustration of guessing, when solving analytically.

Duncan McDougall has been teaching for 27 years, including 13 years in the public school systems of Quebec, Alberta, and British Columbia. During the past 15 years, he has taught mathematics to high school and university students and to elementary school teachers. He owns and operates TutorFind Learning Centre, in Victoria, BC.

Ryan Willoughby is a professional tutor of mathematics and physics who is a certificated teacher in the province of British Columbia. His varied research interests include advanced calculus, physics, astronomy and statistics. He is a graduate of the University of Victoria and has his own full-time tutoring practice (TutorFind Learning Centre) for senior high school and university students in Victoria, BC.

A Shape Quest Through Story

Chantel Mulder and Gladys Sterenberg

Once upon a Time

The popularity of teaching with picture books in elementary mathematics classrooms is increasing as teachers make connections between literature and mathematics. Since picture books can be read in one sitting, they can offer a springboard into mathematical inquiry. Textbook series and teaching resources provide many pedagogical ideas for integrating picture books into mathematics classrooms and some teachers are introducing picture books into their classroom in an attempt to garner interest, increase motivation and improve mathematical sense-making. The visual nature of these books has aesthetic appeal for many students.

This article explores how students in a Grade 1 classroom interacted with the picture book *The Greedy Triangle,* written by Marilyn Burns (1994). As teachers, we were interested in how the story could be used to investigate different polygons, how the students would engage in the story and the colourful illustrations, and how related tasks could enhance mathematical understanding for students. Chantel developed a lesson plan and Gladys expanded on this work when she taught the lesson to a class of Grade 1 students as a visiting professor. What follows is a summary of the book, a description of the tasks offered to students and a reflection on the mathematical learning of the students as they engaged in the tasks.

The Storyline

The Greedy Triangle, written by Marilyn Burns and illustrated by Gordon Silveria, is about a triangle that is unsatisfied with its shape and tired of having only three sides. Initially, the triangle goes to a shapeshifter to be changed into a quadrilateral. Then the triangle asks the local shapeshifter to add more sides and angles until it doesn't know which side is up. After many trips to the shapeshifter, the triangle learns that being a triangle is the best after all.

Storytelling

We started by considering the general and specific outcomes listed in the Alberta Program of Studies. The ones we chose to work on focused on exploring and classifying circles, triangles, and rectangles according to their properties. Since we wanted to emphasize process skills of communication and reasoning, we decided to design student tasks in relation to the book *The Greedy Triangle*.

Gladys began the lesson began by playing a game. Students were asked to guess an answer to the statement, "I spy with my little eye something that is a triangle." This sparked a rich discussion on the attributes of classroom objects. The first suggestion was made by a student who tried to form a triangle with his fingers. I asked him to explain why he thought this shape was a triangle. Another pointed to designs on the carpet. Yet another identified the wings of a butterfly in a poster. At each suggestion, Gladys asked students to explain why they thought the shapes were triangles. When one student mentioned that the points on the maple leaf on the Canadian flag could make a triangle, other students were quick to dispute this conjecture. This led to much discussion about the characteristics of triangles. This initial task seemed to provide students with an opportunity to identify and describe triangles in their classroom environment.

During this game, Gladys was sitting with her hand on her hip, thus forming a triangle with her arm. None of the students guessed her answer and when she offered it to them, they recognized the shape, and many tried to form a triangle using their arms, legs, and fingers. This activity helped prepared them for the description in the book of the triangle's favourite thing to do. As a triangle, it could "slip into place when people put their hands on their hips."

After this game, Gladys read the story with the students. The illustrations were instrumental in fostering conversations about the shapes and their attributes. Students were asked to predict what shape the triangle might change into. Much discussion occurred when new terms were introduced. The students especially enjoyed the section on the quadrilateral's experiences and were quick to point out many examples of quadrilaterals around the classroom. Students remained motivated to interact with the story throughout the entire reading of it.

Following the story, students were given a sheet of paper with a triangle glued to it. These triangles were arranged in a variety of orientations. Chantel had noticed that many students thought about shapes in a particular way. For example, when asked to draw a triangle, students often crafted an equilateral triangle with a horizontal base. A misconception can occur when students see shapes "on their side" or "upside down" and believe that these shapes are somehow "wrong." Chantel wanted the students to draw on their experiences of reading the book and realize that if they saw a triangle "upside down," it was not incorrect and, indeed, it was the same object. Students were instructed to draw a picture incorporating the shape and use as many triangles, quadrilaterals, and other shapes as possible in their drawings.

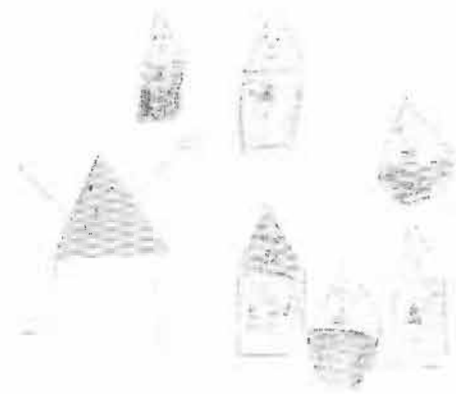
So the Story Goes

Students were quite engaged in this task and created interesting drawings. However, the most important part of this task was how students talked about their shapes. Gladys was able to ask students to classify their shapes and explain their thinking. Again, this fostered rich conversations and allowed Gladys to assess student understanding of the attributes of the shapes.

Jared was very articulate about his drawing:

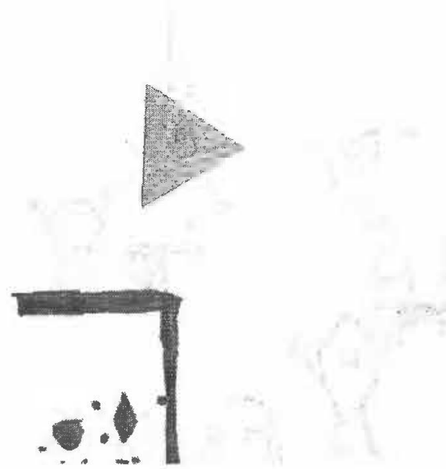
He talked about where he saw triangles in places outside of the classroom and was particularly proud of showing Gladys that shapes can exist inside other shapes. He was able to tell her why the triangles that looked different were still called triangles. While Gladys missed it at the time, she could have asked him about the 3-D box in the centre of the picture. This might have extended his understanding about the difference between 2-D shapes and 3-D objects.

Shondra emphasized that she was drawing friends for her triangle. She wanted these friends to be different so she made them with triangular faces but added quadrilaterals and "partial" circles to her shapes.



Gladys talked with her about semi-circles and squares and rectangles, and Shondra was beginning to make mathematical distinctions between these shapes.

Andrew wanted to include as many shapes as he could in his picture. He especially liked hexagons and enjoyed talking about how he was making the sides join together so that they were "even." He described the L shape as a half a square, and we talked a bit about what he was seeing. His drawing showed his experimentation with drawing the shapes, but this was a mere artifact of a much richer conversation about shapes.

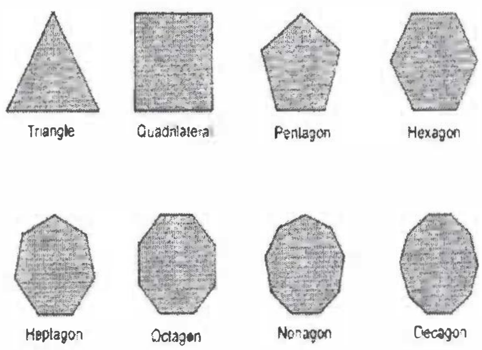


As students completed their drawings, they were offered an opportunity to choose another sheet with a shape and draw a picture using only this particular shape. Chantel wanted students to be able to identify hexagons, pentagons, and rectangles in their environment and distinguish these from triangles. This was quite effective.

This lesson was extremely successful in motivating students to talk and think like mathematicians. During a 45-minute period, these students remained captivated by the tasks and the story. Their excitement was contagious.

As we reflect on this lesson, we believe that follow-up lessons might be fruitful. These lessons could include

- using geoboards to experiment with different shapes,
- examining the different polygons presented in the book in three dimensions,
- making a collage of shapes found in magazines,
- identifying shapes at home by completing an activity with a parent (eg, have a worksheet with the following shapes on it and ask parents to help their child find the shapes in their home and draw them) or



- integrating the story with other content areas such as health (eg, extending this assignment to teach our students about self-acceptance and how important it is for students to accept themselves and others for their uniqueness).

For us, the most exciting parts of this lesson were the conversations that the students engaged in. *The Greedy Triangle* provided a context for students to see geometric shapes all around the world and to investigate the concept of angles and sides. The story was especially helpful in connecting geometric shapes to shapes found in the students' environment. The integration of literature encouraged students to share their understandings of mathematical concepts. Student engagement and learning were enhanced.

References

Burns, M. 1994. *The Greedy Triangle*. New York: Scholastic.

Chantel Mulder is an education student at the University of Lethbridge. She has a passion for high school mathematics and will pursue a lifetime of learning through teaching mathematics.

Gladys Sterenberg is an assistant professor in the Faculty of Education at the University of Lethbridge and the editor of delta-K. *She is interested in narrative ways of knowing mathematics and is currently researching Aboriginal stories of mathematics.*

Diophantine Polynomials

Duncan E McDougall

What is a Diophantine Polynomial? It is a polynomial of degree 2, 3 or 4 which is factorable in the set of integers and whose derivative is factorable in the set of rational numbers. We want to discuss them to facilitate curve sketching.

The polynomials that we are about to examine can be used for both Grade 11 and calculus students, because the intercepts are easy to find and the y-values for the maxima and minima are shared among the families of curves. For example, we can ask a Grade 11 student to sketch $y = x^3 + x^2 - 16x - 16$ by finding both the x and y intercepts. We can use the very same polynomial for the Calculus 12 student who can find the intercepts easily and more readily find the x-values for both maxima and minima because the derivative is easy to factor.

My belief is that students should learn a complicated algorithm in simple progressive steps using straightforward numbers. Diophantus worked with integers and rational numbers only. Pedagogically, Diophantus was really onto something because he created methods that involved a lot of processing and sequencing while focusing on whole numbers. The distractions I refer to in curve sketching are complex and irrational numbers. It is difficult enough to learn some five to eight steps gathering enough data to accurately sketch a cubic, quartic, or quintic polynomial and/or a rational expression that may involve a diagonal asymptote without difficult-to-work-with numbers. If the student has the burden (when first learning the process) of working with irrational or complex numbers, along with concentrating on the behaviour of the curve and concavity, then he or she might simply declare "whatever" and drop the task. If the numbers are whole or integral (Diophantus), then the focus remains where it should be: on the algorithm. The task of the educator is to demonstrate algorithms in such a way that the student can master the process in sequence. The solution is to stick to the Diophantine process and to model examples that involve process and sequencing without getting tangled up with irrational numbers. To some readers this may be self-evident, but it is not as simple as it sounds to find cubics, or quartics with single integral roots whose derivatives have single rational roots. Finding them involved testing hundreds of polynomials using DERIVE (an algebra software developed by Texas Instruments), as I was determined to find easy-to-calculate polynomials, which would facilitate graphing curves like $y = x^3 + 11x^2 + 24x$ without worrying about irrational and complex numbers. There was another challenge, of course, and that was to keep the constant of the polynomial relatively small so that working without a calculator would not be arduous.

Another aspect of this approach with whole numbers is that when the student knows that the numbers are designed to work, learning of the method or algorithm remains the priority. The student also knows that there is something wrong if the numbers do not work. It is kind of a security blanket for the beginner, but it eliminates doubt, which so often takes away confidence in ability and performance. Later on, after mastering the technique, the student gains confidence through the ease of this, and therefore can tackle problems with both irrational and complex numbers.

It is my objective to propose families of cubics and quartics that are factorable in the integers and whose derivatives are factorable in the set of rational numbers. I will also propose methods using DERIVE by which you can construct your own polynomials. We'll start with the very basic table of linear and quadratic polynomials, then lead up to the cubics and quartics. I will end the paper with a brief discussion of the quintic, which should have worked but did not.

Table I contains all the various linear and quadratic forms along with the general set of cubics.

Using Table I

Take a cubic of the form $(x + a^2)(x + 2ab)(x + b^2)$ whose derivative has rational roots. Choosing any integers, $a = 1$ and $b = 3$ for example, our new polynomial is $(x+1)(x+6)(x+9)$ with roots $-1, -6$, and -9. The differential form is $3x^2 + 32x + 69$, whose roots are -3 and $-\frac{23}{3}$.

Table I

Family Function	**Roots**	**Derivative**	**Roots**	**Conditions on coefficients and constants to have integral roots**
a	none	0	none	not applicable
ax	$x=0$	a	none	not applicable
$ax+b$	$x=\frac{-b}{a}$	a	none	not applicable
$(x+a)^2$	$x=-a$	$2(x+a)$	$x=-a$	none
$x^2+ax=x(x+a)$	$x=0,-a$	$2x+a$	$x=\frac{-a}{2}$	a must be even
$x^2+x(a+b)+ab$ $=(x+a)(x+b)$	$x=-a,-b$	$2x+a+b$	$x=\frac{-a-b}{2}$	a and b are both odd or both even
$acx^2+x(ad+bc)+bd$ $=(ax+b)(cx+d)$	$x=\frac{-b}{a},\frac{-d}{c}$	$2acx+ad+bc$	$x=\frac{-ad-bc}{2ac}$	$a\neq 0, c\neq 0$ $ad+bc$ must either equal ac or be an even multiple of it
$(x+a)^3$	$x=-a$	$3(x+a)^2$	$x=-a$	none
$(x+a)^2(x+b)$	$x=-a$, $x=-b$	$2(x+a)(3x+2b+a)$	$x=-a$ $x=\frac{-2b-a}{3}$	none $2b+a$ is 3 or a multiple of 3
$x(x+a)(x+b)$	$x=0$ $x=-a$ $x=-b$	$3x^2+2x(a+b)+ab$	$x=\frac{-(a+b)\pm\sqrt{a^2-ab+b^2}}{3}$	a^2-ab+b^2 equals zero or a perfect square
$(x+a^2)(x+2ab)(x+b^2)$	$x=-a^2$ $x=-2ab, x=-b^2$	$3x^2+x(2a^2+4ab+2b^2)$ $+ab(2a^2+ab+2b^2)$	$x=-ab$ $x=\frac{-2a^2-ab-2b^2}{3}$	$2a^2+ab+2b^2$ must be 3 or a multiple of 3
$(x+1)(x-a)(x+a)$	$x=-1$ $x=a$ $x=-a$	$3x^2-2x-a^2$	$x=\frac{2\pm\sqrt{4+12a^2}}{6}$	$4+12a^2$ must be a perfect square ($a=0,1,4,16...$)

The polynomials in Table II consist of the particular numerical families with single roots. These are the ones that are ready to use in your classroom today.

As we observe the families in Table II, it is hard not to notice the pattern 8, 15, 21, 30, 35 and 36. It is a quadratic arithmetic sequence whose elements (except for a couple) all work as families of curves.

Table II

Family Function	Roots	Derivative	Roots	Transformation
$x(x+3)(x+8)$ x^3+11x^2+24x	$0,-3,-8$	$(3x+4)(x+6)$ $3x^2+22x+24$	$-\frac{4}{3},6$	$(x\pm k)(x\pm 3a\pm k)(x\pm 8a\pm k)$
$x(x+5)(x+8)$ x^3+13x^2+40x	$0,-5,-8$	$(3x+20)(x+2)$ $3x^2+26x+40$	$-\frac{20}{3},-2$	$(x\pm k)(x\pm 5a\pm k)(x\pm 8a\pm k)$
$x(x+7)(x+15)$ x^3+22x^2+105x	$0,-7,-15$	$(3x+35)(x+3)$ $3x^2+44x+105$	$-\frac{35}{3},-3$	$(x\pm k)(x\pm 7a\pm k)(x\pm 15a\pm k)$
$x(x+8)(x+15)$ x^3+23x^2+120x	$0,-8,-15$	$(3x+10)(x+12)$ $3x^2+46x+120$	$-\frac{10}{3},-12$	$(x\pm k)(x\pm 8a\pm k)(x\pm 15a\pm k)$
$x(x+5)(x+21)$ x^3+26x^2+105x	$0,-5,-21$	$(3x+7)(x+15)$ $3x^2+52x+105$	$-\frac{7}{3},-15$	$(x\pm k)(x\pm 5a\pm k)(x\pm 21a\pm k)$
$x(x+16)(x+21)$ x^3+37x^2+336x	$0,-16,-21$	$(3x+56)(x+6)$ $3x^2+74x+336$	$-\frac{56}{3},-6$	$(x\pm k)(x\pm 16a\pm k)(x\pm 21a\pm k)$
$x(x+26-a)(x+26)$ $x^3+x^2(52-a)+26(26-a)x$	$0,a-26,-26$	$3x^2+52x+26a-a^2$	not rational	
$x(x+14)(x+30)$ x^3+44x^2+420x	$0,-14,-30$	$(3x+70)(x+6)$ $3x^2+88x+420$	$-\frac{70}{3},-6$	$(x\pm k)(x\pm 14a\pm k)(x\pm 30a\pm k)$
$x(x+16)(x+30)$ x^3+46x^2+480x	$0,-16,-30$	$(3x+20)(x+24)$ $3x^2+92x+480$	$-\frac{20}{3},-24$	$(x\pm k)(x\pm 16a\pm k)(x\pm 30a\pm k)$
$x(x+33-a)(x+33)$ $x^3+x^2(66-a)+33(33-a)$	$0,a-33,-33$	$3x^2+2x(66-a)+33(33-a)$	not rational	
$x(x+11)(x+35)$ x^3+46x^2+385x	$0,-11,-35$	$(3x+72)(x+5)$ $3x^2+92x+385$	$-\frac{72}{3},-5$	$(x\pm k)(x\pm 11a\pm k)(x\pm 35a\pm k)$
$x(x+24)(x+35)$ x^3+59x^2+840x	$0,-24,-35$	$(3x+14)(x+15)$ $3x^2+118x+840$	$-\frac{14}{3},-15$	$(x\pm k)(x\pm 24a\pm k)(x\pm 35a\pm k)$
$x(x+36-a)(x+36)$ $x^3+x^2(72-a)+36(36-a)x$	$0,a-36,-36$	$3x^2+2x(72-a)+36(36-a)$	not rational	

Methods Using DERIVE

Regarding methods for single roots, let us begin by entering the form $x(x+a)(x+b)$ into DERIVE. This guarantees a factorable form. Press C for Calculus and differentiate. The resulting form is put in function form as DECLARE. Now we can either guess values and hope that our quadratic is factorable, or fix a value for a, and then guess values for b until the quadratic is factorable. The question is, do we have anything to guide our guessing? In fact, we do. Visually, the values of x for maxima and minima will occur between the first and last x-intercepts. Hence, if we were to choose 0 and 8 as two of our first and last roots, we would know that the third one must come between them. It is just a question of leaving enough room between the roots so that the critical points can occur as integers and/or rational numbers. Algebraically, we enter $x(x-a)(x-8)$ into DERIVE, and then differentiate giving $3x^2+2x(a-8)+8a$. Since we have a quadratic, the discriminant B^2-4AC must equal a perfect square in order to be factorable. Using the command DECLARE, we set $f(a)=4a^2-32a+256=4(a^2-8a+64)$ and evaluate (or use the TI83 where second function gives TABLE and we search it for perfect squares). Both 3 and 5 come up quickly, implying that both $x(x-3)(x-8)$ and $x(x-5)(x-8)$ have derivatives whose roots are rational.

I do not pretend to have all the families, but applying translations to any given family will yield many polynomials. The following is a small sample arrived at by adding a constant to all the terms.

Given family	$x(x-3)(x-8)$
Add 1	$(x+1)(x-2)(x-7)$
Add 2	$(x+2)(x-1)(x-6)$
Add 3	$(x+3)(x)(x-5)$
Add 4	$(x+4)(x+1)(x-4)$
Add 5	$(x+5)(x+2)(x-3)$
Add 6	$(x+6)(x+3)(x-2)$
Add 7	$(x+7)(x+4)(x-1)$
Add 8	$(x+8)(x+5)(x)$, etc.

Interestingly enough, the entire above shares a max height of $\frac{400}{27}=\left(\frac{5}{3}\right)\left(\frac{4}{3}\right)\left(\frac{20}{3}\right)$, and minimum low of -36, and the difference between their corresponding x-coordinates is exactly $\frac{14}{3}$. A linear relationship exists between these values and those found in the quartics. We shall explore this after exploring the quartic family of curves.

Having fully explored the cubic, the quartic family of curves presented quite a challenge because there would be three roots, other than zero, to find. Visually, I opted for a span of 7 (one less than the 8 for cubics), entered $x(x-a)(x-b)(x-7)$ into DERIVE, fixed $a=3$ (only because it had worked with the cubic), took the derivative and evaluated b from 1 to 7 hoping that some value b would work. The derived form was $4x^3+x^2(-3b-30)+x(20b+42)-21b$, and by declaring f as the function I simply tested values for b and systematically factored (pressing F). To my great delight $b=4$ worked, giving $2(x-6)(x-1)(2x-7)$. Observing 3 and 4 together, I acted on a hunch that Pythagorean Triples might work. So, following in the footsteps of Diophantus, I tried triplets beginning with odd numbers and even numbers, and they worked beautifully. An added bonus was those triplets with consecutive legs such as 20-21-29 and 119-120-169, etc, which also worked wonderfully. The patterns appear in Table III. The numerical families of the form $x(x+a)(x+b)(x+c)$ appear in Table IV. The numerical families for the form $x^2(x+a)(x+b)$ appear in Table V.

Table III

Family Function	**Roots**	**Derivative**	**Roots**	**Conditions For Integral Roots**
$(x+a)^4$	$-a$	$4(x+a)^3$	$-a$	no restrictions
$(x+a)^3(x+b)$	$-a,-b$	$(a+b)^2(4x+3b+a)$	$-a$ and $\frac{-3b-a}{4}$	$-3b-a$ must be a multiple of 4 for b=1 a=1,5,9… 4k-3 for b=2 a=2,6,10…4k-2 for b=3 a=3,7,11…4k-1 for b=4 a=4,8,12…4k
$(x+a)^2(x+b)^2$	$-a,-b$	$2(x+a)(x+b)(2x+a+b)$	$-a,-b$ and $\frac{-b-a}{2}$	a and b must be both odd or both even
$(x+a)^2(x+b)(x+c)$	$-a,-b,-c$	$(x+a)\left[4x^2+x(3b+3c+2a)+2bc+ab+ac\right]$	$\frac{-3b-3c-2a\pm\sqrt{4a^2+9b^2-4ab-4ac-14bc}}{8}$	$4a^2+9b+9c$ $-4ab-4ac-14bc$ must be perfect square
The product of (x) and $(x+2n+1)$ and $(x+2n^2+2n)$ and $(x+2n^2+4n+1)$	$-2n-1$, $-2n^2-2n$ and $-2n^2-4n-1$	$2(x+n)(x+2n^2+3n+1)$ $(2x+2n^2+4n+1)$	$-n, -2n^2-3n-1$ and $-2n^2-4n-1$	no conditions
The product of (x) and $(x+2n+1)$ and $(x+2n^2+2n)$ and $(x+2n^2+4n+1)$	$-2n, 1-n^2$, and $-n^2-2n+1$	$2(x+n-1)(x+n^2+n)$ $(2x+n^2+2n-1)$	$1-n, -n^2-n$ and $-n^2-2n+1$	no conditions

Table IV

Family Function	**Roots**	**Derivative**	**Roots**	**Transformation**
Odd Pythagorean Triplets				
$x(x+3)(x+4)(x+7)$	$0,-3,-4,-7$	$2(x+1)(x+6)(2x+7)$	$-1,-6,\frac{-7}{2}$	$(x\pm k)(x\pm 3a\pm k)(x\pm 4a\pm k)(x\pm 7a\pm k)$
$x(x+5)(x+12)(x+17)$	$0,-5,-12,-17$	$2(x+2)(x+15)(2x+17)$	$-2,-15,\frac{-17}{2}$	$(x\pm k)(x\pm 5a\pm k)(x\pm 12a\pm k)(x\pm 17a\pm k)$
$x(x+7)(x+24)(x+31)$	$0,-7,-24,-31$	$2(x+3)(x+28)(2x+31)$	$-3,-28,\frac{-31}{2}$	$(x\pm k)(x\pm 7a\pm k)(x\pm 24a\pm k)(x\pm 31a\pm k)$
				etc
Even Pythagorean Triplets				
$x(x+4)(x+3)(x+7)$	$0,-3,-4,-7$	$2(x+1)(x+6)(2x+7)$	$-1,-6,\frac{-7}{2}$	$(x\pm k)(x\pm 4a\pm k)(x\pm 3a\pm k)(x\pm 7a\pm k)$
$x(x+6)(x+8)(x+14)$	$0,-6,-8,-14$	$4(x+2)(x+7)(x+12)$	$-2,-7,-12$	$(x\pm k)(x\pm 6a\pm k)(x\pm 8a\pm k)(x\pm 14a\pm k)$
$x(x+8)(x+15)(x+23)$	$0,-8,-15,-23$	$2(x+3)(x+20)(x+23)$	$-3,-20,\frac{-23}{2}$	$(x\pm k)(x\pm 8a\pm k)(x\pm 15a\pm k)(x\pm 23a\pm k)$
				etc
Consecutive-Leg Triplets				
$x(x+3)(x+4)(x+7)$	$0,-3,-4,-7$	$2(x+1)(x+6)(2x+7)$	$-1,-6,\frac{-7}{2}$	$(x\pm k)(x\pm 3a\pm k)(x\pm 4a\pm k)(x\pm 7a\pm k)$
$x(x+20)(x+21)(x+41)$	$0,-20,-21,-41$	$2(x+6)(x+35)(2x+41)$	$-6,-35,\frac{-41}{2}$	$(x\pm k)(x\pm 20a\pm k)(x\pm 21a\pm k)(x\pm 41a\pm k)$
$x(x+119)(x+120)(x+239)$	$0,-119,-120,-239$	$2(x+35)(x+204)(2x+239)$	$-35,-204,\frac{-239}{2}$	$(x\pm k)(x\pm 119a\pm k)(x\pm 120a\pm k)(x\pm 239a\pm k)$
				etc

Table V

Family Function	**Roots**	**Derivative**	**Roots**	**Transformation**
$x^2(x+5)(x-7)$	$0,-5,7$	$2x(x-5)(2x+7)$	$0,5,\frac{-7}{2}$	$(x^2\pm k)(x+5a\pm k)(x-7a\pm k)$
$x^2(x+5)(x+2)$	$0,-5,-2$	$x(x+4)(4x+5)$	$0,-4,\frac{-5}{4}$	$(x^2\pm k)(x+5a\pm k)(x+2a\pm k)$
$x^2(x+5)(x+9)$	$0,-5,-9$	$x(x+3)(2x+15)$	$0,-3,\frac{-15}{2}$	$(x^2\pm k)(x+5a\pm k)(x+9a\pm k)$
$x^2(x+7)(x+10)$	$0,-7,-10$	$x(x+4)(4x+35)$	$0,-4,\frac{-35}{4}$	$(x^2\pm k)(x+7a\pm k)(x+10a\pm k)$
$x^2(x+9)(x+14)$	$0,-9,-14$	$x(x+12)(4x+21)$	$0,-12,\frac{-21}{4}$	$(x^2\pm k)(x+9a\pm k)(x+14a\pm k)$

Table VI

Family Function	**Roots**	**Derivative**	**Roots**	**Conditions For Integral Roots**
$(x+a)^5$	$-a$	$5(x+a)^4$	$-a$	none
$(x+a)^4(x+b)$	$-a,-b$	$(x+a)^3(5x+a+4b)$	$-a$ and $\frac{-a-4b}{5}$	$a+4b$ must be 5 or a multiple of 5
$(x+a)^3(x+b)(x+c)$	$-a,-b,$ $-c$	$(x+a)^2\left(5x^2+2x(a+2b+2c)+ab+ac+3bc\right)$	$-a$ and $\frac{-(a+2b+2c)\pm\sqrt{a^2+4b^2-ab-ac-7bc}}{5}$	Conditions: $a^2+4b^2+4c^2-ab-ac-7bc$ must be zero or a perfect square
$(x+a)^3(x+b)^2$	$-a,-b$	$(x+a)^2(x+b)(5x+2a+3b)$	$-a,-b$ and $\frac{-2a-3b}{5}$	$2a+3b$ must be 5 or a multiple of 5
$(x+a)^2(x+b)(x+c)$	$-a,-b,$ $-c$	$(x+b)(x+a)\left[5x^2+x(3a+3b+4c)+ab+2ac+2bc\right]$	$-a,-b,$ and $\frac{-(3a+3b+4c)\pm\sqrt{9a^2+9b^2+16c^2-2ab-16bc-16-c}}{10}$	Conditions: $9a^2+9b^2+16c^2-2ab-16bc-16ac$ must be a perfect square
$x^2(x+a)(x+b)(x+c)$	$0,-a,$ $-b,-c$	$x\left(5x^3+x^2(4a+4b+4c)+x(3ab+3ac+3bc)+2abc\right)$	no rational roots	N/A
$x(x+a)(x+b)(x+c)(x+d)$	$0,-a,$ $-b,-c,$ $-d$	$5x^4+4x^3(a+b+c+d)+3x^2(ab+ac+ad+bc+bd+cd)+2x(abc+abd+acd+bcd)+abcd$	no rational roots	N/A

The Quintic

In terms of multiple roots, the quintic lends itself nicely to easy-to-work-with numbers that are small in quantity. However, for quintics of the form $x(x-a)(x-b)(x-c)(x-d)$, the derivative has no rational roots, primarily because of Fermat's Last Theorem whereby there are no integral values for which $x^4+y^4=z^4$. Having run the computer through thousands of number combinations (just to be sure), no derivative with rational roots could be found. Our Table VI contains multiple roots only. Table VII contains the numerical families for the forms

$$x^3(x+a)(x+b) \text{ and } x^2(x+a)^2(x+b).$$

Table VII

Family Function	Roots	Derivative	Roots	Transformation
$x^3(x+3)(x+4)$ $x^5+7x^4+12x^3$	$0,-3,-4$	$x^2(x+2)(5x+18)$ $5x^4+28x^3+36x^2$	$0,-2,\frac{-18}{5}$	$(x\pm k)^3(x\pm 3a\pm k)(x\pm 4a\pm k)$
$x^3(x+3)(x+11)$ $x^5+14x^4+33x^3$	$0,-3,-11$	$x^2(x+9)(5x+18)$ $5x^4+56x^3+99x^2$	$0,-9,\frac{-18}{5}$	$(x\pm k)^3(x\pm 3a\pm k)(x\pm 11a\pm k)$
$x^3(x+4)(x+7)$ $x^5+11x^4+28x^3$	$0,-4,-7$	$x^2(x+6)(5x+14)$ $3x^4+44x^3+84x^2$	$0,-6,\frac{-14}{5}$	$(x\pm k)^3(x\pm 4a\pm k)(x\pm 7a\pm k)$
$x^3(x+5)(x+12)$ $x^5+17x^4+60x^3$	$0,-5,-12$	$x^2(x+10)(5x+18)$ $5x^4+68x^3+180x^2$	$0,-10,\frac{-18}{5}$	$(x\pm k)^3(x\pm 5a\pm k)(x\pm 12a\pm k)$
$x^2(x-3)^2(x-1)$ $x^5-7x^4+15x^3-9x^2$	$0,3,1$	$x(x-3)(x-2)(5x-3)$ $5x^4-28x^3+45x^2-18x$	$0,3,2,\frac{3}{5}$	$(x\pm k)^2(x-3a\pm k)^2(x-a\pm k)$
$x^2(x-3)^2(x-2)$ $x^5-8x^4+21x^3-18x^2$	$0,3,2$	$x(x-3)(x-1)(5x-12)$ $5x^4-32x^3+63x^2-36x$	$0,3,1,\frac{12}{5}$	$(x\pm k)^2(x-3a\pm k)^2(x-2a\pm k)$
$x^2(x-3)^2(x-7)$ $x^5-13x^4+51x^3-63x^2$	$0,3,7$	$x(x-6)(x-3)(5x-7)$ $5x^4-52x^3+153x^2-126x$	$0,6,3,\frac{7}{5}$	$(x\pm k)^2(x-3a\pm k)^2(x-7a\pm k)$
$x^2(x-3)^2(x+4)$ $x^5-2x^4-15x^3+36x^2$	$0,3,-4$	$x(x-3)(x+3)(5x-8)$ $5x^4-8x^3-45x^2+72x$	$0,3,-3,\frac{8}{5}$	$(x\pm k)^2(x-3a\pm k)^2(x+4a\pm k)$

With all the patterns that do work, it was too tempting not to try to make a linear link among the cubic, quartic, and quintic forms. Let us examine the following facts.

I. Cubic Form $x(x-5)(x-8)$	Smallest Root 0	Largest Root 8	Range 8	Sum of Roots 13
Derivative $(3x-20)(x-2)$	2	$\frac{20}{3}$	$\frac{14}{3}$	$\frac{26}{3}$
II. Quartic Form $x(x-3)(x-4)(x-7)$	Smallest Root 0	Largest Root 7	Range 7	Sum of Roots 14
Derivative $2(x-1)(2x-7)(x-6)$	1	6	$5=\frac{20}{4}$	$\frac{42}{4}$
III. Quintic Form $x(x-2)(x-3)(x-6)$	Smallest Root 0	Largest Root 6	Range 6	Sum of Roots 15
Derivative $5x^4-60x^3+240x^2-360x+144$	.616036...	5.383960...	4.767924...	$\frac{60}{5}$

We realize very quickly that we can come close to rational roots, but cannot obtain them as our constant term would have to be a multiple of 5 in order to be factorable, which is impossible in this situation.

Summary

If nothing else, the reader now has a partial list of cubic and quartic polynomials with multiple or single integral roots whose derivatives have multiple or single rational roots. The quintic avails itself to multiple but not to single roots.

I would never have attempted all this work without the user-friendly program DERIVE, as I was able to test many polynomials in seconds and quickly find derivative and corresponding factored forms. The same Diophantine process can be applied to rational forms, making life a little easier for the curve sketcher.

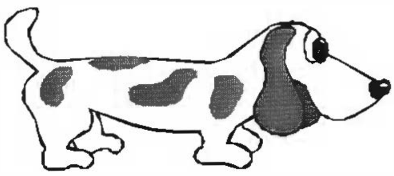
Duncan McDougall has been teaching for 27 years, including 13 years in the public school systems of Quebec, Alberta, and British Columbia. During the past 15 years, he has taught mathematics to high school and university students and to elementary school teachers. He owns and operates TutorFind Learning Centre, in Victoria, British Columbia.

A Page of Problems

A Craig Loewen

High School

A dog is tied outside with a 50 m rope at the corner of a 25 m square building. What is the size of the area the dog can reach?



By how much does the area increase if the dog is given a 60 m rope?

Adapted from Kantecki, C, and L E Yunker. 1982. "Problem Solving for the High School Mathematics Student." *Math Monograph* 7: 49–60.

Middle School

Students in a physical education class are spaced evenly around a circle, and then they count off. Student 15 is directly opposite student 49. How many students are in the class?

Mathematics Teacher 83, no 4: 290–91.

Junior High

You have only one 5-litre container and one 3-litre container. How can you measure out exactly 4 litres of water if neither container is marked for measuring?

Find strategies to measure out any number of litres of water from 1 to 20.

Billstein, R, S Libeskind and J W Lott. 1987. *A Problem Solving Approach to Mathematics for Elementary School Teachers*. 3rd ed. Menlo Park, Cal: Benjamin/Cummings.

Elementary

A frog fell into a well that was 20 metres deep. Each day he climbed 3 metres up the well's sides. At night he slid back down 1 metre. How many days did it take him to climb out of the well?

www.myteacherpages.com/webpages/ LWoods/ index.cfm?subpage=266012

MCATA Executive 2006/07

President
Janis Kristjansson
Bus (403) 777-6690
jkristja@shaw.ca

Past President
Daryl Chichak
Bus (780) 989-3022
mathguy@telusplanet.net

Vice-President
Sharon Gach
Bus (780) 467-0044
sharon.gach@ei.educ.ab.ca

Vice-President and Conference Director 2007
Martina Metz
Bus (403) 282-2890
martina17@shaw.ca

Secretary
Donna Chanasyk
Bus (780) 459-4405
donnajc@telus.net

Treasurer
Elaine Manzer
Bus (780) 624-4221
manzere@prsd.ab.ca

NCTM Representative
Lisa Hauk-Meeker
lhaukmeeker@telus.net

Publications Director
Susan Ludwig
Bus (780) 989-3021
ludwigs@ecsd.net

Awards and Grants Director
Carmen Wasylynuik
Bus (780) 674-8522
cwasylynuik@phrd.ab.ca

Newsletter Editor
Anne MacQuarrie
Bus (403) 777-6390
anne.macquarrie@shaw.ca

Journal Editor
Gladys Sterenberg
Bus (403) 329-2013
gladys.sterenberg@uleth.ca

Dr Arthur Jorgensen Chair
Nicole Patrie
Bus (403) 843-3761
npatrie@wolfcreek.ab.ca

Membership Director
Daryl Chichak
Bus (780) 989-3022
chichakd@ecsd.net

Webmaster
Robert Wong
Bus (780) 413-2211
robert.wong@epsb.ca

Alberta Education Representative
Jennifer Dolecki
Bus (780) 427-5628
jennifer.dolecki@gov.ab.ca

Faculty of Education Representative
Elaine Simmt
Bus (780) 492-3674
elaine.simmt@ualberta.ca

Postsecondary Mathematics Representative
Indy Lagu
Bus (403) 440-6154
ilagu@mtroyal.ca

PEC Liaison
Carol D Henderson
Bus (403) 938-6666
carol.henderson@ata.ab.ca

ATA Staff Advisor
Lisa M Everitt
Bus (780) 447-9463
or 1-800-232-7208
lisa.everitt@ata.ab.ca

Professional Development Director
Rosaline Carson
Bus (403) 945-4000
rcarson@rockyview.ab.ca

Director at Large
Rebecca Steel
Bus (403) 816-6710
rdsteel@cbe.ab.ca

ISSN 0319-8367
Barnett House
11010 142 Street NW
Edmonton AB T5N 2R1