Algebra Through Symmetry

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Algebra Through Symmetry is a visual method that was prevalent at the dawn of the mathematical sciences. At the University of Alexandria, in 310 BC, Euclid's students used symmetry in the solution of equations. In the equation $2x = 8$, each term was presumed to be an *area* and a geometric diagram was constructed. Considering Euclid's fourth axiom, "Things which coincide with one another are equal to one another," a student would conclude that equality requires that the two rectangular areas are equal; therefore, x must coincide with 4.

This article documents and reformulates the method in modern notation. Symmetry in algebra is produced when an equation is reformulated such that the form of the equation is the same on both sides of the equality. The natural concept of symmetry (human and animal form) implies one-to-one matching and is an organic part of human understanding, is ingrained very early in life, and is independent of one's intelligence or mathematical ability.

The axiomatic method relies on the concept of formal proof, assumption--deduction--conclusion, to reduce and simplify an asymmetric equation to its final symmetrical form:

$ax + b = c$ Apply the axioms to achieve final reflective symmetry.

$x = \frac{c-b}{a}$ Variable = Value, by reflective symmetry.

The visual method relies on the natural concept of symmetry. The special properties of 0 and 1 allow the reformulation of an asymmetric equation to its final symmetrical form:

$ax + b = c$ Apply the axioms to reformulate and achieve final translational symmetry.

$ax + b = a(\frac{c-b}{a}) + b$ Variable = Value, by a one-to-one matching of terms.

$ax + b = b + (\frac{c-b}{a})\,a$ Reflective symmetry is a possible reformulation, but not likely.

The visual method is the inverse of the axiomatic method in the sense that both achieve symmetry in the final equation, one through abstract symbolic logic and the other through a process of reformulation.

Define: Axiomatic Method

Apply the axioms to both sides of an equation to simplify and reduce to an equivalent equation. Solve for the variable explicitly and check the solution in the original equation.

Axiomatic Mantra: ***Whatever you do to one side of an equation you must do to the other.***

Define: Visual Method

Apply the special properties of 0 and 1 to obtain symmetry in an equation. Solve for the variable implicitly using a one-to-one matching of terms. Check any modified expression.

Visual Mantra: ***Rewrite the equation until both sides look the same.***

Linear Equations

Linear equations are particularly well suited for a solution by the visual method. The method enables the student to determine by inspection the terms required to obtain symmetry. Generally, the abstract portion of the equation is kept unmodified, and the real portion is reformulated such that the form of the equation is the same on both sides.

L1) Visual solution:

$$3x+5=11 \;\Rightarrow\; 3x+5=(11-5)+5$$

$$3x+5=3(\frac{11-5}{3})+5 \;\Rightarrow\; x=\frac{11-5}{3}, \text{ from one-to-one matching}$$

Alternative Visual: $3x+5=11 \;\Leftarrow$ formulate and add a symmetrical equation

$$3(\frac{11-5}{3})=3x \;\Rightarrow\; x=\frac{11-5}{3}, \text{ from one-to-one matching}$$

$$3x+11=3x+11 \;\Leftarrow\; \text{Symmetry verifies that: } x=\frac{11-5}{3}=2$$

$3x+5=3(2)+5$ Translational symmetry—the more likely reformulation

$3x+5=5+(2)3$ Reflective symmetry—the less likely reformulation

$x=2$ Reflective symmetry—the axiomatic solution

The unique value that produces symmetry in an equation verifies that value as the solution; there is no need to check this value in the original equation. No learning takes place by evaluating the final expression, except when the variable has a tangible meaning, because that hides the terms used in the reformulation, negates the visual check of the solution, requires the tedium of arithmetic and is a source of errors.

L2) A salesperson earns a salary of $125 per week plus a commission of $2.40 for each item sold. Find the number of items sold if the earnings were $528.20 in one week.

$$2.40n+125=528.20 \;\Rightarrow\; 2.40n+125=2.40(\frac{528.20-125}{2.40})+125$$

$$2.40n+125=2.40(168)+125 \quad n=168 \text{ items sold, from one-to-one matching}$$

L3) The cost to rent an automobile is $37 per day plus $0.21 per mile. If the final bill is $61.15, how many miles were driven?

$$0.21m+37=61.15 \;\Rightarrow\; 0.21m+37=0.21(\frac{61.15-37}{0.21})+37$$

$$0.21m+37=0.21(115)+37 \quad m=115 \text{ miles, from one-to-one matching}$$

Systems of Linear Equations

In systems of linear equations, where the variables have a tangible meaning, the equations will be reformulated to obtain symmetry or to contain a collinear equation. Examples S1 to S6 introduce problem solving in two variables. The visual method will be used to obtain the solutions; current textbooks would use the axiomatic method exclusively.

S1) There are two supplementary angles in which one angle is 12 degrees less than 3 times the other. What is the measure of the angles?

$$x+y=180 \;\Rightarrow\; x+y=192-12 \;\Rightarrow\; x+y=4(48)-12$$

$$y=3x-12 \;\Rightarrow\; x+y=x+3x-12 \;\Rightarrow\; x+y=4x-12$$

$$x=48^\circ \text{ from one-to-one matching} \qquad y=180-48=132^\circ$$

S2) Cool Mitts, Inc, sold 20 pairs of gloves. Plain leather gloves sold for \$24.95 per pair and gold-braided gloves sold for \$37.50 per pair. The company took in \$687.25. How many of each kind were sold?
Reformulate to contain a collinear equation with ratio = 1/24.95; 20(24.95) = 499.00

$$p+g=20 \quad \Rightarrow \quad p+g = 20$$

$$24.95p+37.50g=687.25 \quad \Rightarrow \quad 24.95p+24.95g+12.55g=499.00+188.25$$

$$g=\frac{188.25}{12.55} \quad \Rightarrow \quad g=15 \text{ gold-braided gloves}$$

$$p=20-15 \quad \Rightarrow \quad p=5 \text{ plain leather gloves}$$

S3) Antifreeze solution A is 2% methanol, and solution B is 6% methanol. Auto-Parts, Inc, wants to mix the two to get 60 litres of a solution that is 3.2% methanol. How many litres of each solution are required?
Reformulate to contain a collinear equation with ratio = 1/2%

$$A+B=60 \quad \Rightarrow \quad A+B=60$$

$$2\%A+6\%B=3.2\%(60) \quad \Rightarrow \quad 2\%A+2\%B+4\%B=2\%(60)+1.2\%(60)$$

Equality requires that $B=\frac{1.2\%(60)}{4\%}=18$ litres of 6% methanol,

and that A is the 4's complement of 1.2, $A=\frac{(4-1.2)\%(60)}{4\%}=42$ litres of 2% methanol,

or, using the first equation $A+18=60 \Rightarrow A=42$ litres of 2% methanol.

S4) A \$4800 investment in two corporate bonds earns \$412 in interest the first year. The bonds pay interest of 8% and 9% per annum. Find the amount invested at each rate of interest.
Reformulate to contain a collinear equation with ratio = 1/0.08; 0.08(4800) = 384

$$E+N=4800 \quad \Rightarrow \quad E+N=4800$$

$$0.08E+0.09N=412 \quad \Rightarrow \quad 0.08E+0.08N+0.01N=384+28$$

$$N=\frac{28}{0.01}=\$2800 \text{ at } 9\% \qquad E=\$4800-\$2800=\$2000 \text{ at } 8\%$$

In systems of abstract linear equations, one equation is kept unmodified while the other equations are reformulated by multiplication and the addition of zeros to obtain the matching terms. The procedure eliminates the entire unmodified equation and yields a solution to the remaining variables. In the examples, the coefficients and constants are prime numbers or fractions so that the solution is difficult by the axiomatic method.

S5) Visual solution, system of two linear equations:

$$A: 11x-13y=17 \quad \Rightarrow \quad 11x-13y=17$$

$$B: 7x+5y=23 \quad \Rightarrow \quad (\frac{11}{7}B):\ 11x-13y+13y+\frac{55}{7}y=17-17+\frac{253}{7}$$

$$17+(\frac{7(13)+55}{7})y=17+(\frac{7(-17)+253}{7}) \quad \Rightarrow \quad \frac{146}{7}y=\frac{134}{7} \quad \Rightarrow \quad y=\frac{134}{146} \quad [y=\frac{67}{73}]$$

$$B: 7x+5(\frac{67}{73})=23(\frac{73}{73}) \quad \Rightarrow \quad x=\frac{23(73)-5(67)}{7(73)} \quad [x=\frac{192}{73}]$$

$$\text{Check } A:\ 11(\frac{192}{73})-13(\frac{67}{73})=17 \Rightarrow \frac{2112}{73}-\frac{871}{73}=17 \Rightarrow \frac{1241}{73}=17 \quad 17=17$$

$$\text{Check } B:\ 7(\frac{192}{73})+5(\frac{67}{73})=23 \Rightarrow \frac{1344}{73}+\frac{335}{73}=23 \Rightarrow \frac{1679}{73}=23 \quad 23=23$$

S6) Visual solution, system of two linear equations with fractions as coefficients:

$A: \frac{1}{11}x - \frac{1}{13}y = 17 \qquad \frac{1}{11}x - \frac{1}{13}y = 17$

$B: \frac{1}{7}x + \frac{1}{5}y = 23 \Rightarrow (\frac{7}{11}B): \frac{1}{11}x - \frac{1}{13}y + \frac{1}{13}y + \frac{7}{55}y = 17 - 17 + \frac{161}{11}$

$17 + (\frac{55+13(7)}{13(55)})y = 17 + (\frac{11(-17)+161}{11}) \Rightarrow \frac{146}{13(55)}y = \frac{-26}{11} \Rightarrow y = -\frac{13(55)(26)}{146(11)} \quad [y = -\frac{845}{73}]$

$B: \frac{1}{7}x + \frac{1}{5}(-\frac{845}{73}) = 23(\frac{73}{73}) \Rightarrow \frac{1}{7}x - \frac{169}{73} = \frac{1679}{73} \Rightarrow x = \frac{7(1679+169)}{73} \quad [x = \frac{12936}{73}]$

Check A: $\frac{1}{11}(\frac{12936}{73}) - \frac{1}{13}(-\frac{845}{73})y = 17 \Rightarrow \frac{1176}{73} + \frac{65}{73}y = 17 \Rightarrow \frac{1241}{73} = 17 \quad 17 = 17$

Check B: $\frac{1}{7}(\frac{12936}{73}) + \frac{1}{5}(-\frac{845}{73})y = 23 \Rightarrow \frac{1848}{73} - \frac{169}{73}y = 23 \Rightarrow \frac{1679}{73} = 23 \quad 23 = 23$

Linear Inequalities

Linear inequalities can be reformulated by keeping the abstract portion of the inequality unmodified. Add the equivalent of zeros to obtain the matching terms, and then multiply by the equivalent of one to obtain a matching coefficient. If the coefficient of the variable term is negative, the sense of the inequality must be reversed at the final step.

LI1) Visual solution, variable with positive coefficient:

$-3 \le 2x + 4 \le 10 \Rightarrow 4 - 4 - 3 \le 2x + 4 \le 10 + 4 - 4 \Rightarrow -7 + 4 \le 2x + 4 \le 6 + 4$

$\frac{2}{2}(-7) + 4 \le 2x + 4 \le \frac{2}{2}(6) + 4 \Rightarrow 2(\frac{-7}{2}) + 4 \le 2x + 4 \le 2(3) + 4$

Range of the variable: $(-3.5) \le x \le 3$ from one-to-one matching.

LI2) Visual solution, variable with negative coefficient:

$-3 \le -2x + 4 \le 10 \Rightarrow 4 - 4 - 3 \le -2x + 4 \le 10 + 4 - 4 \Rightarrow -7 + 4 \le -2x + 4 \le 6 + 4$

$\frac{-2}{-2}(-7) + 4 \le 2x + 4 \le \frac{-2}{-2}(6) + 4 \Rightarrow -2(\frac{-7}{-2}) + 4 \le -2x + 4 \le -2(-3) + 4$

Reverse sense of the inequality. Range: $(3.5) \ge x \ge -3$ from one-to-one matching.

Rational Equations

Rational equations are particularly difficult to comprehend precisely because they contain abstract fractions. The visual solution generally keeps the more difficult side of the equation unmodified and rewrites the less difficult side to obtain symmetry. In some cases partial symmetry is created, followed by division. Division generally produces remainders on both sides of an equation. The remainders must be equated repeatedly to reduce the original rational equation to a linear form.

R1) Visual solution: create partial symmetry to avoid the binomial multiplications.

$\frac{x+3}{x+5} = \frac{2x-7}{2x+11} \Rightarrow \frac{x+5-2}{x+5} = \frac{2x+11-18}{2x+11} \Rightarrow 1 + \frac{-2}{x+5} = 1 + \frac{-18}{2x+11} \Rightarrow \frac{-2}{x+5}(\frac{9}{9}) = \frac{-18}{2x+11}$

$9x + 45 = 2x + 11 \Leftarrow$ Formulate and add a symmetrical equation

$-7x = -7(\frac{34}{-7}) \Rightarrow x = -\frac{34}{7}$ from one-to-one matching

$2x + 45 = 2x + 45 \Leftarrow$ Symmetry verifies that $x = -\frac{34}{7}$; no check is needed.

R2) Visual solution: rational equation that contains multiple fractions

$$\frac{x+5}{2}+\frac{1}{2}=2x-\frac{x-3}{8}$$

$$2x-\frac{x-3}{8}-2x+\frac{x-3}{8}+\frac{x+5+1}{2}=2x-\frac{x-3}{8} \quad \Leftarrow \text{ LHS: add } 0=\text{RHS}-\text{RHS}$$

$$2x-\frac{x-3}{8}-2x(\frac{8}{8})+\frac{x-3}{8}+\frac{x+6}{2}(\frac{4}{4})=2x-\frac{x-3}{8}$$

$$2x-\frac{x-3}{8}+\frac{-16x}{8}+\frac{x-3}{8}+\frac{4x+24}{8}=2x-\frac{x-3}{8}$$

$$2x-\frac{x-3}{8}+\frac{-11x+21}{8}=2x-\frac{x-3}{8}$$

Unmatched term (numerator) = zero; reformulate to contain an inverse coefficient.

$-11x+21=0 \;\Rightarrow\; -11x+11(\frac{21}{11})=0 \quad x=\frac{21}{11}$ from one - to - one matching.

LHS expression check: is the original expression equivalent to the final expression?

There is no need to use $x=\frac{21}{11}$ for the LHS expression check; $x=0$ will do.

Original expression: $\frac{5}{2}+\frac{1}{2}=\frac{6}{2}=3$ Final expression: $-\frac{-3}{8}+\frac{21}{8}=\frac{+3+21}{8}=\frac{24}{8}=3$

R3) Visual solution: rational expression equal to an abstract fraction

$$\frac{x-4}{3x}+\frac{x-8}{5x}=\frac{-16}{x} \quad \Rightarrow \quad \frac{x-4}{3x}+\frac{x-8}{5x}=\frac{-16}{x}(\frac{15}{15})$$

$$\frac{5}{5}(\frac{x-4}{3x})+\frac{3}{3}(\frac{x-8}{5x})=\frac{5}{5}(\frac{x-4}{3x})+\frac{3}{3}(\frac{x-8}{5x})+\frac{-240-5(x-4)-3(x-8)}{15x}$$

Unmatched term (numerator) = zero: $-240-5x+20-3x+24=0$

$-196-8x=0 \;\Rightarrow\; +8(\frac{-196}{+8})-8x=0 \;\Rightarrow\; x=-\frac{49}{2}$ from one-to-one matching

This unique value produces symmetry: $\frac{5}{5}(\frac{x-4}{3x})+\frac{3}{3}(\frac{x-8}{5x})=\frac{5}{5}(\frac{x-4}{3x})+\frac{3}{3}(\frac{x-8}{5x})$

in the final equation and thereby verifies that the solution is $x=-\frac{49}{2}$. Check the modified

expression: does final RHS = original RHS? Do not use $x=-\frac{49}{2}$ to check, $x=1$ will do.

$$\frac{5(1-4)+3(1-8)}{15(1)}+\frac{-240-5(1-4)-3(1-8)}{15(1)}=\frac{-16}{1} \quad \Rightarrow \quad \frac{-240}{15(1)}=\frac{-16}{1} \quad \Rightarrow \quad \frac{-16}{1}=\frac{-16}{1}$$

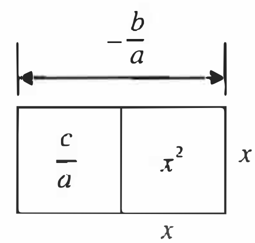
Note: in R2 and R3, the axiomatic method (multiply through by 8 and 15*x*, and so on) requires 26 and 24 arithmetic operations respectively, including the check, versus 11 and 9 for the visual method.

Quadratic Equations

In quadratic equations, completing the square is currently in vogue as the only general solution of these equations. A geometric–numerical method, formulated at the dawn of the mathematical sciences, is also a general solution. A quadratic equation can be represented by a geometric diagram, which, in turn, leads to a simple numerical solution.

Generalized Geometric–Numerical Method

$$f(x) = ax^2 + bx + c = 0 \quad \Rightarrow \quad x^2 + \frac{c}{a} = -\frac{b}{a}x \quad \Rightarrow$$



The diagram is symmetrical when the equation is a trinomial square, that is, area $\frac{c}{a}$ = area x^2.

The redundant roots are : $x = [-\frac{b}{2a}, -\frac{b}{2a}]$. Assuming that every quadratic equation has these roots would produce an error, $|e^2| \geq 0$. The deviation from a trinomial square can be computed.

$$e^2 = -x^2 - \frac{b}{a}x - \frac{c}{a} \quad \Rightarrow \quad e^2 = -(-\frac{b}{2a})^2 - \frac{b}{a}(-\frac{b}{2a}) - \frac{c}{a} \quad \Rightarrow \quad e^2 = -\frac{b^2}{4a^2} + (\frac{b^2}{2a^2}) - \frac{c}{a}$$

$$e^2 = \frac{b^2}{4a^2} - \frac{c}{a} \quad \Rightarrow \quad e^2 = \frac{(\frac{b}{2})^2}{a^2} - \frac{ac}{a^2} \quad \Rightarrow \quad e^2 = \frac{(\frac{b}{2})^2 - ac}{a^2} \qquad f(-\frac{b}{2a}) = \frac{-e^2}{a}$$

Roots of a quadratic equation are $x = \frac{-\frac{b}{2}}{a} \pm \frac{\sqrt{e^2}}{a}$ and the vertex is $v = (-\frac{b}{2a}, \frac{-e^2}{a})$.

Redefine e^2 as the numerator $(\frac{b}{2})^2 - ac$, and express as a determinant to facilitate computation.

Example : $f(x) = 3x^2 - 18x + 10 \qquad e^2 = \begin{vmatrix} -\frac{b}{2} & a \\ c & -\frac{b}{2} \end{vmatrix} = \begin{vmatrix} -\frac{-18}{2} & 3 \\ 10 & -\frac{-18}{2} \end{vmatrix} = \begin{vmatrix} 9 & 3 \\ 10 & 9 \end{vmatrix} = [81] - [30] = 51$

Root: $x = \frac{9}{3} \pm \frac{\sqrt{51}}{3} \quad \Rightarrow \quad x = 3 \pm \frac{\sqrt{51}}{3}$ $\qquad$ Vertex: $v = (3, \frac{-51}{3}) \quad \Rightarrow \quad v = (3, -17)$

The geometric–numerical method requires six arithmetic operations to compute the roots and vertex. Considering that three coordinates can be found quickly using only arithmetic, the method lends itself to graphing quadratic equations. In this example, the quadratic formula requires nine arithmetic operations to compute the roots alone. The geometric–numerical method also calls into question the need to learn the factoring procedures to solve quadratic equations.

General Quadratic Equation: $f(x) = ax^2 + bx + c = 0$

Symmetrical form: $ax(x + \frac{b}{a}) + c = a[-\frac{b}{2a} \pm \frac{\sqrt{e^2}}{a}]([-\frac{b}{2a} \pm \frac{\sqrt{e^2}}{a}] + \frac{b}{a}) + c$

Factored form: $(x + \frac{b}{2a} + \frac{\sqrt{e^2}}{a})(x + \frac{b}{2a} - \frac{\sqrt{e^2}}{a}) = 0$

Geometric–numerical solution of quadratic equations

$$e^2 = \begin{vmatrix} -\frac{b}{2} & a \\ c & -\frac{b}{2} \end{vmatrix}, \quad \text{Roots: } x = \frac{-\frac{b}{2}}{a} \pm \frac{\sqrt{e^2}}{a}, \quad \text{Vertex: } v = (\frac{-\frac{b}{2}}{a}, \frac{-e^2}{a})$$

Trinomial square deviation: e^2 determines the nature of the solutions.

Q 1) $9x^2 - 24x + 16 = 0 \qquad e^2 = \begin{vmatrix} 12 & 9 \\ 16 & 12 \end{vmatrix} = [144] - [144] \qquad e^2 = 0$

e^2 is zero;equation is a trinomial square. Roots: $x = \frac{4}{3} \pm 0$ Vertex: $v = (\frac{4}{3}, 0)$

Q 2) $2x^2 - 6x - 7 = 0 \qquad e^2 = \begin{vmatrix} 3 & 2 \\ -7 & 3 \end{vmatrix} = [9] - [-14] \qquad e^2 = 23$

e^2 is positive; roots are irrational. Roots: $x = \frac{3}{2} \pm \frac{\sqrt{23}}{2}$ Vertex: $v = (\frac{3}{2}, \frac{-23}{2})$

Q 3) $-3x^2 - x + 2 = 0 \quad e^2 = \begin{vmatrix} \frac{1}{2} & -3 \\ 2 & \frac{1}{2} \end{vmatrix} = [\frac{1}{4}] - [-6] \qquad e^2 = \frac{4(6)+1}{4} \qquad e^2 = \frac{25}{4}$

e^2 is a positive square; roots are rational. Roots: $x = -\frac{1}{6} \pm \frac{5}{6}$ Vertex: $v = (-\frac{1}{6}, \frac{25}{12})$

Q 4) $3x^2 - 6x + 13 = 0 \qquad e^2 = \begin{vmatrix} 3 & 3 \\ 13 & 3 \end{vmatrix} = [9] - [39] \qquad e^2 = -30$

e^2 is negative; roots are complex. Roots: $x = 1 \pm \frac{\sqrt{-30}}{3}$ Vertex: $v = (1, 10)$

or Roots: $x = 1 \pm \frac{\sqrt{30}}{3} i$

Note: the second level of abstraction, $i = \sqrt{-1}$, can be avoided by instructing the students that the square of a square root with a negative radicand is a negative number, for example, $(\sqrt{-30})^2 = -30$.

Exponential Equations

Exponential equations with variables or fractions as exponents can be reformulated to obtain symmetry by using the change-of-base formula in logarithms. Symmetrical exponential equations eliminate the need to take the logarithm of both sides of an equation.

E1) Visual solution: an equation containing a variable exponent

$$24 = 5(2)^x \quad \Rightarrow \quad 5(\frac{24}{5}) = 5(2)^x \quad \Rightarrow \quad 5(4.8) = 5(2)^x$$

$$5(2)^{\frac{\log(4.8)}{\log(2)}} = 5(2)^x \qquad 5(2)^{\frac{\mathrm{Ln}(4.8)}{\mathrm{Ln}(2)}} = 5(2)^x$$

$x = \frac{\mathrm{Log}(4.8)}{\mathrm{Log}(2)}$ or $x = \frac{\mathrm{Ln}(4.8)}{\mathrm{Ln}(2)}$ from one-to-one matching.

E2) Visual solution: an equation containing an irrational exponent

$$3x^{\pi} - 7 = 4 \quad \Rightarrow \quad 3x^{\pi} - 7 = 3(\frac{4+7}{3}) - 7 \quad \Rightarrow \quad 3x^{\pi} - 7 = 3x^{\frac{\mathrm{Ln}(\frac{11}{3})}{\mathrm{Ln}(x)}} - 7$$

$$\pi = \frac{\mathrm{Ln}(\frac{11}{3})}{\mathrm{Ln}(x)} \text{ from one-to-one matching. } \mathrm{Ln}(x) = \frac{\mathrm{Ln}(\frac{11}{3})}{\pi} \quad \Rightarrow \quad x = 1.51221$$

E3) Visual solution: an equation containing a variable exponent and base

$$y = x^{x} \quad \Rightarrow \quad x^{\frac{\ln y}{\ln x}} = x^{x} \quad \Rightarrow \quad \frac{\ln y}{\ln x} = x \text{ from one-to-one matching}$$

$$\ln y = x \ln x \quad \Rightarrow \quad \text{limit as } x \Rightarrow 0; \quad \ln y = 0 \quad \Rightarrow \quad y = 1 \quad \Rightarrow \quad \therefore 0^{0} = 1$$

Note that this statement or a similar one appears in all of the algrebra textbooks:

For any real number a, $a \neq 0$, $a^{0} = 1$.

Anynonzero number raised to the zero power is 1.

This is the eternal textbook error as shown in example E3. While the error is minor, the statement is false and does not belong in an algebra textbook.

Conclusion

There is substantial experimental evidence that all human abilities are normally distributed. The centre point mean of a normal distribution implies that one-half of the student population will be below average in mathematical ability. Every student, to some degree, has a measure of mathematical ability that appears to depend on the degree to which one accumulates the mental representation of mathematical objects whose properties are reproducible. Intuition is the faculty by which one can consider or examine the mathematical objects that are stored in a mental set of neurons. When contemplating a problem, internal mathematical intuition is the faculty of browsing in one's neuron library until a new insight or connection between the objects is found. When contemplating a diagram that has a quantitative connotation, visual mathematical intuition is the faculty that enables the brain to discover the mathematical truth revealed in the diagram. The axiomatic method, the basis of which is the concept of formal proof (assumption–deduction–conclusion), has reigned too long as the exclusive basis of mathematics pedagogy, to the detriment of those students who lack the intuition necessary to comprehend the concept. Because visual learning is the dominant mode, visual mathematical intuition is also the dominant mode. This is why teaching the axiomatic and visual methods in parallel, both in the classroom and in the textbooks, would create a more powerful pedagogy that more closely corresponds to the mathematical abilities of the typical class in algebra. Such pedagogy will demonstrate that the visual method is the inverse of the axiomatic method in the sense that both achieve the unique solution, one through an axiomatic reformulation to translational symmetry and the other through an axiomatic reduction to reflective symmetry. Moreover, the visual method may be the only means available to rescue those students whose circumstances place them at high risk of failure in abstract mathematics.

Bill Mandras is a retired teacher and aerospace engineer. He has extensive experience as a college and university professor. His current interests include mathematics education and quantum physics. At age 75, he is delighted with our decision to publish his first manuscript in delta-K *and hopes that the ideas presented here will accrue to the benefit of thousands of students in the Alberta school system over time.*