Alberta High School Mathematics Competition 2006/07

Andy Liu

The Alberta High School Mathematics Competition (AHSMC) is a two-part competition taking place in November and February of each school year. Book prizes are awarded for Part I (multiple-choice questions), and cash prizes and scholarships for Part II (extended-response questions).

The AHSMC is sponsored and helped by the following:

- ConocoPhillips of Canada
- Peter H Denham Memorial Fund
- A K Peters, Ltd
- Greenwoods' Bookshoppe
- Canadian Mathematical Society
- Mathematics Council of the Alberta Teachers' Association
- Pacific Institute for the Mathematical Sciences
- University of Calgary
- University of Alberta

For more information about the competition, visit www.math.ualberta.ca/~ahsmc/.

Report on Part I

Individual Results

Part I of the 51st AHSMC was written on November 21, 2006, by 731 students (290 girls and 441 boys). The number of students in Grades 8, 9, 10, 11 and 12 were, respectively, 1, 23, 192, 292 and 223.

Here are the top individual students:

Rank	Score	Student	School
1	100	Jarno Sun	Western Canada High School, Calgary
		Boris Braverman	Sir Winston Churchill High School, Calgary
		Jeffrey Mo	William Aberhart High School, Calgary
4	95	Jerry Lo	Ross Sheppard High School, Edmonton
5	91	Hunter Spink	Calgary Science School, Calgary
6	90	Danny Shi	Sir Winston Churchill High School, Calgary
		Gary Yang	Sir Winston Churchill High School, Calgary
		Andrew Zheng	Western Canada High School, Calgary
9	88	Wen Wang	Western Canada High School, Calgary
10	86	Yu Xiang Liu	Western Canada High School, Calgary
		Michael Wong	Tempo School, Edmonton
12	85	Melissa Chung	Harry Ainlay High School, Edmonton
		Linda Zhang	Western Canada High School, Calgary
14	84	Di Mo	Queen Elizabeth Junior/Senior High School, Calgary
15	83	William Wong	Ross Sheppard High School, Edmonton
16	81	Cindy Qian	Harry Ainlay High School, Edmonton
		Alex Sampaleanu	Saint Francis High School, Calgary
		David Ting	William Aberhart High School, Calgary
19	80	Philip Hogg	William Aberhart High School, Calgary
		Ye Jay	Henry Wise Wood High School, Calgary
		Yu Liu	Western Canada High School, Calgary
22	79	David Szepesvari	Harry Ainlay High School, Edmonton
		Lillian Wang	Western Canada High School, Calgary
		Kyle Boone	Western Canada High School, Calgary
		Victor Feng	Sir Winston Churchill High School, Calgary
		Victor Zheng	Harry Ainlay High School, Edmonton
27	78	Alex Chen	Sir Winston Churchill High School, Calgary
		Cindy Liu	Old Scona Academic High School, Edmonton
29	77	Li Han	Western Canada High School, Calgary
		Hyunbin Jeong	Old Scona Academic High School, Edmonton
31	76	Dustin Styner	Queen Elizabeth Junior/Senior High School, Calgary
		Kevin Trieu	Ross Sheppard High School, Edmonton
		Prudence Wu	William Aberhart High School, Calgary
34	75	Brett Baek	Western Canada High School, Calgary
		Patrick Pringle	Crowsnest Consolidated High School, Coleman
		Jie Yu	Western Canada High School, Calgary
37	74	Sophia Zhang	Western Canada High School, Calgary
		Stephen Portillo	Old Scona Academic High School, Edmonton
		Simon Sun	Sir Winston Churchill High School, Calgary
40	73	Tom Liu	Western Canada High School, Calgary
		Shervin Ghafouri	Western Canada High School, Calgary
		Matthew He	Western Canada High School, Calgary
		Graham Hill	Sir Winston Churchill High School, Calgary
		Sumayr Sekhon	Tempo School, Edmonton
45	72	Edward Choi	Sir Winston Churchill High School, Calgary
		David Liu	Winston Churchill High School, Lethbridge
		Darren Xu	Sir Winston Churchill High School, Calgary
		Jason Shin	William Aberhart High School, Calgary
		Travis Woodward	Sir Winston Churchill High School, Calgary
		Linda Yu	Western Canada High School, Calgary

Team Results

The contest was written by students from 33 schools, four of which did not enter a team. There were twelve schools from Zone I (Calgary), with 434 students; seven schools from Zone II (Southern Alberta), with 54 students; nine schools from Zone III (Edmonton), with 128 students; and five schools from Zone IV (Northern Alberta), with 115 students.

Here are the top teams:

Rank	Score	School and Team Members	Manager
1	280	Sir Winston Churchill High School, Calgary, with Boris Braverman, Danny Shi and Gary Yang	Mr Patrick Ancelin
2	278	Western Canada High School, Calgary, with Jarno Sun, Andrew Zheng and Wen Wang	Ms Renata Delisle
3	261	William Aberhart High School, Calgary, with Jeffrey Mo, David Ting and Philip Hogg	Mr Jim Kotow
4	254	Ross Sheppard High School, Edmonton, with Jerry Lo, William Wong and Kevin Trieu	Mr Jeremy Klassen
5	245	Harry Ainlay High School, Edmonton, with Melissa Chung, Cindy Qian and David Szepesvari	Ms Jacqueline Coulas
6	231	Queen Elizabeth Junior/Senior High School, Calgary, with Di Mo, Dustin Styner and R Wang/M Wanless	Ms Sharon Reid
7	229	Old Scona Academic High School, Edmonton, with Cindy Liu, Hyunbin Jeong and Stephen Portillo	Mr Lorne Pasco
8	227	Tempo School, Edmonton, with Michael Wong, Sumayr Sekhon and Maninder Longowal	Mr Lorne Rusnell
9	215	Henry Wise Wood High School, Calgary, with Ye Jay, Javier Romualdez and Patricia Rohs	Mr Michael Retallack
10	204	Bishop Carroll High School, Calgary, with Sean Heisler, Connor Kjersteen and Allison Yuen	Ms Susan Osterkamp

The other participating schools (and team managers) were as follows:

Zone I

- Calgary Science School, Calgary (Ms Martina Metz)
- Central Memorial High School, Calgary (Mr Gerald Krabbe)
- John G Diefenbaker High School, Calgary (Mr Terry Loschuk)
- John Ware School, Calgary (Ms Gail Slen)
- Rundle College High School, Calgary (Ms Rachel Hinz)
- Saint Francis High School, Calgary (Ms Allison van de Laak)

Zone II

- Canmore Collegiate High School, Canmore (Ms Patti Fairhart-Jones)
- Cardston High School, Cardston (Ms Debbie Fletcher)
- Crowsnest Consolidated High School, Coleman (Mr Bruce Kutcher)
- Foothills Composite High School, Okotoks (Ms Audra Schneider)
- Prairie Christian Academy, Three Hills (Mr Robert Hill)
- Strathcona-Tweedsmuir School, Okotoks (Ms Nola Adam)
- Winston Churchill High School, Lethbridge (Ms Terri Yamagashi)

Zone III

- Archbishop MacDonald High School, Edmonton (Mr John Campbell)
- Concordia High School, Edmonton (Ms Jenny Kim)
- Jasper Place High School, Edmonton (Ms Nadine Molnar)
- McNally High School, Edmonton (Mr Neil Peterson)
- St Francis Xavier High School, Edmonton (Ms Joanne Stepney)

Zone IV

- Archbishop Jordan Catholic High School, Sherwood Park (Ms Marge Hallonquist)
- École Secondaire Sainte Marguerite d'Youville, St Albert (Ms Lisa La Rose)
- Father Patrick Mercredi Community High School, Fort McMurray (Mr Ted Venne)
- Leduc Composite High School, Leduc (Ms Corlene Balding)
- Paul Kane High School, St Albert (Mr Percy Zalasky)

Report on Part II

Individual Results

Part II of the 51st AHSMC was written on February 7, 2007, by 65 students representing 13 schools. Here are the top performers:

Rank	Student	School
1	Jeffrey Mo	William Aberhart High School, Calgary
2	Jerry Lo	Ross Sheppard High School, Edmonton
3	Boris Braverman	Sir Winston Churchill High School, Calgary
4	Jarno Sun	Western Canada High School, Calgary
5	Linda Zhang	Western Canada High School, Calgary
6	Danny Shi	Sir Winston Churchill High School, Calgary
7	Tony Zhao	Sir Winston Churchill High School, Calgary
	Brett Baek	Western Canada High School, Calgary
9	Sherwin Ghafouri	Western Canada High School, Calgary
	Dustin Styner	Queen Elizabeth Junior/Senior High School, Calgary
11	Matthew Wang	Western Canada High School, Calgary
	Darren Xu	Sir Winston Churchill High School, Calgary
	Simon Sun	Sir Winston Churchill High School, Calgary
14	Kyle Boone	Western Canada High School, Calgary
	Annie Xu	Old Scona Academic High School, Edmonton
16	Yu Xiang Liu	Western Canada High School, Calgary
	Chong Shen	Sir Winston Churchill High School, Calgary
	Michael Wong	Tempo School, Edmonton
	Graham Hill	Sir Winston Churchill High School, Calgary
	Stephanie Li	Sir Winston Churchill High School, Calgary
	Cindy Qian Harry	Harry Ainlay High School, Edmonton
	David Ting	William Aberhart High School, Calgary

Congratulations to the above students, their schools and their teachers!

Problems and Solutions

Part I

1. The value of $2^4 4^8 8^{16}$ is
   (a) 2^{16}
   (b) 2^{52}
   (c) 2^{68}
   (d) 2^{84}
   (e) none of these

2. The number of noncongruent rectangles with integer sides and area $2006 = 2 \times 17 \times 59$ is
   (a) 3
   (b) 4
   (c) 6
   (d) 8
   (e) none of these

3. The number of pairs (m, n) of positive integers such that $m^4 + n = 100{,}000{,}001$ is
   (a) 100
   (b) 101
   (c) 200
   (d) 201
   (e) none of these

4. In a city, all streets run north–south or east–west, dividing the city into squares. A, B, C and D are four students who live at four street intersections that define a rectangle, with A and C at opposite corners of this rectangle. They all go to the same school, which is at some street intersection within this rectangle. Each goes to school by the most direct route along streets. A travels 10 blocks,

B travels 20 blocks and C travels 50 blocks. The number of blocks D travels is

(a) 10
(b) 20
(c) 30
(d) 40
(e) 50

5. When $(1 + x)(1 + x^2)(1 + x^3)(1 + x^4)(1 + x^5)$ is expanded, the coefficient of the term x^9 is

(a) 0
(b) 1
(c) 2
(d) 3
(e) 4

6. On a plane are two points, A and B, at a distance 5 apart. The number of straight lines in this plane that are at a distance 2 from A and at a distance 3 from B is

(a) 0
(b) 1
(c) 2
(d) 3
(e) 4

7. A country has three provinces, each province has three cities, each city has three wards and each ward has three electors. In a two-way election, a candidate wins a ward by getting more votes in the ward, wins a city by winning more wards in the city, wins a province by winning more cities in the province and wins the election by winning more provinces. Only electors may vote, and they must vote. The minimum number of votes needed to guarantee winning the election is

(a) 41
(b) 54
(c) 66
(d) 81
(e) none of these

8. The teacher asked, "What is the largest possible diameter of a circular coin of negligible thickness that may be stored in a rectangular box with inner dimensions $7 \times 8 \times 9$?" Ace said less than 8, Bea said 8, Cec said strictly between 8 and 9, Dee said 9 and Eve said more than 9. The one who was right was

(a) Ace
(b) Bea
(c) Cec
(d) Dee
(e) Eve

9. If a and b are two real numbers that satisfy $a + b - ab = 1$ and a is not an integer, then b

(a) is never an integer
(b) must be some positive integer
(c) must be some negative integer
(d) must be equal to 0
(e) may be either an integer or a noninteger

10. The nonzero numbers a, b, c, x, y and z are such that

$$\frac{x}{a} = \frac{y}{b} = \frac{z}{c}.$$

The value of

$$\frac{xyz(a+b)(b+c)(c+a)}{abc(x+y)(y+z)(z+x)}$$

is

(a) 0
(b) $\frac{1}{3}$
(c) 1
(d) 3
(e) dependent on the common value of $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$

11. If k is a rational number such that $\sqrt[3]{9\sqrt{3} - 11\sqrt{2}} = \sqrt{3} + k\sqrt{2}$, then the value of k is

(a) −2
(b) −1
(c) 1
(d) 2
(e) none of these

12. All positive integers that can be expressed as a sum of one or more different integer powers of 5 are written in increasing order. The first three terms of this sequence are 1, 5 and 6. The 50th term is

(a) 3751
(b) 3755
(c) 3756
(d) 3760
(e) 3761

13. Consider all polynomials whose coefficients are all integers, whose roots include $\sqrt{3}/2$ and $\sqrt{2}/3$, and whose degree is as small as possible. Among the coefficients of these polynomials, the smallest positive coefficient is

(a) 1
(b) 6
(c) 35
(d) 36
(e) none of these

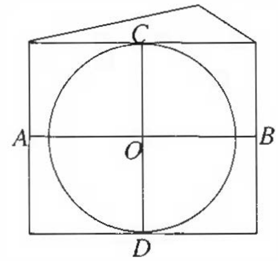
14. Colleen used a calculator to compute $(a + b)/c$, where a, b and c are positive integers. She pressed the buttons **a**, **+**, **b**, **/**, **c** and **=** (in that order) and got the answer 11. When she pressed **b**, **+**, **a**, **/**, **c** and **=** (in that order), she was surprised to get a different answer—14. Then she realized that the calculator performed the division before the addition. So she pressed **(**, **a**, **+**, **b**, **)**, **/**, **c** and **=** in that order. She finally got the correct answer, which is
   (a) 4
   (b) 5
   (c) 20
   (d) 25
   (e) none of these

15. The base of a tetrahedron is an equilateral triangle of side 1. The fourth vertex is at a distance 1 above the centre of the base. The radius of the sphere that passes through all four vertices of the tetrahedron is
   (a) $\frac{1}{3}$
   (b) $\frac{\sqrt{3}}{6}$
   (c) $\frac{1}{2}$
   (d) $\frac{\sqrt{3}}{3}$
   (e) $\frac{2}{3}$

16. If $p(x)$ is a polynomial of degree 4 such that $p(-1) = p(1) = 5$ and $p(-2) = p(0) = p(2) = 2$, then the maximum value of $p(x)$ is
   (a) 5
   (b) 6
   (c) 7
   (d) 10
   (e) none of these

Solutions

1. The value of $2^4 4^8 8^{16}$ is $2^{4+8\times2+16\times3} = 2^{68}$. The answer is **(c)**.
2. There are four such rectangles, namely 1×2006, 2×1003, 17×118 and 34×59. The answer is **(b)**.
3. Since $100^4 = 100{,}000{,}000$, $(100, 1)$ is a desired pair. In fact, m can be any positive integer up to 100, and there is a unique positive integer n such that $m^4 + n = 100{,}000{,}001$ for this particular m. Hence, the total number of pairs is 100. The answer is **(a)**.
4. A and C together travel as many north–south blocks as B and D together. A and C together also travel as many east–west blocks as B and D together. Hence, A and C together travel as many blocks as B and D together. It follows that the number of blocks D travels is $10 + 50 - 20 = 40$. The answer is **(d)**.
5. We are counting the number of ways to make 9 from 1, 2, 3, 4 and 5, using each number no more than once. There are three ways, namely $1 + 3 + 5$, $2 + 3 + 4$ and $4 + 5$. The answer is **(d)**.
6. Draw a circle with centre A and radius 2. Draw another circle with centre B and radius 3. The lines we seek are the common tangents of these two circles, of which there are three. The answer is **(d)**.
7. We can steal the election by winning as few as 16 votes. We can win two provinces, two cities within each of those two provinces, two wards within each of those four cities and two electors within each of those eight wards. Hence, $(81 - 16) + 1 = 66$ votes are required to guarantee winning the election. The answer is **(c)**.
8. Let A and B be the respective midpoints of two opposite edges of length 9. Let C and D be the respective centres of the two 7×8 faces. (See the diagram below.) Then, AB and CD intersect at the centre O of the box. Place a coin of diameter 9 on the plane determined by AB and CD with centre O. Its circumference will pass through C and D, but A and B are not covered up since $AB = \sqrt{7^2 + 8^2} > 9$. Rotate the coin about AB so that it just comes off C and D. We can then expand the coin slightly and still have it fit inside the box. The answer is **(e)**. *Note: We do not know the actual maximum value. That is why the question is phrased in its current form.*



9. Solving for b, we have $b(1 - a) = 1 - a$. Since a is not an integer, $1 - a \neq 0$ and can be cancelled. Hence, $b = 1$ and is a positive integer. The answer is **(b)**.
10. Let

$$\frac{x}{a} = \frac{y}{b} = \frac{z}{c} = r.$$

Then,

$$\frac{xyz}{abc} = r^3, x = ra, y = rb \text{ and } z = rc.$$

It follows that $x + y = r(a + b)$, $y + z = r(b + c)$ and $z + x = r(c + a)$, so that

$$\frac{x+y}{a+b} = \frac{y+z}{b+c} = \frac{z+x}{c+a} = r.$$

The given expression is equal to

$$r^3\left(\frac{1}{r}\right)^3 = 1.$$

The answer is **(c)**.

11. We have $9\sqrt{3} - 11\sqrt{2} = \left(\sqrt{3} + k\sqrt{2}\right)^3 = 3\sqrt{3} + 9k\sqrt{2} + 6k^2\sqrt{3} + 2k^3\sqrt{2}$. From $9 = 3 + 6k^2$, we have $k = \pm 1$. From $-11 = 9k + 2k^3$, $k < 0$. Hence, $k = -1$. The answer is **(b)**.

12. We change the base-10 number 50 to base-2 and obtain the number 110,010, representing $2^5 + 2^4 + 2^2$. We now interpret this as a base-5 number, representing $5^5 + 5^4 + 5^2$. Changing this number to base-10, we obtain 3755. The answer is **(b)**.

13. Since $\sqrt{2}/3$ and $\sqrt{3}/2$ are roots, we must also have $-\left(\sqrt{2}/3\right)$ and $-\left(\sqrt{3}/2\right)$ as roots in order to have rational coefficients. Thus, the degree of the polynomial cannot be less than 4, but 4 is sufficient. Of these polynomials, the one with 1 as the leading coefficient is

$$\left(x-\frac{\sqrt{2}}{3}\right)\left(x+\frac{\sqrt{2}}{3}\right)\left(x-\frac{\sqrt{3}}{2}\right)\left(x+\frac{\sqrt{3}}{2}\right) = \left(x^2-\frac{2}{9}\right)\left(x^2-\frac{3}{4}\right)$$

$$= x^4 - \frac{35}{36}x^2 + \frac{1}{6}.$$

Since we want integral coefficients, we clear the denominators to obtain $36x^4 - 35x^2 + 6$. Since the coefficients are relatively prime (though not pairwise relatively prime) and the smallest absolute value among the coefficients is 6, the smallest positive coefficient is also 6. The answer is **(b)**.

14. We have

$$a + \frac{b}{c} = 11 \text{ and } b + \frac{a}{c} = 14.$$

Adding the two equations yields

$$(a+b)\frac{c+1}{c} = 25 \text{ or } (a + b)(c + 1) = 25c.$$

Since $c + 1$ and c are relatively prime, $c + 1$ must divide 25. Hence, $c = 4$ or $c = 24$. If $c = 24$, then $a + b = 25$, but at least one of a/c and b/c is not an integer. Hence, $c = 4$, $a + b = 20$ and $(a + b)/c = 5$. The answer is **(b)**.

15. Let the base of the pyramid be the equilateral triangle ABC. Let D be the midpoint of BC. Then,

$$AD = \sqrt{AB^2 - BD^2} = \frac{\sqrt{3}}{2}.$$

Let G be the centre of ABC. Then, G lies on AD and

$$AG = \frac{2AD}{3} = \frac{\sqrt{3}}{3}.$$

Let V be the fourth vertex of the pyramid, and let O be the centre of the sphere passing through all four vertices. Then, O lies on VG, and let $r = VO = OA$ be the radius of the sphere. Since $VG - VO = OG = \sqrt{OA^2 - AG^2}$, we have

$$1 - r = \sqrt{r^2 - \frac{1}{3}}.$$

Squaring both sides, we have $1 - 2r + r^2 = r^2 - \frac{1}{3}$. It follows that $r = \frac{2}{3}$. The answer is **(e)**.

16. Note that $p(-2) < p(-1) > p(0) < p(1) > p(2)$. Hence, the graph of this fourth-degree polynomial opens down. Let $p(x) = -x^4 + ax^3 + bx^2 + cx + d$. Then, $d = p(0) = 2$. From $p(\pm 2) = 2$, we have $-16 \pm 8a + 4b \pm 2c + 2 = 2$. Hence, $4b - 16 = \pm(8a + 2c)$. This is only possible if both sides are equal to 0, so that $b = 4$. Similarly, from $p(\pm 1) = 5$, we have $25a + 5c = 0$. Combining with $8a + 2c = 0$, we have $a = c = 0$, so that $p(x) = -x^4 + 4x^2 + 2 = 6 - (x^2 - 2)^2$. It follows that the maximum value of $p(x)$ is 6, occurring when $x^2 - 2 = 0$ or $x = \pm\sqrt{2}$. The answer is **(b)**.

Part II

Problem 1

Determine all positive integers n such that n is divisible by any positive integer m that satisfies $m^2 + 4 \le n$.

Problem 2

The numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14 and 15 are arranged to form a 5×3 table in each of the 15! possible ways. For each table, we compute the sum of the three numbers in each row, and record in a list the largest and the smallest of these sums. Determine the sum of the $2 \times 15!$ numbers on our list.

Problem 3

One angle of a triangle is 36° while each of the other two angles is also an integral number of degrees. The triangle can be divided into two isosceles triangles by a straight cut. Determine all possible values of the largest angle of this triangle.

Problem 4

Let a, b and c be distinct nonzero real numbers such that

$$\frac{1-a^3}{a} = \frac{1-b^3}{b} = \frac{1-c^3}{c}.$$

Determine all possible values of $a^3 + b^3 + c^3$.

Problem 5

A survey in Alberta was sent to some teachers and students, a total of $2006 = 2 \times 17 \times 59$ people. Exactly $a\%$ of the teachers and exactly $b\%$ of the students responded, yielding an overall response rate of exactly $c\%$, where a, b and c are integers satisfying $0 < a < c < b < 100$. For each possible combination of values of a, b and c, determine the total number of teachers and the total number of students who responded to the survey.

Solutions and Comments

Problem 1

This problem really consists of two parts, finding values for n and proving that there are no more. Most contestants got somewhere with the first part, but many faltered in the second.

For $n = 1, 2, 3$ and 4, there are no positive integers m such that $m^2 + 4 \le n$. Hence, these four values have the desired property vacuously. While not an essential part of the problem, these values should be included for completeness.

If the maximum value of m is 1, then $1^2 + 4 \le n < 2^2 + 4$ and $n = 5, 6$ or 7. Since 1 divides all of them, these three values have the desired property. If the maximum value of m is 2, then $2^2 + 4 \le n < 3^2 + 4$ and $n = 8, 9, 10, 11$ or 12. Of these, only 8, 10 and 12 are divisible by both 1 and 2. If the maximum value of m is 3, then $3^2 + 4 \le n < 4^2 + 4$ and $n = 13, 14, 15, 16, 17, 18$ or 19. Of these, only 18 is divisible by all of 1, 2 and 3. If the maximum value of m is 4, then $4^2 + 4 \le n < 5^2 + 4$ and $n = 20, 21, 22, 23, 24, 25, 26, 27$ or 28. Of these, only 24 is divisible by all of 1, 2, 3 and 4. It will turn out that no positive integers other than 1, 2, 3, 4, 5, 6, 7, 8, 10, 12, 18 and 24 have the desired property.

For the second part, **Jeffrey Mo,** of William Aberhart High School, argued as follows. Suppose the maximum value of m is k for some integer $k \ge 5$. Then, $k^2 + 4 \le n < (k + 1)^2 + 4$. In order for n to be divisible by just $k - 1$ and k, it has to be a multiple of $k(k - 1)$ since $k - 1$ and k are relatively prime. Now $k(k - 1) < k^2 + 4$, while $2k(k - 1) - [(k + 1)^2 + 4] = k^2 - 4k - 5 = (k + 1)(k - 5) \ge 0$ for $k \ge 5$. Hence, n cannot be a multiple of $k(k - 1)$, so that there are no solutions for $m \ge 5$.

Jerry Lo, of Ross Sheppard High School, argued as follows. Suppose we have solutions n for some $m \ge 5$. Then, n must be a multiple of m. Now $m^2 + 4 < m^2 + m < m^2 + 2m < (m + 1)^2 + 4 \le m^2 + 3m$, with equality holding in the last case only for $m = 5$. If $n = m^2 + 3m$, then we must have $m = 5$ so that $n = 40$, but 40 is not divisible by 3. Hence, $n = m(m + 1)$ or $m(m + 2)$. Note that n must also be a multiple of $m - 1$. Note that $m - 1$ and m are relatively prime. If $n = m(m + 1)$, then $m - 1$ must divide $m + 1 = (m - 1) + 2$. Hence, it must divide 2, so that $m \le 3$. If $n = m(m + 2)$, then $m - 1$ must divide $m + 2 = (m - 1) + 3$. Hence, it must divide 3, so that $m \le 4$. Either case contradicts $m \ge 5$. Hence, there are no solutions n for $m \ge 5$.

Problem 2

Far too many contestants did not know that the total number of tables is 15!. For those who did, most merely observed that the maximum sum of a row is $13 + 14 + 15 = 42$ and the minimum sum is $1 + 2 + 3 = 6$. From these, they concluded that the total of the two sums must be 48, in that if the maximum sum dropped, the minimum sum would rise and compensate. While this may be a loose description of what is the case, it does not explain why this is the case. The argument really rests on one simple fact. The following solution, by **Linda Zhang,** of Western Canada High School, is typical of those of the top contestants.

For each table A, there is a table B that may be obtained from A by subtracting each number in A from 16. Note that A and B are distinct tables. Now the row in A with the largest sum turns into the row in B with the smallest sum, and the row in A with the smallest sum turns into the row in B with the largest sum. The largest row sum of A plus the smallest row sum of B is 48, as is the largest row sum of B plus the smallest row sum of A. Since the 15! tables may be divided into 15!/2 such pairs, the sum of the $2 \times 15!$ numbers on our record is $48 \times 15!$.

Problem 3

This turned out to be the problem in which most contestants could make some progress. However, many approached it haphazardly and managed to find only some of the answers. Others found all the answers but did not prove that there are no more. The following is the solution by **Jarno Sun,** of Western Canada High School.

Let ABC be the triangle. Let $\angle ABC = 36°$. We may assume that $\angle CAB \ge \angle BCA$. Then,

$$\angle CAB \ge \frac{180° - 36°}{2} = 72° > \angle ABC.$$

In order for ABC to be divided into two triangles with a straight cut, the cut must pass through a vertex. We consider three cases:

Case 1. The cut passes through B.

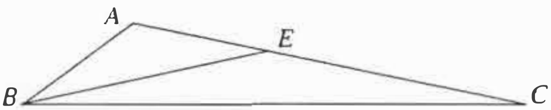
Let the cut meet CA at E. Since $\angle CAB > \angle ABC > \angle ABE$, $\angle BEA$ must be one of the equal angles in triangle BEA. It follows that $\angle BEA$ is acute so that $\angle BEC$ is obtuse. (See the diagram below.) Let $\angle EBC = \angle BCA = x°$. Then, $\angle BEA = 2x°$. We consider two subcases:

SUBCASE 1a. $\angle BEA = \angle CAB$.

Then, $\angle ABE = 180° - 4x°$ and $36° = \angle ABC = (180° - 4x°) + x°$. This yields $x = 48$, but then $\angle ABE = -12°$. This is impossible.

SUBCASE 1b. $\angle BEA = \angle ABE$.

Then, $\angle ABE = 2x°$ and $36° = \angle ABC = 2x° + x°$. This yields $x = 12$. It follows that ABC is a (132°, 36°, 12°) triangle.



CASE 2. The cut passes through A.

Let the cut meet BC at D. We consider three subcases:

SUBCASE 2a. $\angle BDA = \angle ABC = 36°$.

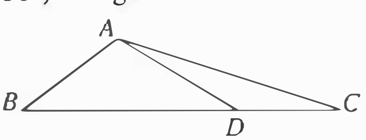
Then, $\angle ADC$ is obtuse. (See the first diagram below.) We must have $\angle BCA = \angle CAD = 36°/2 = 18°$, so that ABC is a (126°, 36°, 18°) triangle.

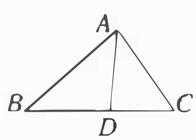
SUBCASE 2b. $\angle BAD = \angle ABC = 36°$.

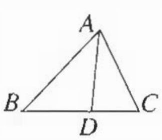
Then, $\angle BDA = 108°$. If $AD = CD$, then $\angle DAC = \angle BCA = 108°/2 = 54°$ and $\angle CAB = 90°$. (See the second diagram below.) It follows that ABC is a (90°, 54°, 36°) triangle. If $AD = AC$, then $\angle BCA = \angle ADC = 72°$ and $\angle CAB = 180° - 36° - 72° = 72°$. (See the third diagram below.) It follows that ABC is a (72°, 72°, 36°) triangle. Finally, if $AC = CD$, then $\angle CAD = \angle ADC = 72°$. Hence, $\angle BCA = 36°$ and $\angle ABC = 108°$. (See the fourth diagram below.) It follows that ABC is a (108°, 36°, 36°) triangle.

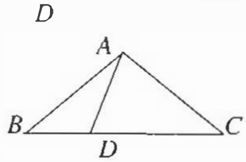
SUBCASE 2c. $\angle BAD = \angle BDA = 72°$.

Then, $\angle ADC$ is obtuse. We must have $\angle BCA = \angle CAD = 72°/2 = 36°$, and ABC is again a (108°, 36°, 36°) triangle.









CASE 3. The cut passes through C.

Let the cut meet AB at F. Since $\angle CAB \geq \angle BCA > \angle ACF$, $\angle AFC$ must be one of the equal angles in triangle AFC. It follows that $\angle CFB$ is obtuse. It follows that $\angle BCF = \angle ABC = 36°$ and $\angle AFC = 72°$. Since $\angle CAB \geq \angle BCA$, ABC is again a (72°, 72°, 36°) triangle.

In summary, the largest angle of ABC, namely $\angle CAB$, may be 72°, 90°, 108°, 126° or 132°.

Problem 4

With greater reliance on graphing calculators and computer software, most students nowadays are uncomfortable with algebraic manipulations. A problem such as this has become inaccessible to most contestants. **Brett Baek,** of Western Canada High School, took the following approach.

From

$$\frac{1-a^3}{a} = \frac{1-b^3}{b},$$

we have $b - a^3b = a - ab^3$. Hence, $a - b = ab(b^2 - a^2)$ so that $ab(a + b) = -1$. Similarly, $bc(b + c) = -1$. From these two equations, we have $a^2 - c^2 = bc - ab$ or $(a + c)(a - c) = -b(a - c)$. Since $a \neq c$, we have $a + b + c = 0$. Hence, $abc(a + b) = -c = a + b$. Since $a + b + c = 0$ but $c \neq 0$, $a + b \neq 0$ and we have $abc = 1$. Now,

$$\begin{aligned} 0 &= (a + b + c)^3 \\ &= a^3 + 3a^2(b + c) + 3a(b + c)^2 + (b + c)^3 \\ &= (a^3 + b^3 + c^3) + 3(a + b + c)(bc + ca + ab) - 3abc \\ &= (a^3 + b^3 + c^3) + 0 - 3. \end{aligned}$$

It follows that the only possible value of $a^3 + b^3 + c^3$ is 3.

For those with more knowledge of algebra, this problem was practically trivial. **Jerry Lo** had the most succinct write-up.

The given conditions show that a, b and c are roots of the equation $x^3 + kx - 1 = 0$, where k is the common value of the three fractions. Hence,

$$\begin{aligned} x^3 + kx - 1 &= (x - a)(x - b)(x - c) \\ &= x^3 - (a + b + c)x^2 + (bc + ca + ab)x - abc. \end{aligned}$$

It follows that $a + b + c = 0$, $abc = 1$ and $a^3 + b^3 + c^3 = (a + b + c)(a^2 + b^2 + c^2 - bc - ca - ab) + 3abc = 3$.

Problem 5

This problem, which looks easy, is very annoying. Only one contestant gave a complete argument, and a handful of others came close. There is a relatively easy part of the problem that many contestants got—that is, showing that $c = 50$. Suppose z people in all responded to the survey. Then, $z/2006 = c/100$, or $50z = 1003c$. Since 50 and 1003 are relatively prime, c must be a multiple of 50. Since we are given that $0 < a < c < b < 100$, the only possible value is $c = 50$. After this, things get messy. What follows is the approach taken by **Jeffrey Mo**.

Let the total number of teachers be d and the number of those teachers who responded be x. Then, $x/d = a/100$ and $(1003 - x)/(2006 - d) = b/100$. From the first, we have $ad = 100x$. From the second, we have $2006b - bd = 100{,}300 - 100x = 100{,}300 - ad$. This may be rewritten as $(b - a)d = 2006(b - 50)$. It follows that $1003 = 17 \times 59$ divides $(b - a)d$. Now $b - a < 100 < 1003$, and we also have $d < 1003$ since $a < c = 50$. Hence, there are two cases.

Case 1. 59 divides $b - a$.

This means, of course, that $b - a = 59$ and $d = 2 \times 17(b - 50)$. Hence, $50x = ad/2 = 17a(a + 9)$. Since 25 is relatively prime to 17 and to at least one of a and $a + 9$, it must divide either $a < 50$ or $a + 9 < 59$. We consider three subcases.

Subcase 1a. $a = 25$.

We have $b = 25 + 59 = 84$, $x = [17 \times 25(25 + 9)]/50 = 289$ and $d = 2 \times 17(84 - 50) = 1156$. Hence, $1003 - 289 = 714$ and $2006 - 1156 = 850$. It follows that 289 of 1156 teachers and 714 of 850 students responded to the survey.

Subcase 1b. $a + 9 = 25$.

We have $a = 25 - 9 = 16$, $b = 16 + 59 = 75$, $x = (17 \times 16 \times 25)/50 = 136$ and $d = 2 \times 17(75 - 50) = 850$. Hence, $1003 - 136 = 867$ and $2006 - 850 = 1156$. It follows that 136 of 850 teachers and 867 of 1156 students responded to the survey.

Subcase 1c. $a + 9 = 50$.

We have $a = 50 - 9 = 41$ and $b = 41 + 59 = 100$. This contradicts $b < 100$, and there are no solutions in this subcase.

Case 2. 17 divides $b - a$.

Then, $b - a = 17n$, where $n \leq 5$. We have $nd = 2 \times 59(b - 50)$. Hence, $50nx = (and)/2 = 59a(17n - 50 + a)$.

We consider five subcases, none of which yield additional solutions.

Subcase 2a. $n = 1$.

We have $50x = 59a(a - 33)$, and 25 must divide either a or $a - 33$. The former means $a = 25$, but then $a - 33 < 0$. The latter means $a - 33 \geq 25$, but then $a \geq 58 > 50 = c$. Both lead to contradictions.

Subcase 2b. $n = 2$.

We have $100x = 59a(a - 16)$, and 25 must divide either a or $a - 16$. The former means that $a = 25$, but then $59a(a - 16)$ is odd. The latter means $a - 16 = 25$, but then $59a(a - 16)$ is again odd.

Subcase 2c. $n = 3$.

We have $150x = 59a(a + 1)$, and 25 must divide either a or $a + 1$. The former means that $a = 25$, but then $59a(a + 1)$ is not divisible by 3. The latter means $a + 1 = 25$ or 50. If $a = 49$, $59a(a + 1)$ is again not divisible by 3. If $a = 24$, then $x = 236$, but then $d = (100 \times 236)/24$ is not an integer.

Subcase 2d. $n = 4$.

We have $200x = 59a(a + 18)$, and 25 must divide either a or $a + 18$. The former means that $a = 25$, but then $59a(a + 18)$ is odd. The latter means $a + 18 = 25$ or 50. If $a = 7$, $59a(a + 18)$ is again odd. If $a = 32$, then $b = 32 + 68 = 100$, and this contradicts $b < 100$.

Subcase 2e. $n = 5$.

We have $250x = 59a(35 + a)$, and 25 must divide either a or $35 + a$. However, this means that $a \geq 25$ and $b = a + 85 > 100$, a contradiction.

A shorter approach goes as follows. Let s be the number of students and t be the number of teachers in the survey. Then, $s + t = 2006$, both $(ta)/100$ and $(sb)/100$ are integers, and

$$\frac{ta}{100} + \frac{sb}{100} = 1003.$$

Note that 5 cannot divide both s and t. If 5 does not divide s, then b must be a multiple of 25. Since $50 < b < 100$, we must have $b = 75$. If 5 does not divide t, then a must be a multiple of 25. Since $0 < a < 50$, we must have $a = 25$. If 5 does not divide either s or t, then $a = 25$ and $b = 75$, and we have $t + 3s = 4012$. Subtract from this $s + t = 2006$, and we have $2s = 2006$, so that $s = t = 1003$. However, neither $(ta)/100$ nor $(sb)/100$ is an integer. Henceforth, we assume that 5 divides exactly one of s and t. We consider two cases. Suppose 5 divides t. Then, $b = 75$ and we have $ta + 75s = 2006 \times 50$. Subtracting this from $75t + 75s = 2006 \times 75$, we get $(75 - a)t = 2006 \times 25$. Since 5 divides t, $75 - a$ divides $2 \times 17 \times 59 \times 5$. Since $0 < a < 50$, $25 < 75 - a < 75$. Hence, we must have $75 - a = 34$ or 59. If $a = 41$, both t and a are odd, and $(ta)/100$ will not be an integer. If $a = 16$, we have $t = 850$. This leads to $s = 1156$. Thus, 867 students and 136 teachers responded to the survey, yielding a total of 1003, as required by $c = 50$. Finally, since $(100 - a)t + (100 - b)s = 100(s + t) - (at + bs) = 200{,}600 - 100{,}300 = 100{,}300$, the only other solution is $b = 100 - 16 = 84$ and $a = 100 - 75 = 25$, as indicated above since now 5 divides s instead of t. Solving $s + t = 2006$ and $84s + 25t = 100{,}300$, we have $s = 850$ and $t = 1156$. Thus, 714 students and 289 teachers responded to the survey, again yielding the desired total of 1003.