One Way to Avoid Using a Graphing Calculator

Indy Lagu

I was recently looking through a calculus text, and a question caught my eye. The question is quite common; I suspect that it appears in most calculus books. It concerns the optimal distance for viewing a statue. Variations of the problem abound: where to sit in a movie theatre and from where to attempt an extra point in rugby come to mind.

Imagine that you are standing and looking at a statue. Let b be the distance from your eyes to the base of the statue, and let a be the distance from your eyes to the top of the statue. If the distance from you to the statue is x, then the angular size of the statue is given by

$$\theta = \tan^{-1}\left(\frac{a}{x}\right) - \tan^{-1}\left(\frac{b}{x}\right).$$

The task is to determine x to maximize θ.

The approach taken in a calculus class is to take a derivative:[1]

$$\frac{d\theta}{dx} = \frac{-a/x^2}{1+\left(a^2/x^2\right)} - \frac{-b/x^2}{1+\left(b^2/x^2\right)}.$$

A bit (all right, a lot) of algebra reveals

$$\frac{d\theta}{dx} = \frac{-a}{x^2+a^2} + \frac{b}{x^2+b^2},$$

which is 0 when $x = \sqrt{ab}$. Note that $\theta \to 0$ as $x \to 0$ and as $x \to \infty$, so that this x gives a maximum.

A lower-tech approach (mathematically and intellectually) is to plug the equation into a graphing calculator and eyeball the maximum.

I do not mind the first approach, but I abhor the graphing calculator approach. Now, I think my resistance is not to the use of technology but to the *misuse* of technology. One objection is, of course, that eyeballing a maximum is not mathematically correct—indeed, the power of mathematics is that we can find the exact answer here and not resort to eyeballing. However, my objection is deeper: the act of finding an answer to this question misses the point entirely. Surely the point of solving a problem such as this is not merely to get an answer but to *learn* something.

To that end, I offer a mathematically correct approach that provides an opportunity to use some facts about trigonometric functions.

Recall that we wish to find x to maximize θ, where

$$\theta = \tan^{-1}\left(\frac{a}{x}\right) - \tan^{-1}\left(\frac{b}{x}\right).$$

Taking the tangent of both sides yields

$$\tan(\theta) = \tan\left[\tan^{-1}\left(\frac{a}{x}\right) - \tan^{-1}\left(\frac{b}{x}\right)\right]$$

$$= \frac{a/x - b/x}{1+\left(ab/x^2\right)}$$

$$= \frac{a-b}{x+\left(ab/x\right)},$$

where we have used the identity

$$\tan\left(A-B\right) = \frac{\tan A - \tan B}{1+\tan A \tan B}.$$

Now, since $\tan(\theta)$ increases with θ, and $a - b$ is a constant, it is enough to maximize the function

$$\frac{1}{x+\left(ab/x\right)},$$

and therefore it is enough to minimize the function

$$f(x) := x + \frac{ab}{x}.$$

Even now, the temptation to stop thinking and use a graphing calculator can be avoided. Since x is increasing and ab/x is decreasing, a good guess is that $f(x)$ is minimized when $x = ab/x$. That is, at $x = \sqrt{ab}$.

Since $f(\sqrt{ab}) = 2\sqrt{ab}$, the question now is to try to show that

$$x + \frac{ab}{x} \geq 2\sqrt{ab},$$

and even this can be done without resorting to calculus or calculators. I will leave this as an exercise for the reader. Your hint is to start with the fact that

$$\left(\sqrt{x} - \sqrt{\frac{ab}{x}}\right)^2 \geq 0.$$

Incidentally, the quantity $\sqrt{ab}$ is called the *geometric mean of a and b*,[2] and it has an interesting geometric interpretation. Imagine a rectangle with sides a and b. Its area is ab. Now imagine a square with the same area. The sides of the square must be $\sqrt{ab}$.

I guess that the approach you favour in solving this problem depends on your goal. If your goal is to estimate the answer, and avoid any depth of thought, then using a graphing calculator is all right, I suppose. If your goal is to obtain the correct answer and to flex your calculus muscles, then differentiation is the way to go.

My goal would be to see the utility of trigonometry; hence, I would avoid using a graphing calculator, or even calculus, and use my head. What makes a problem good is not the realism or practicality in it but, rather, where it takes you.

Notes

1. If you have never taken a calculus class, skip this part. The headache is not worth it.

2. The arithmetic mean—$(a + b)/2$—is probably familiar to most readers and is often called *the average*.

Indy Lagu completed his PhD in mathematics at the University of Calgary in 1996. His interests moved from numerical analysis to the state of mathematics education in Alberta. He has been a member of the MCATA executive for seven years and has been working with teachers and students in the province for much longer. Currently, Indy is chair of the Department of Mathematics, Physics and Engineering at Mount Royal College in Calgary. He spends his free time learning Spanish, playing golf and wondering when the Montreal Canadiens will win another Stanley Cup.