Textbooks in Mathematics Learning: The Potential for Misconceptions

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Introduction

As textbook authors ourselves[1] (for example, Kajander 2007; Lovric 2007a) we have found ourselves face to face with the tension between the historical use of the text as the mathematical authority and the notions of mathematics reform as described in the *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics [NCTM] 2000), which position the learners as prominent participants in their own learning. "Rather than the textbook and the teacher acting as major sources of authority, this intended curriculum encourages students to rely on their own mathematical reasoning and evidence when discussing mathematical solutions" (Herbel-Eisenmann 2007, 345).

As we began to critically examine sample secondary and postsecondary mathematics texts, we saw evidence of textbook formats that might have been intended to simplify the learning process for students. In our examination we found many examples of such attempts that, in fact, potentially introduced mathematical misconceptions. This article summarizes some of the types of issues we have found related to mathematical exposition, and suggests areas requiring attention in textbook writing, design and use.

Background

Both older and more recent research seems to highly privilege the role of the textbook in classroom mathematics learning. McKnight, Crosswhite and Dossey (1987) reported that more than 95 per cent of Grade 12 teachers indicated that the textbook was their most commonly used resource, and recent research in Ontario indicates that this situation has changed little (Kajander and Mason, forthcoming). More generally, "Commercially published, traditional textbooks dominate mathematics curriculum materials in US classrooms and to a great extent determine teaching practices" (Clements 2007, 55). Yet even with all this emphasis on textbooks in learning, we are becoming convinced that research about the textbooks themselves, and particularly the implications for mathematical content interpretation they impart, is modest at best.

Some effort has been put into content analysis and exploring the ways in which textbooks are used in classrooms and beyond (for example, Love and Pimm 1996; McCrory 2006). Recent work (Herbel-Eisenmann and Wagner 2007; Herbel-Eisenmann 2007) has examined how the textbook positions the learner, particularly with respect to discourse. However, market research, rather than research based on mathematics education sources, is usually used to determine textbook content and approach (Clements 2007).

Project 2061 is a long-term project, aimed at evaluating resources in science and mathematics, supported by the American Association for the Advancement of Science. According to one of their studies, "the majority of textbooks used for algebra ... have some potential to help students learn, but they also have serious weaknesses" (Project 2061 2000). More than half of the 12 textbooks evaluated by the project team were considered adequate, but none were rated highly; three were rated as highly inadequate for student learning. According to the Project 2061 findings, authors of textbooks generally ignore the research on how students acquire new mathematical ideas and concepts as described in the *Standards* (NCTM 2000).

Our work is grounded in the theory of *conceptual change* (Davis 2001; Biza, Souyoul and Zachariades 2005), which provides a framework for the study of potential student misconceptions related to learning from textbooks. The theory describes learning processes of adults as well as children and hence is appropriate in addressing high school and university students.

The Study

We have begun to take a closer look at textbooks commonly used at the secondary and first-year-university levels, particularly in Ontario, to determine how mathematics textbooks might contribute to the development of students' conceptions and misconceptions about mathematics, particularly with respect to the development of conceptual understanding. We begin by providing our current working model, which may be helpful to others wishing to undertake critical mathematical examinations of texts, and follow this with an example.

Our Working Model

As we worked through a number of textbook examples, we found issues that appeared to fall into a number of somewhat overlapping groupings. We have attempted to make sense of these by offering the following framework for such analyses, which has emerged from our work thus far. Roughly speaking, these groupings are

1. the use of colloquial, reader-friendly language beyond an intuitive introduction, and in places where mathematical precision was warranted;
2. incorrect generalizations, often taken out of context;
3. diagrams as sources of misconceptions (sometimes supporting oversimplified definitions);
4. oversimplification, often leading to inaccuracy;
5. discussion of concepts not yet properly defined; and
6. design issues such as summary boxes.

We are continuing to use and refine this rough framework as we examine more sources.

An Example

As one of many examples we have encountered, consider the concept of the tangent line to the graph of the function $y = f(x)$. From our own experience, common ideas held by many students entering first-year university include incomplete or inaccurate conceptions, such as "the tangent is the line that touches the graph of $y = f(x)$ at only one point," or "a tangent line cannot cross the graph of $y = f(x)$." Accordingly, we began exploring how such conceptions might arise from previously studied written sources. When we examined a Grade 12 calculus textbook currently in use in some Ontario schools, we easily found evidence that could support (or at best fail to correct and clarify) such misconceptions. For example, in one source, an initial explanation near the beginning of the chapter on tangents states clearly (with accompanying diagrams), "In the graphs of the circle and the parabola, a tangent line touches exactly one point of the graph, *P*. For other curves, such as the one in the third diagram [an example of a tangent that also crosses the curve at two other points] a tangent line touches the graph at the point of tangency, *P*, but may pass through other points on the graph as well" (Kirkpatrick et al 2002, 183).

While this explanation is thorough, a subsequent coloured summary box contains the incomplete definition of a tangent as follows: "A tangent is a line that touches exactly one point on the graph of a relation" (Kirkpatrick et al 2002, 190).

In fact, this oversimplified statement is not true in all cases, and emphasizing it in a summary box may increase the likelihood of its being the definition remembered by the student, rather than the concept being deeply understood. This oversimplification is an example of Issue 4 in our working model, and is further emphasized by the design issue (Issue 6 in the model) of placing the text in a coloured summary box on the page. While the summary is meant to make things easier for students, highlighting such oversimplifications, *which are not always true*, introduces the potential for incomplete understanding and misconceptions. In a traditional classroom environment, definitions are often presented by the teacher rather than developed by the students themselves as meaning is made. If the definition used by a teacher is the page 190 summary version on its own, and students are not encouraged to investigate the concept more fully with suitable prompts and questions, students will likely be left with an incomplete understanding. These and other examples made it painfully obvious to us that students' common misconceptions in first-year calculus may often have their roots in previously studied materials.

Similarly, standard university calculus texts contain numerous illustrations of tangent lines. However, in a majority of cases, the tangent is shown in the generic position where it touches the curve at one point and does not cross it. The concept of *touching* might be further suggested in examples where students are given the graph of a function $f(x)$ and are asked to sketch the graph of its derivative $f'(x)$. Although the tangent is defined as the limit of secant lines, these examples do not attempt to encourage drawing the tangent or thinking about it (either in illustration or in accompanying text) as limiting the position of secant lines.

Continuing with this example, to address this potential misconception, a textbook (or an instructor) could ask students to create illustrations that show relationships between curves and lines and identify which are (or are not) tangents (Lovric 2007b). These illustrations should include cases such as a tangent that crosses the graph at the point of tangency; a tangent line that touches the graph at more than one point; a line that touches the graph, does not cross it, but is not a tangent (cusp); and so on. By analyzing such situations, students could gain a more accurate understanding.

Discussion

Given that textbooks appear to remain the resource of choice for many teachers, we believe that more attention needs to be focused on the quality of their content. To date, "we know little about the impact of curriculum materials on the ... relationships developed between these readers and the curriculum materials themselves ... [thus] we must first examine the relationships that the textbook materials themselves encourage" (Herbel-Eisenmann 2007, 345–46).

In our work to date, we have been struck by how the issues we identified as problematic for learning from textbooks—such as generalizations that were incomplete and that lacked connections to intuitive understanding, ideas that were separated from connections to context, and incomplete summaries placed in specially designed coloured boxes implying that they were to be remembered—echoed a traditional pedagogy. On the other hand, the treatments that we found more effective—based on a gradual shift from informal language, good visuals and identification of misconceptions to more formal, rigorous and precise generalizations—echoed a more reform-based philosophy as defined in the *Standards* (NCTM 2000).

We believe that as students move through the secondary curriculum and into the postsecondary realm, they might obtain deeper and more accurate conceptual understandings of fundamental concepts if more attention were paid to textbooks that support rather than undermine such growth. Rather than criticizing the predominant use of textbooks by teachers, better success might be achieved by actively ensuring the quality of these texts in promoting deep and accurate student understanding.

Note

1. Ann Kajander has written books of activities for elementary and early secondary students, while Miroslav Lovric is the author of undergraduate textbooks.

References

Biza, I, A Souyoul and T Zachariades. 2005. "Conceptual Change in Advanced Mathematical Thinking." Discussion paper presented at the Fourth Congress of the European Society for Research in Mathematics Education, Sant Feliu de Guíxols, Spain, February 17–21.

Clements, D H. 2007. "Curriculum Research: Toward a Framework for 'Research-Based Curricula.'" *Journal for Research in Mathematics Education* 38, no 1: 35–70.

Davis, J. 2001. "Conceptual Change." In *Emerging Perspectives on Learning, Teaching, and Technology*. Ed M Orey. www.coe.uga.edu/epltt/ConceptualChange.htm (accessed February 6, 2008).

Herbel-Eisenmann, B. 2007. "From Intended Curriculum to Written Curriculum: Examining the 'Voice' of a Textbook." *Journal for Research in Mathematics Education* 38, no 4: 344--69.

Herbel-Eisenmann, B, and D Wagner. 2007. "A Framework for Uncovering the Way a Textbook Might Position the Mathematics Learner." *For the Learning of Mathematics* 27, no 2: 8–14.

Kajander, A. 2007. *Big Ideas for Growing Mathematicians*. Chicago: Zephyr.

Kajander, A, and R Mason. Forthcoming. "Examining Teacher Growth in Professional Learning Groups for In-Service Teachers of Mathematics." *Canadian Journal of Mathematics, Science and Technology Education.*

Kirkpatrick, C, R McLeish, R Montesanto, C Suurtamm, S Trew and D Zimmer. 2002. *Advanced Functions and Introductory Calculus.* Scarborough, Ont: Nelson.

Love, E, and D Pimm. 1996. "'This Is So': A Text on Texts." In *International Handbook of Mathematics Education: Part One*, ed A J Bishop, M A Clements, C Keitel, J Kilpatrick and C Laborde. 371--409. Dordrecht, Netherlands: Kluwer Academic.

Lovric, M. 2007a. *Vector Calculus.* Hoboken, NJ: Wiley and Sons.

———. 2007b. "Mathematics Textbooks and Promotion of Conceptual Understanding." Presentation to the Education Forum of the Fields Institute for Mathematical Sciences, Toronto, Ont, January 20.

McCrory, R. 2006. "Mathematicians and Mathematics Textbooks for Prospective Elementary Teachers." *Notices of the American Mathematical Society* 53, no 1: 20–29.

McKnight, C, F J Crosswhite and J A Dossey. 1987. *The Underachieving Curriculum.* Champaign, Ill: Stipes.

National Council of Teachers of Mathematics (NCTM). 2000. *Principles and Standards for School Mathematics.* Reston, Va: NCTM.

Project 2061. 2000. "Algebra for All—Not with Today's Textbooks, Says AAAS." Press release. April 26. www.project2061.org/about/press/pr000426.htm (accessed January 22, 2008).

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