delta-k

JOURNAL OF THE
MATHEMATICS COUNCIL
OF THE ALBERTA
TEACHERS' ASSOCIATION

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Volume 45, Number 2 June 2008

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Guidelines for Manuscripts

delta-K is a professional journal for mathematics teachers in Alberta. It is published twice a year to

- promote the professional development of mathematics educators, and
- stimulate thinking, explore new ideas, and offer various viewpoints.

Submissions are requested that have a classroom as well as a scholarly focus. They may include

- personal explorations of significant classroom experiences;
- descriptions of innovative classroom and school practices;
- reviews or evaluations of instructional and curricular methods, programs or materials;
- discussions of trends, issues or policies;
- a specific focus on technology in the classroom; or
- a focus on the curriculum, professional and assessment standards of the NCTM.

Suggestions for Writers

1. *delta-K* is a refereed journal. Manuscripts submitted to *delta-K* should be original material. Articles currently under consideration by other journals will not be reviewed.
2. If a manuscript is accepted for publication, its author(s) will agree to transfer copyright to the Mathematics Council of the Alberta Teachers' Association for the republication, representation and distribution of the original and derivative material.
3. All manuscripts should be typewritten, double-spaced and properly referenced. All pages should be numbered.
4. The author's name and full address should be provided on a separate page. If an article has more than one author, the contact author must be clearly identified. Authors should avoid all other references that may reveal their identities to the reviewers.
5. All manuscripts should be submitted electronically, using Microsoft Word format.
6. Pictures or illustrations should be clearly labelled and placed where you want them to appear in the article. A caption and photo credit should accompany each photograph.
7. References should be formatted consistently using *The Chicago Manual of Style*'s author–date system.
8. If any student sample work is included, please provide a release letter from the student's parent/guardian allowing publication in the journal.
9. Articles are normally 8–10 pages in length.
10. Letters to the editor or reviews of curriculum materials are welcome.
11. Send manuscripts and inquiries to the editor: Gladys Sterenberg, 195 Sheep River Cove, Okotoks, AB, T1S 2L4; email gladyss@ualberta.ca

MCATA Mission Statement

Providing leadership to encourage the continuing enhancement of teaching, learning and understanding mathematics.

delta-k

Volume 45, Number 2 June 2008

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EDITORIAL

This year has continued to bring opportunities for learning about mathematics education. With the implementation of the new program of studies in the fall, we find ourselves making many decisions that we know will affect our classrooms for years to come. This issue of *delta-K* responds to changing times by presenting research and teaching ideas that will inspire, challenge and motivate you to continue to develop professionally as a math teacher.

One major event of the Mathematics Council of the Alberta Teachers' Association (MCATA) is the annual conference. Included in this issue are photographs of participants and presenters who attended the 2007 conference, in Edmonton. I invite you to consider attending the 2008 conference, in Jasper, as a way of engaging in ongoing conversations about curriculum and pedagogical changes. The report from Alberta Education contains updated information about these changes.

Several awards are presented each year by MCATA, and the recipients are briefly introduced in this issue. As you can see, we are proud to be part of a strong community of mathematics educators. Art Jorgensen, a long-time member of this community, shared his thoughts about math education in a conversation with me. The Dr Arthur Jorgensen Chair Award honours him for the work he has done in this field. I encourage each of you to honour those leaders who have made a difference in your professional lives by nominating them for one of these awards. Application forms are available on MCATA's website at www.mathteachers.ab.ca.

Textbook purchases are priorities in upcoming budgets. The letter to the editor by Mandras and the feature article by Kajander and Lovric seek to provoke thoughtfulness about these decisions.

Throughout this process of change, we must reflect on our pedagogical practices. Kajander and Zuke present ideas for working with at-risk students; Mandras suggests that there are alternative ways of considering algebra. Avramovic and Oddy remind us to play with mathematics. A page of problems is also included for this purpose.

Be curious, be inspired. Have a wonderful summer.

Gladys Sterenberg

From the President's Pen

Another school year completed; where do they go? Whether this was your first year, your last year or somewhere in between, I hope that it brought you something new and exciting. One of my students asked me how I could keep teaching the same thing year in and year out. I told her that in 25 years of teaching high school mathematics, I have yet to teach the same thing two years in a row. Every year, our students change and our methodology changes. Whether it is new curriculum, new courses, new grade levels, new schools or just a new question that leads us in a new direction, we are always evolving. Often the newest trend in education is an old favourite with a new label that rejuvenates it and reminds us of things we haven't done in awhile.

The latest new curriculum may not seem that different at the surface level, but the depth of mathematical understanding is critical. Whether you took the opportunity of optional implementation of kindergarten, Grade 1, 4 or 7 this year or will be jumping into it in the fall, please take the time to discuss the changes with your colleagues. Between now and full implementation, in 2012, we must keep talking to each other and drawing on the amazing expertise of Alberta teachers in developing best practices. Whether you attend workshops, summer institutes or conferences, form professional learning communities, or meet informally with colleagues to discuss the changes, it is important that we keep the dialogue open and remain focused on the reasons for change—our students.

Alberta students have always done very well on international mathematics tests because of the guidance of their teachers. We have developed students who are critical thinkers, who can show logical reasoning and follow algorithmic methods to solve a multitude of problems. Through the past 25 years, I have watched the type of student change and have seen a few different curriculums. The topics we teach, the order in which we teach them and the depth to which we teach all change, but our passion for teaching does not. When we watch the light bulb turn on or see the student who has struggled graduate and move on, we know it is worth it.

I have had the pleasure and the challenge of discussing the new curriculum with many people over the past few years. There are many different viewpoints about what is important, what should and should not be in the curriculum and the best delivery methods. Debate is healthy, and at the end of the day we do not have to be happy about all decisions made. Because we are so passionate about our subject, it would be impossible for all math teachers to be happy, but the one thing that I know for certain is that we will all do the best job we can to ensure that our students are prepared for what lies ahead.

Enjoy your summer, take time to bask in the successes of the last year and recharge your batteries for the excitement of the next year. I look forward to seeing you in Jasper in October.

Sharon Gach

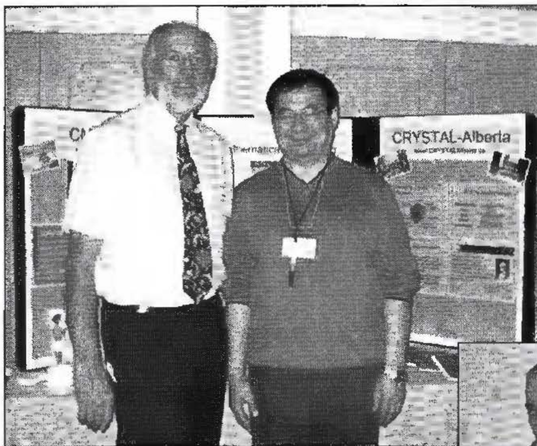
Photographic Memories

MCATA Conference 2007

Symposium

Lynn McGarvey—
Algebraic Reasoning

Conference Planners and Award Recipients



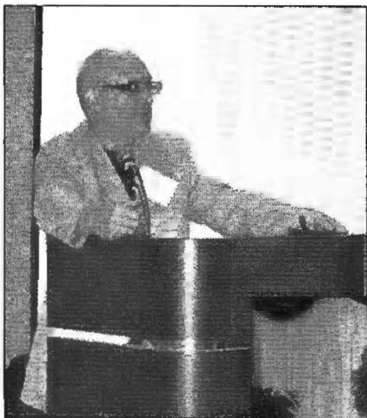
MCATA Displays

Friends of MCATA 2007—Tom Janzen, Eithne Keegan (missing) (presented by Janis Kristjansson)

Registration Desk

Door prizes presented by Sharon Gach

Presenters



Ross Marian

Irene Meglis and Elizabeth Mowat

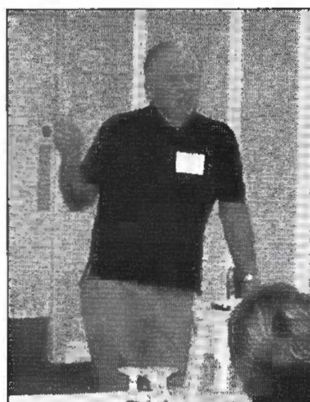
Garry Bell

Ellen Johnson

Mary Beisiegel

Sherry Talbot

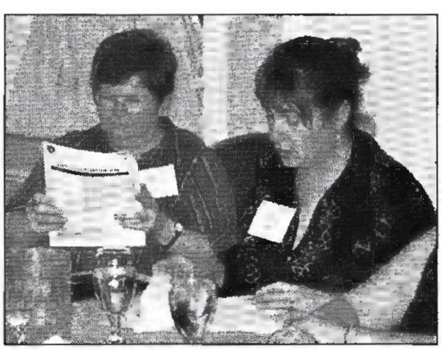
Frank Jenkins



Ron Larson

Julie Long

Participants



Dr Arthur Jorgensen Chair Award 2007: Tom Janzen

Every year, MCATA presents an award to a student in a degree program at a faculty of education in Alberta who has demonstrated academic excellence and a clear commitment to mathematics education. The recipient of the Dr Arthur Jorgensen Chair Award for 2007 is Tom Janzen.

It has been my great pleasure to meet Tom and I've been impressed by his commitment and desire to provide excellence in mathematics education. As an introduction into the community of mathematics teachers in Alberta, I present his thoughts on his background, preparation and goals for teaching mathematics.

—Editor

In school I was a strong mathematics student, and math was always my favourite subject. Following my high school graduation I was offered a job at my high school as a math assistant, which led to a job as an educational assistant at the local outreach school as their math specialist. This is what led me down the path towards pursuing mathematics education as my area of study and career. I have thoroughly enjoyed the opportunities I have had to work with students and share some of my love of the subject with them. I am excited by the challenges I face as a math educator and am discovering new ways to address them. At the outreach school I quickly discovered that many students had huge deficits in mathematics, which

Tom Janzen (left) receives the Dr Arthur Jorgensen Chair Award from Dr Jorgensen.

created a passion in me to find ways to open the doors to mathematics for students who have shut it out or been shut out.

I believe that mathematics education needs to be made visible and accessible to everyone in the school. This could take the form of math contests, fairs or family evenings, displays on school bulletin boards, or a math club. These are all opportunities I would like to be a part of if they already exist or initiate if they do not. I think it is very important that mathematics go beyond the math classroom. I think it would be very exciting and tremendously beneficial to work with colleagues in other departments to set up cross-curricular activities that would not only incorporate math into other subjects and the whole school community but also allow for more diverse activities in the math classroom. Networking with other schools could unite the school community. Partnering with local schools or doing something online with schools further away, even internationally, could get students very excited about math. Mathematics education needs to be seen as more than a necessary evil, and these are just a few ways that I believe that that can be achieved.

With respect to students, I believe that one of the biggest challenges continues to be a negative attitude towards mathematics. I think this is a societal issue that needs to be combatted at that level. It is very detrimental to have a socially acceptable distaste for mathematics. This can lead to students having preconceived notions of both the content and difficulty of the subject. I think that that is why many students give up on the subject before they even give it a chance. I also believe that this is one of the causes of the numerical literacy concerns in mathematics today. While working at the outreach school, I experienced the deficits in this area first-hand with junior high and high school students. For whatever reason, some students had been promoted from level to level without gaining the necessary skills to move forward. This sets them up for frustration and failure and contributes to negative attitudes towards math. I believe that numerical literacy is an issue that needs to be addressed at all levels of mathematics education, because it is such a key component of students' education and preparation for today's world. Another issue in mathematics education is the disconnect between math and the social-development nature of education. I do not believe that mathematics needs to be separated from the school's goal of preparing students to be contributing citizens to society. This is where projects like my work on developing global awareness and citizenship in the math classroom can be very beneficial. There are many opportunities to incorporate these different issues, and I do not believe that enough math teachers are taking advantage of them.

I feel that to foster positive attitudes I need to lead by example. As both a teacher and a person, I need to maintain my positive attitude toward the subject. Fostering a positive attitude also needs to be a priority for everyone, including colleagues and the students' families; everybody needs to be on board. In the classroom, I think giving students something different and unexpected could excite them and allow them to see math in a new light. Diversifying instruction and assessment strategies and incorporating ideas and issues from outside of the curriculum could help achieve this goal. I think this could make math class something fun and interesting that students would look forward to.

Mathematics Educator of the Year Award 2007: Sandra Unrau

MCATA's Mathematics Educator of the Year Award seeks to honour those who have made exceptional contributions to the professional development of teachers at the school, local, provincial or national levels and who have demonstrated leadership in encouraging the continuing enhancement of teaching, learning and understanding of mathematics in Alberta.

This year the award was presented to Sandra Unrau, a worthy recipient. When I joined MCATA, Sandra had just completed her presidency and become the past president. In our interactions, it was evident that her commitment to mathematics education was strong. Those of you who know her will attest to her dedication to excellence. If you have not had the pleasure of meeting or working with her, below is a brief introduction.

—Editor

Sandra Unrau (left) receives the Mathematics Educator of the Year Award from Janis Kristjansson.

Sandra is a long-time leader in the mathematics education community at the local, municipal, provincial and national level. She is incredibly active in the Calgary area. She was a "Math Their Way" instructor for many years, a math consultant for Calgary Board of Education, and an AISI coordinator for a math project that involved forty schools and their administrators. She also coordinated a Cycle 2 AISI project involving 20 Calgary schools interested in developing the mathematical literacy of students and teachers. She is on the executive of the Calgary ATA local and is a well-respected principal for the Calgary Board of Education.

In addition to her work with local and municipal teachers, she has served many terms as an executive member of MCATA. She has chaired both MCATA and National Council of Teachers of Mathematics (NCTM) conferences, including the NCTM Canadian regional conference, hosted in Edmonton in 2003, and the NCTM conference in St Louis, Missouri, and has organized math leadership symposia for MCATA. Sandra listens carefully to others and offers thoughtful responses that at once honour, challenge and provide alternatives to other points of views. These qualities were also very evident in her work with MCATA.

Sandra is passionately dedicated to improving the classroom mathematics experiences of children. She believes deeply that children's school experiences must respect their independence and intelligence, and she works hard to ensure that the mathematics presented to students is relevant and meaningful. This is evident in her active involvement in the way mathematics is taught in her school: she works closely with teachers both in and out of the classroom. It is also evident in her work on mathematics curriculum documents. She helped author the original Western Canadian Protocol for Mathematics, and she provided extensive and ongoing feedback for the new Western and Northern Canadian Protocol.

It is Sandra's unique way of being with people and sharing her passion for mathematics that makes her such an influential and inspiring mathematics educator, fully deserving of the title of Mathematics Educator of the Year.

The Right Angle: Report from Alberta Education

In the last instalment of the "Right Angle," we examined the curriculum development cycle. The K–9 mathematics program is currently in the optional implementation phase. Provincial implementation of K, 1, 4 and 7 begins in September 2008.

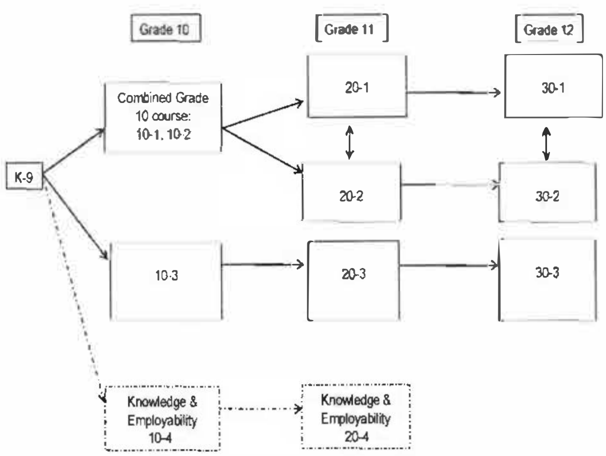
	September 2007	September 2008	September 2009	September 2010
Optional Implementation	K, 1, 4, 7	2, 5, 8	3, 6, 9	—
Provincial Implementation	—	K, 1, 4, 7	2, 5, 8	3, 6, 9

Implementation of the revised program has been supported through workshops offered by the Alberta Regional Professional Development Consortia (ARPDC) and through a summer institute offered by Alberta Education. The 2008 summer institute will focus on the optional implementation of Grades 2, 5 and 8 and on the provincial implementation of K, 1, 4 and 7. We anticipate that we will again have more than 300 teachers participating in this institute.

The high school mathematics program is currently in the final development phase. Along with our Western and Northern Canadian Protocol (WNCP) partners, we have revised the consultation draft of the Common Curriculum Framework (CCF) based on feedback from our stakeholders. The CCF was finalized and posted on the WNCP website in January 2008. There are three course sequences in the CCF: Apprenticeship and Workplace Mathematics, Foundations of Mathematics, and Pre-Calculus. Alberta will use two of the three sequences but will replace Foundations of Mathematics with its own Mathematics 20-2 and 30-2. The Alberta Mathematics 20-2 and 30-2 courses have been approved and are posted on the Alberta Education website.

The structure of the Alberta senior high school mathematics program will be as follows:

Structure of Alberta Senior High School Mathematics



Provincial implementation of the Grade 10 courses is scheduled for September 2010, followed by the Grade 11 courses in 2011 and the Grade 12 courses in 2012. (Note: Knowledge and Employability Mathematics 10-4 and 20-4 were provincially implemented in September 2006.) It is important to note that there is no optional implementation year for the senior high school mathematics programs. Due to the significantly reduced number of outcomes at all grade levels, students who study the new Grade 9 program in its optional year will be at a disadvantage in taking one of the current Grade 10 programs. To ensure that this does not occur, provincial implementation of Grade 9 and Grade 10 will occur in the same year.

There have been initial meetings with publishers to answer questions about the program of studies, its rationale and philosophy, and resource needs. English and French resources will be available for implementation in 2010.

Consultation with postsecondary institutions continues. We anticipate greater acceptance of the Mathematics 20-2 and 30-2 course sequence for postsecondary programs than the current applied mathematics program receives. Institutions have been asked to clarify their positions well in advance of 2013, when the first students will graduate from the new program.

Alberta Education would like to thank all the stakeholders who have provided input on the programs of study. Your input ensures strong programs that will meet the needs of our students.

Kathy McCabe

Letter to the Editor

Dear Editor

I am aware that many of the teachers in your province are in the process of selecting textbooks to use when implementing the new curriculum. I have two issues with current textbooks that I feel teachers should be aware of: the eternal textbook error and the eternal textbook omission. I've described each of these below.

The Eternal Textbook Error

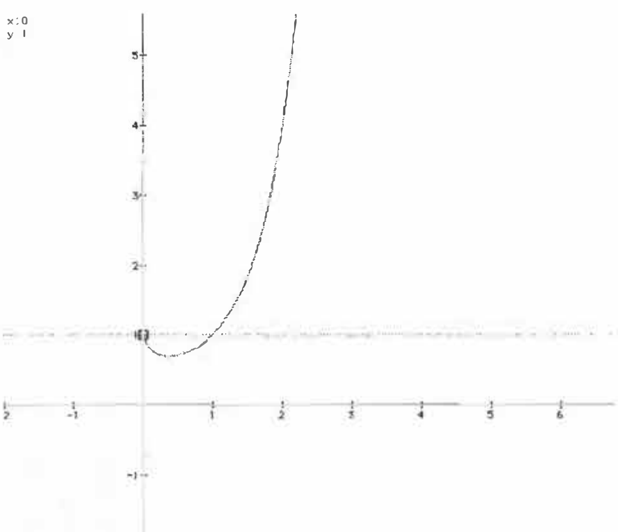
This statement, or a facsimile thereof, has appeared in every algebra textbook I have ever seen or used:

For any real number a, $a \neq 0, a^0 = 1$ (Any nonzero number raised to the zero power is 1.)

Consider the following function:

$$y = x^x$$
$$Ln\, y = x Ln\, x$$
$$Limit\ as\ x \Rightarrow 0,\ Ln\, y = 0$$
$$y = 1, \quad \therefore 0^0 = 1$$
$$Calculator\ test:\ 0.000001^{0.000001} = 0.99999$$

While the statement is a fundamental error that may have minor consequences, it does not belong in a textbook on algebra. A graph of the function is shown below.



The Eternal Textbook Omission

It is a mystery why anyone would write a textbook on algebra and routinely fail to define the subject. So what is algebra? Algebra is the language of higher mathematics, but it is not just another language. Algebra is an abstract symbolic language plus reasoning; it is a language with the rules of logic built in. The structure of a language with the embedded rules of logic has the form *assumption–deduction–conclusion*. An equation in algebra is equivalent to a sentence in a written language that has a quantitative connotation; such sentences can be translated into equations. For example, the statement "John is five years older than Mary," translates into $J = 5 + M$, where the literals, J and M, are John and Mary's ages, the verb *is* translates to an equals sign (=), and the words *five years older* translate to 5 and the addition operator (5 +). Algebra is a tool for reasoning in that it enables the connection of one statement to another. The following statement provides the additional information needed to proceed: Two years ago, John was twice as old as Mary. Both statements can now be connected through a process of reformulation, and the unknown ages can be deduced from the one-to-one matching of terms.

First statement: $J = 5 + M >>$
$J - 2 = 5 + M - 2 + 2 - 2 >>$
$J - 2 = 7 + M - 4$

Second statement: $J - 2 = 2(M - 2) >>$
$J - 2 = M - 2 + M - 2 >>$
$J - 2 = M + M - 4$

The above method does not solve for the variable or use any abstract symbolic logic, but reformulates and compares the equations to deduce that Mary's age is 7, and John is 5 + 7 or 12. Two years ago John was 10 and Mary was 5; that is, John was twice as old as Mary.

The language analogy requires the following definitions: the set of real numbers is the *nouns* of the language. The set is infinite and unique in the sense that it is completely devoid of duplicate numbers. The set contains two numbers, 0 and 1, that have special properties that enable the process of reformulation.

The difficult, if not impossible, task of trying to define the numbers can be avoided by postulating that the numbers, whatever they are, satisfy certain basic assumptions or laws. These assumptions or laws are the foundation of algebra and are therefore the *grammar and syntax* of the language. Finally, the operators, $+, -, \times, \div, =, \sqrt{}$ and so on, tell us what action to take, so they play the role of *verbs* of the language.

Algebra is a human invention that deals with quantity (matter), space and time; like all languages, it is composed of concepts that attempt to capture the essence of reality. A concept is the basic element of thought; consequently, concepts such as the equator, the North Pole, the number 5 or any number do not exist in the physical reality. The most useful application of algebra is in the fundamental laws of nature. These laws can and must be expressed in abstract form so that the field of applicability is entirely unrestricted. In *The Mathematical Principles of Natural Philosophy*, by Isaac Newton, published by Cambridge University in 1686, the second law of motion is expressed in the form of analytical description and without any mathematical formulation: "The alteration of the quantity of motion is ever proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed." The same law in algebraic form: $f = \frac{d}{dt}(mv)$.

Science is analytical description—it is not mathematics; mathematics is a more precise and concise way of stating a principle. The equation states that the impressed force *(f)* is (=) ever proportional to the time rate of change (d/dt) in the linear momentum (mv). Newton's definition of momentum: "The quantity of motion is the measure of the same, arising from the velocity and quantity of matter conjunctly"; hence, the statement of the law and the algebraic equation have exactly the same meaning. Newton knew how to differentiate *(d/dt),* having invented fluxions (calculus) before his 25th birthday. Newton's book is completely devoid of his mathematical invention because he wanted people to understand what he wrote. Historically, this was the first book on physics that was correct; it was the catalyst for the Industrial Revolution.

I hope your readers will consider textbook omissions and errors when making important decisions about the textbooks they will use.

William G Mandras

Choosing Resources

Janis Kristjansson

The new *Program of Studies for Mathematics K–9* was signed by the Minister of Education, in April 2007. Optional implementation for K and Grades 1, 4, and 7 began in September 2007, with mandatory implementation to follow a year later. The program of studies is available on the government website at www.education.gov.ab.ca/k_12/curriculum/bysubject/math/Kto9Math.pdf (accessed April 15, 2008).

The new program of studies is significantly different from the one that preceded it. Some estimates say that more than 50 per cent of the program is new in content, grade assignment or both. This means that resources that matched the old program do not match the new one.

Successful implementation of the new mathematics curriculum depends on teacher knowledge and strong resources that support appropriate teaching of the curriculum outcomes, concepts and skills. Teacher knowledge should be addressed through widely available professional development and through support provided in the authorized resources.

During the past few years many schools have purchased unauthorized resources that do not cover all the objectives of the program of studies and that, therefore, may have an inappropriate approach. In addition, they do not provide the required context, problem solving and support for teacher learning.

Soon schools and school districts will be choosing new resources for the new curriculum. Many of you will be asked for your input. What are the criteria that you should use in making your decision?

The first and most important criterion is to select a WNCP- and Alberta-authorized resource. These resources have been assessed as being a perfect match to the program of studies. In the Learning Resources Centre catalogue, these resources have an Alberta flag and WNCP symbol in the provincial status column. They also have a publication date of 2007 or later. One of the authorized resources also has a warning that it does not adequately address problem solving. Schools would be ill-advised to select such a limited resource.

When you have narrowed down your choices to authorized resources, what questions should you ask so that you can make a final decision? An excellent resource not only covers curriculum objectives but also promotes understanding and a positive attitude to mathematics. It includes contexts that are relevant to young learners and problems that are engaging. The following questions, taken from the MCATA brochure on mathematical literacy, may help you make your final choice.

Do the textbooks and other resources

- ❑ use models and demonstrations that support conceptual understanding?
- ❑ model multiple solution paths for both students and teachers?
- ❑ require the justification of answers and processes?
- ❑ require students to understand concepts and explain their thinking in order to do well on assessment tasks?
- ❑ suggest ways to have students share their reasoning and communicate their work in different ways?
- ❑ provide opportunities for students to work on problems that are engaging and interesting to them?
- ❑ include activities that require the critiquing of real-world mathematical information?
- ❑ include mathematical situations and tasks that would be encountered in everyday life?
- ❑ provide explicit suggestions for links to other subject areas?
- ❑ provide support for teachers to help students to learn to think rather than focus on procedures?
- ❑ suggest explicit ways for teachers to build students' confidence in mathematics?
- ❑ provide tasks based on the diverse ways that students learn mathematics?
- ❑ contain tasks based on sound and significant mathematics?
- ❑ provide practice in context such as investigating significant problems, exploring mathematical patterns or developing game strategies?
- ❑ use technology in ways that expand what the student is able to understand and do?

If the resources that you are considering meet all or most of these criteria, you can be confident that they will support you and your students as you work together on the new program of studies.

A Conversation with Dr Jorgensen

Dr Arthur Jorgensen has greatly influenced mathematics education in our province. He has been a principal and teacher in various schools in northern Alberta, an instructor at Grande Prairie College and the University of Lethbridge, a consultant in Jamaica, a Justice of the Peace, and an assistant superintendent. He served on the MCATA executive as director, secretary, vice-president, journal editor and newsletter editor, and most recently has been a contributor to delta-K. *MCATA's annual Dr Arthur Jorgensen Chair Award is created to honour his work in education and his passion for mathematics. I wanted to know what he was thinking about current educational issues in mathematics. Here are excerpts from our conversation.*

—Gladys Sterenberg

G: Tell me a little bit about who you are and your background.

Dr Jorgensen: I've really been interested in mathematics ever since I was just a child. In elementary school the teacher and I got along very well. I often had ideas, and I really supported her. Then when I got into high school, I often helped many of the students with their math. And then of course I got into university and I took some math there. As far as training is concerned, I got my BEd, which was general; I got a bachelor of arts in psychology and I got my master's in education, which dealt primarily with administration. And then my doctorate in curriculum and instruction in mathematics.

I have really, really been concerned about how children are taught mathematics. I don't think, historically, children have been taught mathematics well. They are so turned off because the teachers themselves are turned off. When people say, "Well, I teach mathematics," that's a disaster. You have to teach *children* mathematics and when you are teaching children, you've got to realize they are individuals. They don't all learn at the same pace, and they don't all learn in the same way. And as a result you have to treat them as individuals.

Mathematics should be enjoyable for children and for anybody who's taking it. Some people would say to me, "It's fun," and I'd say, "No, I wouldn't say it's fun, I'd say it's enjoyable." I don't consider it fun but I consider it enjoyable, and that's the way it should be for children. But that doesn't happen sometimes. I think of giving a test to a little boy in Grade 3. I tell him on Friday that on Monday there will be a test. He knows he's going to fail the test, his friend knows he's going to fail the test, his teacher knows he's going to fail the test, his parents know he's going to fail the test. And what kind of weekend does he have? I myself know that if I have a test tomorrow, I won't sleep very well tonight. And the same with this little boy; he's not going to have a very good weekend. So what really was the point of the test? And if this little boy had worked hard, it's like running a race. Somebody will be first and somebody will be last. And the one that was last maybe worked harder than the one that was first. And the same thing applies to mathematics. It really disturbs me. I know of a professor. He knew some math but he didn't have any teaching skills, and students had taken his course and they were crying. I wish I could have helped them in the classroom.

G: Talk a little bit about how you envisioned a classroom that would be different from that.

Dr Jorgensen: First of all, I get to know my students right from the start. From Grade 1, 2, 3, I would want them to become involved with mathematics. What we do now is get them to regurgitate answers. 6×56. All we want is an answer. ***I*** want that child to actually get involved with 6×56—I can teach a dog 6×56. Let's look at something like $2 + 3$. What's $2 + 3$?

G: Five.

Dr Jorgensen: Is that the only answer that you'll accept?

G: What would you suggest?

Dr Jorgensen: I say that there are many answers to that question: $3 + 3 = 6$, and $6 - 1 = 1+1+1+1+1$. You get all kinds of answers. Something like $11 + 2$ is 13. But what if I say $11 + 2$ is 1? And we do it every day on a clock. Everybody knows 11 and 2 is 13. But 1 and 1 can be 0 if I'm working in base 2. I think what's important is that children get an opportunity to see how math works in all these different ways. And I think then they will enjoy mathematics.

G: So, how can we work with teachers?

Dr Jorgensen: I think that teachers should have some good workshops on how to teach children mathematics. When I was in Jamaica, I had teachers who were able to run workshops in mathematics, good workshops, and feel confident in doing it. But we've got to spend more time on teaching teachers how to teach children. People think there's just one answer to mathematics. There are all kinds of answers to mathematics. Let's get our teachers involved in creativity and get them involved with things.

Get the teachers involved with things and show them different ways of doing things. And when you ask our teachers why they did something, they might not know. All they know is that it works. Well, show them *why* something works and show them other ways to make it work. It would make more sense to me to teach children that way. We want our teachers to enjoy the process because many teachers today do not enjoy teaching kids mathematics.

G: You said earlier that you've enjoyed math for a very long time. How did you grow to enjoy it?

Dr Jorgensen: I guess we've all got our areas of interest, but right from the start I could work with numbers, from the time I was very young. And a teacher I had in school would often ask me, "How did you do this?" So I showed her how I would do things. When I got into high school, I often helped other students.

G: Tell me what you would do with teachers that don't have that natural ability or that natural interest. You said you would get them involved, but often they really struggle with their own confidence and with their own abilities and past experiences.

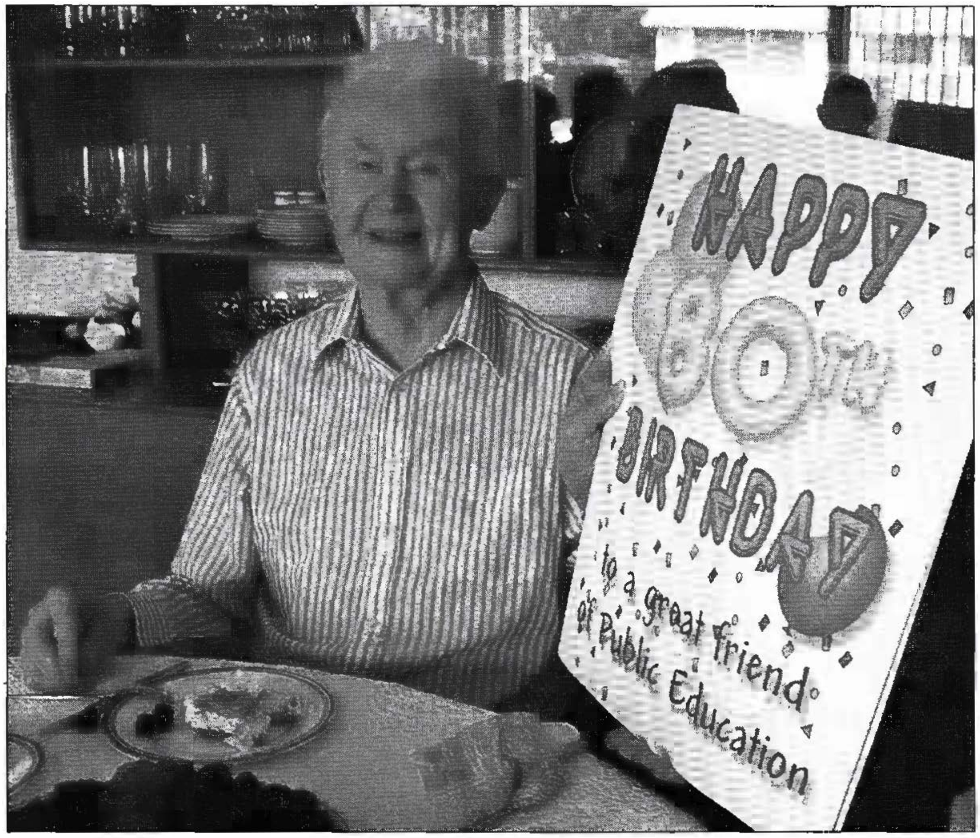
Dr Jorgensen: I don't think all teachers should be expected to teach children mathematics any more than they expect to teach them art or physical education, for example. I think we should have teachers that teach children mathematics who have a good understanding of it. I think children should adjust. Often one teacher is expected to teach children all the subjects, and I disagree with that because there are some teachers who will not be able to teach children mathematics.

G: So, you are kind of making an argument that we should have specialists in our elementary schools.

Dr Jorgensen: Yes, I am. Absolutely. Who will teach piano lessons? You expect teachers in school to do this but not all teachers can play the piano. If I'm having problems with my heart, I go to a heart specialist, and if I'm having problems with my brain, I go to a brain specialist. These doctors don't know everything about our health, and we don't expect them to. And I think we can do the same thing for children in school. I think we should have people who are really interested in mathematics teaching math.

G: How did you become involved with MCATA?

Dr Jorgensen: I became involved with MCATA, I guess 45 years ago. I felt this council made a difference. And I guess I kept bumping into some very interesting people in MCATA. I was hoping that I could have some influence on how to teach children mathematics.



Textbooks in Mathematics Learning: The Potential for Misconceptions

Ann Kajander and Miroslav Lovric

Introduction

As textbook authors ourselves[1] (for example, Kajander 2007; Lovric 2007a) we have found ourselves face to face with the tension between the historical use of the text as the mathematical authority and the notions of mathematics reform as described in the *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics [NCTM] 2000), which position the learners as prominent participants in their own learning. "Rather than the textbook and the teacher acting as major sources of authority, this intended curriculum encourages students to rely on their own mathematical reasoning and evidence when discussing mathematical solutions" (Herbel-Eisenmann 2007, 345).

As we began to critically examine sample secondary and postsecondary mathematics texts, we saw evidence of textbook formats that might have been intended to simplify the learning process for students. In our examination we found many examples of such attempts that, in fact, potentially introduced mathematical misconceptions. This article summarizes some of the types of issues we have found related to mathematical exposition, and suggests areas requiring attention in textbook writing, design and use.

Background

Both older and more recent research seems to highly privilege the role of the textbook in classroom mathematics learning. McKnight, Crosswhite and Dossey (1987) reported that more than 95 per cent of Grade 12 teachers indicated that the textbook was their most commonly used resource, and recent research in Ontario indicates that this situation has changed little (Kajander and Mason, forthcoming). More generally, "Commercially published, traditional textbooks dominate mathematics curriculum materials in US classrooms and to a great extent determine teaching practices" (Clements 2007, 55). Yet even with all this emphasis on textbooks in learning, we are becoming convinced that research about the textbooks themselves, and particularly the implications for mathematical content interpretation they impart, is modest at best.

Some effort has been put into content analysis and exploring the ways in which textbooks are used in classrooms and beyond (for example, Love and Pimm 1996; McCrory 2006). Recent work (Herbel-Eisenmann and Wagner 2007; Herbel-Eisenmann 2007) has examined how the textbook positions the learner, particularly with respect to discourse. However, market research, rather than research based on mathematics education sources, is usually used to determine textbook content and approach (Clements 2007).

Project 2061 is a long-term project, aimed at evaluating resources in science and mathematics, supported by the American Association for the Advancement of Science. According to one of their studies, "the majority of textbooks used for algebra ... have some potential to help students learn, but they also have serious weaknesses" (Project 2061 2000). More than half of the 12 textbooks evaluated by the project team were considered adequate, but none were rated highly; three were rated as highly inadequate for student learning. According to the Project 2061 findings, authors of textbooks generally ignore the research on how students acquire new mathematical ideas and concepts as described in the *Standards* (NCTM 2000).

Our work is grounded in the theory of *conceptual change* (Davis 2001; Biza, Souyoul and Zachariades 2005), which provides a framework for the study of potential student misconceptions related to learning from textbooks. The theory describes learning processes of adults as well as children and hence is appropriate in addressing high school and university students.

The Study

We have begun to take a closer look at textbooks commonly used at the secondary and first-year-university levels, particularly in Ontario, to determine how mathematics textbooks might contribute to the development of students' conceptions and misconceptions about mathematics, particularly with respect to the development of conceptual understanding. We begin by providing our current working model, which may be helpful to others wishing to undertake critical mathematical examinations of texts, and follow this with an example.

Our Working Model

As we worked through a number of textbook examples, we found issues that appeared to fall into a number of somewhat overlapping groupings. We have attempted to make sense of these by offering the following framework for such analyses, which has emerged from our work thus far. Roughly speaking, these groupings are

1. the use of colloquial, reader-friendly language beyond an intuitive introduction, and in places where mathematical precision was warranted;
2. incorrect generalizations, often taken out of context;
3. diagrams as sources of misconceptions (sometimes supporting oversimplified definitions);
4. oversimplification, often leading to inaccuracy;
5. discussion of concepts not yet properly defined; and
6. design issues such as summary boxes.

We are continuing to use and refine this rough framework as we examine more sources.

An Example

As one of many examples we have encountered, consider the concept of the tangent line to the graph of the function $y = f(x)$. From our own experience, common ideas held by many students entering first-year university include incomplete or inaccurate conceptions, such as "the tangent is the line that touches the graph of $y = f(x)$ at only one point," or "a tangent line cannot cross the graph of $y = f(x)$." Accordingly, we began exploring how such conceptions might arise from previously studied written sources. When we examined a Grade 12 calculus textbook currently in use in some Ontario schools, we easily found evidence that could support (or at best fail to correct and clarify) such misconceptions. For example, in one source, an initial explanation near the beginning of the chapter on tangents states clearly (with accompanying diagrams), "In the graphs of the circle and the parabola, a tangent line touches exactly one point of the graph, *P*. For other curves, such as the one in the third diagram [an example of a tangent that also crosses the curve at two other points] a tangent line touches the graph at the point of tangency, *P*, but may pass through other points on the graph as well" (Kirkpatrick et al 2002, 183).

While this explanation is thorough, a subsequent coloured summary box contains the incomplete definition of a tangent as follows: "A tangent is a line that touches exactly one point on the graph of a relation" (Kirkpatrick et al 2002, 190).

In fact, this oversimplified statement is not true in all cases, and emphasizing it in a summary box may increase the likelihood of its being the definition remembered by the student, rather than the concept being deeply understood. This oversimplification is an example of Issue 4 in our working model, and is further emphasized by the design issue (Issue 6 in the model) of placing the text in a coloured summary box on the page. While the summary is meant to make things easier for students, highlighting such oversimplifications, *which are not always true*, introduces the potential for incomplete understanding and misconceptions. In a traditional classroom environment, definitions are often presented by the teacher rather than developed by the students themselves as meaning is made. If the definition used by a teacher is the page 190 summary version on its own, and students are not encouraged to investigate the concept more fully with suitable prompts and questions, students will likely be left with an incomplete understanding. These and other examples made it painfully obvious to us that students' common misconceptions in first-year calculus may often have their roots in previously studied materials.

Similarly, standard university calculus texts contain numerous illustrations of tangent lines. However, in a majority of cases, the tangent is shown in the generic position where it touches the curve at one point and does not cross it. The concept of *touching* might be further suggested in examples where students are given the graph of a function $f(x)$ and are asked to sketch the graph of its derivative $f'(x)$. Although the tangent is defined as the limit of secant lines, these examples do not attempt to encourage drawing the tangent or thinking about it (either in illustration or in accompanying text) as limiting the position of secant lines.

Continuing with this example, to address this potential misconception, a textbook (or an instructor) could ask students to create illustrations that show relationships between curves and lines and identify which are (or are not) tangents (Lovric 2007b). These illustrations should include cases such as a tangent that crosses the graph at the point of tangency; a tangent line that touches the graph at more than one point; a line that touches the graph, does not cross it, but is not a tangent (cusp); and so on. By analyzing such situations, students could gain a more accurate understanding.

Discussion

Given that textbooks appear to remain the resource of choice for many teachers, we believe that more attention needs to be focused on the quality of their content. To date, "we know little about the impact of curriculum materials on the ... relationships developed between these readers and the curriculum materials themselves ... [thus] we must first examine the relationships that the textbook materials themselves encourage" (Herbel-Eisenmann 2007, 345–46).

In our work to date, we have been struck by how the issues we identified as problematic for learning from textbooks—such as generalizations that were incomplete and that lacked connections to intuitive understanding, ideas that were separated from connections to context, and incomplete summaries placed in specially designed coloured boxes implying that they were to be remembered—echoed a traditional pedagogy. On the other hand, the treatments that we found more effective—based on a gradual shift from informal language, good visuals and identification of misconceptions to more formal, rigorous and precise generalizations—echoed a more reform-based philosophy as defined in the *Standards* (NCTM 2000).

We believe that as students move through the secondary curriculum and into the postsecondary realm, they might obtain deeper and more accurate conceptual understandings of fundamental concepts if more attention were paid to textbooks that support rather than undermine such growth. Rather than criticizing the predominant use of textbooks by teachers, better success might be achieved by actively ensuring the quality of these texts in promoting deep and accurate student understanding.

Note

1. Ann Kajander has written books of activities for elementary and early secondary students, while Miroslav Lovric is the author of undergraduate textbooks.

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Listening for Their Voices: A Study of Intermediate Students at Risk in Mathematics

Ann Kajander and Carly Zuke

Introduction and Rationale

During the 2005/06 school year, Grades 7–10 teachers at an Ontario school board were invited to participate in voluntary professional learning groups for mathematics, which met for a half-day a month (Kajander and Mason, forthcoming). During one of these meetings the topic of students at risk arose, and teachers expressed the desire for a better understanding of the needs of such students. This teacher interest spearheaded our work, and a study was planned to take place during the 2006/07 academic year. This article reports on the outcomes of that work.

The authors of this article (Ann Kajander, a mathematics educator and experienced classroom teacher, and Carly Zuke, a graduate student and experienced tutor of students at risk) comprised the research team, and planned the study with significant participant teacher input. Our research questions were as follows:

1. What does the literature tell us about the needs of students at risk in mathematics, and about best practices in teaching them?
2. What are at-risk students' apparent needs?
3. What are the perceptions of teachers regarding these students' needs?
4. What subsequent classroom practices do teachers adopt with such students?

Literature Review

While space does not permit a full discussion,[1] the characteristics and needs of students at risk are well documented in the literature. The success of students at risk may relate to their motivation, which in turn potentially influences their behaviour (Hannula 2006). Hence, it is beneficial for teachers to understand students' motives if they are to fully understand their actions (Hannula 2006). The often-unsatisfactory levels of participation of at-risk students may be related to their needs (Sullivan, Tobias and McDonough 2006).

Intermediate students in particular have the need for identity, independence and social acceptance (Sullivan, Tobias and McDonough 2006). If the teaching methodology is one in which much routine and rote learning takes place, the need for independence cannot be met. In classrooms where teachers do not encourage communication and provide interactive activities, students cannot fulfill their need for socialization or develop a sense of acceptance (Hannula 2006).

The success of mathematics students can also be related to how much control they have over their own learning and the mathematical identity they are able to formulate (Sullivan, Tobias and McDonough 2006). This is important, because the attitude of a mathematics student is so influential in determining whether or not he or she will succeed; this is especially true for those who have been placed at risk (Sullivan, Tobias and McDonough 2006).

If students feel that the curriculum does not meet their needs, they tend not to participate (Daniels and Arapostathis 2005). Students who are unwilling to participate in learning mathematics often have the ability to be successful when offered a more relevant curriculum with less focus on extrinsic rewards (Daniels and Arapostathis 2005).

At-risk students have typically been subjected to previous negative school experiences, and such experiences are likely to affect their present and future learning of mathematics (McFeetors and Mason 2005). The negative experience has the potential to begin a cycle that is difficult to break (Marchesi 1998) because lack of confidence may be reinforced by continued poor performance, so the perception becomes self-sustaining.

Intermediate students are experiencing changing needs along with their concerns about social acceptance and independence (Fleener, Westbrook and Rogers 1995). Students will benefit the most from hands-on, active learning where they are free to explore and manipulate objects while solving problems they see as relevant (Fleener, Westbrook and Rogers

1995; Van de Walle and Folk 2005). Because socialization has become very important to students, they need to learn in a social environment, communicating with other students about mathematics to increase their own learning (Fleener, Westbrook and Rogers 1995). It seems reasonable that social needs can be met better in small group environments typical of the reform-based approach.

Much research indicates that students at risk will not benefit from rote learning or procedural practice alone (Fleener, Westbrook and Rogers 1995; Huhn, Huhn and Lamb 2006; Van de Walle and Folk 2005). Instead, their successful learning is dependent upon manipulating concrete objects, exploration and active problem solving (Fleener, Westbrook and Rogers 1995; National Council of Teachers of Mathematics [NCTM] 2000). Intermediate students in particular need to be actively involved in their learning, and be provided with many hands-on, relevant and engaging learning experiences to better support the development and retention of knowledge (NCTM 2000). The content and associated practice work not only need to be appealing and mathematically rich, but also must be connected with the student's real world. At-risk students may find learning only basic fundamental skills to be boring; students are more likely to practise incorrect methods with this type of rote learning, especially if they are working individually (Woodward and Brown 2006). Thus, more practice and volume alone do *not* guarantee success in mathematics (Woodward and Brown 2006).

To effectively learn in a group environment, students need some common ground to refer to and must be able to relate to one another socially (Wood, Williams and McNeal 2006). This implies that the teacher should allow some time at the beginning of the school year for social connections to be made through group activities and games. Acceptable ways of interaction and means to support or refute mathematical arguments must be developed and supported. In reform-based classroom environments, students are typically required to support their answers with an oral presentation defending the strategy they chose, which underscores the importance of being able to interact effectively in a social setting (Wood, Williams and McNeal 2006). Ignoring the development of such "social math norms" may seriously impair the success of an isolated reform-based lesson. The development of group and problem-solving skills may take several months. Students are not able to switch back and forth between a student-centred and a teacher-centred classroom; the transition to a student-centred learning environment must be gradual and consistent (Huhn, Huhn and Lamb 2006).

Hence, based on the literature, it would be fair to conclude that traditional practices do not provide strong support for the learning of students at risk. Practices that support active, interactive learning in environments that are interesting for students and support their needs for social contact, independence and self-concept development are beneficial. The support for such learning environments can be a considerable challenge for classroom teachers, especially those trained in more traditional methodologies.

The Study

We wanted to examine students at risk in detail, using a case-study approach to gain as much insight about them as possible, but we also wanted to be sure that our study examined the issues more broadly. Hence we adopted an intensive case-study approach, supported by a written survey of a broader sample of more than 60 teachers. Teachers in the professional learning group that inspired the study opened their classrooms to us, and an initial cohort of 15 students in four different classrooms of Grades 7–9 were chosen for study based on teacher recommendation. Six of these students were studied in depth, and were observed for their entire mathematics class three times a week for four months. As well, a larger sample of teachers received a written survey that contained questions about their perceptions of students at risk and the choices they made in teaching such students.

Details of the case studies and survey analysis may be found in the full research report;[2] space here permits only a summary of our results.

Results and Discussion

The case-study approach adopted with the students proved particularly revealing. The classroom researcher (Zuke) worked individually with the study students, supporting them, asking them questions and giving them extra help. As the research progressed, the students appeared to trust her more and more, and opened up to her to a significant degree. Although both of us had worked as teachers with at-risk students in classrooms in the past, we could not help but be surprised at the level of insecurity, low self-esteem and overall poor self-concept we found in all the case-study students as we got to know them. While individual issues surfaced with some students (poor reading ability, home issues such as parental health or drug dependency issues, a great number of different schools attended, or attendance issues), all of the students we studied seemed uninterested in the material, and seemed to feel that they wouldn't be able to do it anyway, even if they tried. Most students, in fact, were highly unmotivated and almost completely disengaged during mathematics. Most avoided asking the teacher questions at all costs, even when she came by to ask them if they had any. But after the first few weeks of the study, they would turn to the researcher and tentatively ask her for help. While we knew that self-esteem was an issue for these students, neither of us realized the overwhelming significance it appeared to have. While we had observed other such students sitting quietly during our own math classes, we had been unaware of just how disconnected from our lessons they really were.

The teacher survey also revealed some surprises. Again, a more formal statistical analysis is available in the research report cited in Note 2. Briefly, however, teachers seemed well aware of the issues for students at risk, and generally cited the same issues related to students' difficulties as outlined in the literature and as observed in our study. These included motivation and behaviour issues, attendance, home issues, reading issues and so forth. However, when teachers were asked to describe how they worked with students at risk in their classrooms, significant differences from the best practices described in the literature emerged, consistent with the case-study classroom teacher observations.

While we did observe a few attempts by the teachers of the case-study students to use alternative lessons, these tended to be occasional, with no prior development of problem-solving skills or group processes. As a result, the teachers became discouraged, and did not try hands-on lessons again.

Significantly, the survey data showed that the teachers in the sample used traditional learning strategies to help students at risk. Teacher-directed instruction was the most favoured approach for teaching a student at risk, and more than 50 per cent of teachers reported using teacher-directed instruction more for students at risk than for other students. Fewer than 5 per cent of teachers reported using direct instruction less for at-risk students. Extra help during seatwork was selected by 82 per cent of the respondents as "usually" used to help a student at risk in mathematics, and the low variance found for this selection indicated general agreement on this strategy. Also, most teachers reported that they used rich tasks and projects either less than or to the same degree as they would for a student who is not at risk. The vast majority—more than 70 per cent—reported using tasks less, and fewer than 5 per cent of teachers reported using rich tasks or projects more frequently for students at risk.

The significant disconnect between best practices as described in the literature and teachers' everyday realities is a significant area of concern, and points to the need for considerable professional development. When teachers see their initial attempts at alternative lessons as unsuccessful, they tend to fall back on those practices that are most comfortable and with which they are experienced. Clearly, support is needed to develop the kinds of classroom practices and environments described in the literature, especially since initiating such practices with deeply mathematically disengaged students is a significant challenge for any teacher. Based on our work in this study, we have developed the fervent opinion that for some students' mathematical development, teacher training in best practices may be the only hope.

Conclusions and Recommendations

We believe that our study underscores the urgency of the situation, and provides evidence that the status quo is simply not working for many students. Hence we end by making some direct and practical recommendations for individual teachers working with such students, even if professional development is not immediately available. These recommendations are based on examples of actual classroom practices (Kajander 2002).

Many at-risk students have become so because of particular procedural weaknesses; thus, an important first step is to remember that the visual element may be more powerful than a verbal or symbolic format for such students. Because of the typically short attention span and lack of interest of these students, any teacher-directed instruction needs to be kept to a minimum and focused, if at all possible, on examples based on highly engaging contexts and using concrete materials or real-world examples.

Beyond the briefest of teacher introductions, we believe that most mathematics must be learned in a hands-on, investigative manner. University-bound students may be willing to accept the explanation that content will be important for them in the future as a reason to learn it now; at-risk students are not. They need to be exposed to highly engaging contexts that will entice them to study right now—today. Time invested by the teacher in determining the interests of her students and then in using these interests as the basis of classroom mathematics tasks will more than pay off in the long run.

At-risk students who are not engaged will achieve little, as evidenced by the significant off-task behaviour we observed in the research. Therefore, the first step is to create tasks with which the students will engage. Often such hands-on investigative tasks are great starting points for learning by all students, and thus they can be assigned to the entire class. Beginning with a learning task is a good idea; once it is completed it can be submitted to the teacher for constructive feedback (not marks). Having the students work individually or in pairs or groups at their own pace allows the teacher to circulate and work individually with students as needed. Then, after the students finish the learning task (while these can be done in pairs or small groups, we suggest individual write-ups) and discuss their work individually with the teacher, they can move on to an assessment task. Students move on only after completing the learning task, discussing their work with the teacher and receiving detailed feedback. Using a different context but the same curriculum expectations in both tasks subsequently allows the student to improve on the assessment task. The assessment task is then graded according to a provided rubric, and can be used to provide a significant portion of the student's grade. After completing the assessment task, faster students can be assigned the regular textbook homework that gives them the procedural fluency they need; this can be followed up with a quiz at the end of the unit that assesses procedural skill. In my (Kajander) experience, at-risk students who are working slowly on the tasks (and are often absent) may not get to do much of the procedural practice. However, based on what we saw and described in the research, in many cases they weren't doing it anyway! If the grading emphasizes the assessment task, and students get a good grade there, then often, even with a poorer quiz grade, a passing mark can still be achieved. But here's the really wonderful part that I have observed with my own students (Kajander 2002). Students at risk, who begin (often for the first time in a very long time) to engage with, understand and show success (in terms of marks) on mathematical tasks, *begin to see the value of developing better procedural skills*. They may start to realize that honing those skills will allow them to work more efficiently on the tasks. They may start to actually *do* their homework!

Above all, we must remember that students who are off-task are not learning. So an important starting point is to engage these students in mathematical contexts that have the potential to interest them. Only then can we hope to better support their learning.

Notes

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1. The full research report can be found at http://noelonline.ca/depo/fdfiles/KajanderNOEL%20FINAL%20report%20APRIL%2030%202007.pdf.

2. See Note 1.

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Universals That Are Not: Learning About the Language of Mathematics Around the Dinner Table

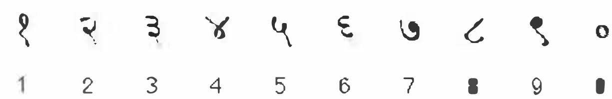
by Christine McCuaig

My revelation began this summer as I watched the bright young girl from Kuwait before me. Diligently, she completed the tasks that were asked of her. She was working on an assessment package at the Riverside Reception Centre. The centre welcomes new students and families to the Calgary Board of Education, provides information about the school system, reviews boundary maps to establish which school each child will attend and provides a basic assessment during the intake, which involves reviewing prior school records and current proficiencies. I had the privilege of working alongside the talented staff at Riverside Bungalow for four weeks last summer.

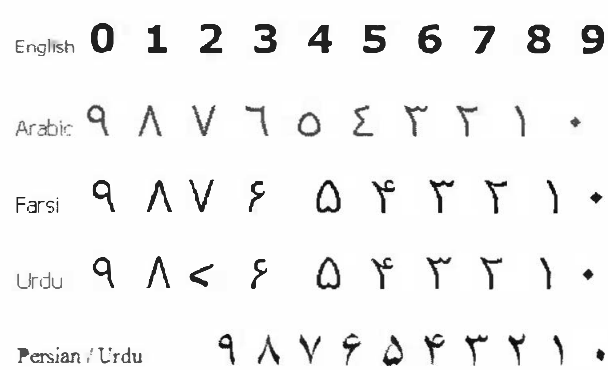
What struck me about this particular high school student was how slowly she worked through the math problems we had given her. She made errors in minor calculations that were housed within more challenging problems that she was able to understand and work on. Her report cards indicated that she was a strong student in all areas and had been educated as a bilingual student in English and Arabic. I asked her why she seemed to be struggling with the math. Her answer came as a great surprise to me. I was humbled to learn that what I considered to be a universal—the use of the number system 1, 2, 3, 4, 5, 6, 7, 8, 9, 10—was not! This young girl explained to me that in Kuwait she had been taught a different system to represent the base 10 numbers. I was amazed at this and then realized that an ESL learner may need to translate not only in reading but also in mathematics. I had always assumed that all students were united in their use of common numbers.

The other night in the dining hall here in my University of Toronto residence, I struck up a conversation with three young men at my table whose first languages were Japanese, Hindi and Persian. In the course of our conversation I brought up my surprise at learning of alternative number representations, particularly as an ESL teacher. To my delight the young men added to both my knowledge and my reflection on this topic.

In India, I was told, numbers may be taught in Hindi, especially in rural schools. The numerals would appear as follows[1]:



Another young man informed me that he had used a Persian system of numerical representation throughout school, in fact right into his university-level math courses. Below are the various number systems taught and used in the Middle East.[2] The first is our familiar number system.



To write numbers up to 10 in traditional Japanese Kanji, a system derived from Chinese is used.

I wonder, if language and vocabulary shape our perspectives and our thinking, would the use of an alternative number system influence or enhance our mathematical thinking and reasoning? The students I was sitting with wondered how many people were aware of these different number systems. We mused over the complications of a nonalphabetic language, such as Persian, that requires speakers to write words from right to left but numbers from left to right.

Learning about alternative numerals used in modern cultures has been a revelation for me and has encouraged me to reflect, as I hope you will, also.

Think about the assumptions you hold about what is universal, and contemplate how these assumptions may influence your teaching of and your interactions with students from diverse language and cultural backgrounds. You may be surprised what you learn.

Note

1. Used with permission from www.krysstal.com/writing_hindi.html.

2. The illustrations for English numerals, Arabic numerals, Farsi numerals and Urdu numerals have all been used with permission from www.microsoft.com/middleeast/arabicdev/windows/winxp/DigitsSupport.aspx. The illustration for Persian/Urdu numerals has been used with permission from www.bellaonline.com/articles/art46096.asp.

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A Radian of Pie

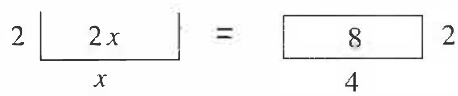
Sanja Avramovic and Renée Oddy

This fall, Sanja Avramovic and Renée Oddy were enrolled in a Pure Math 30 class at Central Memorial High School, in Calgary. Their teacher, Gerald Krabbe, introduced the radian measure and definition in class and challenged his students to "make sure you get your radian-sized piece of pumpkin pie at the Thanksgiving table!" They did just that.

Algebra Through Symmetry

William G Mandras

Algebra Through Symmetry is a visual method that was prevalent at the dawn of the mathematical sciences. At the University of Alexandria, in 310 BC, Euclid's students used symmetry in the solution of equations. In the equation $2x = 8$, each term was presumed to be an *area* and a geometric diagram was constructed. Considering Euclid's fourth axiom, "Things which coincide with one another are equal to one another," a student would conclude that equality requires that the two rectangular areas are equal; therefore, x must coincide with 4.



This article documents and reformulates the method in modern notation. Symmetry in algebra is produced when an equation is reformulated such that the form of the equation is the same on both sides of the equality. The natural concept of symmetry (human and animal form) implies one-to-one matching and is an organic part of human understanding, is ingrained very early in life, and is independent of one's intelligence or mathematical ability.

The axiomatic method relies on the concept of formal proof, assumption--deduction--conclusion, to reduce and simplify an asymmetric equation to its final symmetrical form:

$ax + b = c$ Apply the axioms to achieve final reflective symmetry.

$x = \frac{c-b}{a}$ Variable = Value, by reflective symmetry.

The visual method relies on the natural concept of symmetry. The special properties of 0 and 1 allow the reformulation of an asymmetric equation to its final symmetrical form:

$ax + b = c$ Apply the axioms to reformulate and achieve final translational symmetry.

$ax + b = a(\frac{c-b}{a}) + b$ Variable = Value, by a one-to-one matching of terms.

$ax + b = b + (\frac{c-b}{a})\,a$ Reflective symmetry is a possible reformulation, but not likely.

The visual method is the inverse of the axiomatic method in the sense that both achieve symmetry in the final equation, one through abstract symbolic logic and the other through a process of reformulation.

Define: Axiomatic Method

Apply the axioms to both sides of an equation to simplify and reduce to an equivalent equation. Solve for the variable explicitly and check the solution in the original equation.

Axiomatic Mantra: ***Whatever you do to one side of an equation you must do to the other.***

Define: Visual Method

Apply the special properties of 0 and 1 to obtain symmetry in an equation. Solve for the variable implicitly using a one-to-one matching of terms. Check any modified expression.

Visual Mantra: ***Rewrite the equation until both sides look the same.***

Linear Equations

Linear equations are particularly well suited for a solution by the visual method. The method enables the student to determine by inspection the terms required to obtain symmetry. Generally, the abstract portion of the equation is kept unmodified, and the real portion is reformulated such that the form of the equation is the same on both sides.

L1) Visual solution:

$$3x+5=11 \Rightarrow 3x+5=(11-5)+5$$

$$3x+5=3(\frac{11-5}{3})+5 \Rightarrow x=\frac{11-5}{3}, \text{ from one-to-one matching}$$

Alternative Visual: $3x+5=11 \Leftarrow$ formulate and add a symmetrical equation

$$3(\frac{11-5}{3})=3x \Rightarrow x=\frac{11-5}{3}, \text{ from one-to-one matching}$$

$$3x+11=3x+11 \Leftarrow \text{ Symmetry verifies that: } x=\frac{11-5}{3}=2$$

$3x+5=3(2)+5$ Translational symmetry—the more likely reformulation

$3x+5=5+(2)3$ Reflective symmetry—the less likely reformulation

$x=2$ Reflective symmetry––the axiomatic solution

The unique value that produces symmetry in an equation verifies that value as the solution; there is no need to check this value in the original equation. No learning takes place by evaluating the final expression, except when the variable has a tangible meaning, because that hides the terms used in the reformulation, negates the visual check of the solution, requires the tedium of arithmetic and is a source of errors.

L2) A salesperson earns a salary of \$125 per week plus a commission of \$2.40 for each item sold. Find the number of items sold if the earnings were \$528.20 in one week.

$$2.40n+125=528.20 \Rightarrow 2.40n+125=2.40(\frac{528.20-125}{2.40})+125$$

$$2.40n+125=2.40(168)+125 \quad n=168 \text{ items sold, from one-to-one matching}$$

L3) The cost to rent an automobile is \$37 per day plus \$0.21 per mile. If the final bill is \$61.15, how many miles were driven?

$$0.21m+37=61.15 \Rightarrow 0.21m+37=0.21(\frac{61.15-37}{0.21})+37$$

$$0.21m+37=0.21(115)+37 \quad m=115 \text{ miles, from one-to-one matching}$$

Systems of Linear Equations

In systems of linear equations, where the variables have a tangible meaning, the equations will be reformulated to obtain symmetry or to contain a collinear equation. Examples S1 to S6 introduce problem solving in two variables. The visual method will be used to obtain the solutions; current textbooks would use the axiomatic method exclusively.

S1) There are two supplementary angles in which one angle is 12 degrees less than 3 times the other. What is the measure of the angles?

$$x+y=180 \Rightarrow x+y=192-12 \Rightarrow x+y=4(48)-12$$

$$y=3x-12 \Rightarrow x+y=x+3x-12 \Rightarrow x+y=4x-12$$

$$x=48^\circ \text{ from one-to-one matching} \qquad y=180-48=132^\circ$$

S2) Cool Mitts, Inc, sold 20 pairs of gloves. Plain leather gloves sold for $24.95 per pair and gold-braided gloves sold for $37.50 per pair. The company took in $687.25. How many of each kind were sold?
Reformulate to contain a collinear equation with ratio = 1/24.95; 20(24.95) = 499.00

$$p+g=20 \quad \Rightarrow \quad p+g = 20$$

$$24.95p+37.50g=687.25 \quad \Rightarrow \quad 24.95p+24.95g+12.55g=499.00+188.25$$

$$g=\frac{188.25}{12.55} \quad \Rightarrow \quad g=15 \text{ gold-braided gloves}$$

$$p=20-15 \quad \Rightarrow \quad p=5 \text{ plain leather gloves}$$

S3) Antifreeze solution A is 2% methanol, and solution B is 6% methanol. Auto-Parts, Inc, wants to mix the two to get 60 litres of a solution that is 3.2% methanol. How many litres of each solution are required?
Reformulate to contain a collinear equation with ratio = 1/2%

$$A+B=60 \quad \Rightarrow \quad A+B=60$$

$$2\%A+6\%B=3.2\%(60) \quad \Rightarrow \quad 2\%A+2\%B+4\%B=2\%(60)+1.2\%(60)$$

Equality requires that $B=\frac{1.2\%(60)}{4\%}=18$ litres of 6% methanol,

and that A is the 4's complement of 1.2, $A=\frac{(4-1.2)\%(60)}{4\%}=42$ litres of 2% methanol,

or, using the first equation $A+18=60 \Rightarrow A=42$ litres of 2% methanol.

S4) A $4800 investment in two corporate bonds earns $412 in interest the first year. The bonds pay interest of 8% and 9% per annum. Find the amount invested at each rate of interest.
Reformulate to contain a collinear equation with ratio = 1/0.08; 0.08(4800) = 384

$$E+N=4800 \quad \Rightarrow \quad E+N=4800$$

$$0.08E+0.09N=412 \quad \Rightarrow \quad 0.08E+0.08N+0.01N=384+28$$

$$N=\frac{28}{0.01}=\$2800 \text{ at } 9\% \qquad \text{E}=\$4800-\$2800=\$2000 \text{ at } 8\%$$

In systems of abstract linear equations, one equation is kept unmodified while the other equations are reformulated by multiplication and the addition of zeros to obtain the matching terms. The procedure eliminates the entire unmodified equation and yields a solution to the remaining variables. In the examples, the coefficients and constants are prime numbers or fractions so that the solution is difficult by the axiomatic method.

S5) Visual solution, system of two linear equations:

$$A: 11x-13y=17 \qquad \Rightarrow \qquad 11x-13y=17$$

$$B: 7x+5y=23 \quad \Rightarrow \quad (\frac{11}{7}B): \ 11x-13y+13y+\frac{55}{7}y=17-17+\frac{253}{7}$$

$$17+(\frac{7(13)+55}{7})y=17+(\frac{7(-17)+253}{7}) \quad \Rightarrow \quad \frac{146}{7}y=\frac{134}{7} \quad \Rightarrow \quad y=\frac{134}{146} \quad [y=\frac{67}{73}]$$

$$B: 7x+5(\frac{67}{73})=23(\frac{73}{73}) \quad \Rightarrow \quad x=\frac{23(73)-5(67)}{7(73)} \qquad [x=\frac{192}{73}]$$

$$\text{Check } A: \ 11(\frac{192}{73})-13(\frac{67}{73})=17 \Rightarrow \frac{2112}{73}-\frac{871}{73}=17 \Rightarrow \frac{1241}{73}=17 \quad 17=17$$

$$\text{Check } B: \ 7(\frac{192}{73})+5(\frac{67}{73})=23 \Rightarrow \frac{1344}{73}+\frac{335}{73}=23 \Rightarrow \frac{1679}{73}=23 \quad 23=23$$

S6) Visual solution, system of two linear equations with fractions as coefficients:

$$A:\ \frac{1}{11}x-\frac{1}{13}y=17 \qquad \frac{1}{11}x-\frac{1}{13}y=17$$

$$B:\ \frac{1}{7}x+\frac{1}{5}y=23 \Rightarrow (\frac{7}{11}B):\ \frac{1}{11}x-\frac{1}{13}y+\frac{1}{13}y+\frac{7}{55}y=17-17+\frac{161}{11}$$

$$17+(\frac{55+13(7)}{13(55)})y=17+(\frac{11(-17)+161}{11}) \Rightarrow \frac{146}{13(55)}y=\frac{-26}{11} \Rightarrow y=-\frac{13(55)(26)}{146(11)} \quad [y=-\frac{845}{73}]$$

$$B:\ \frac{1}{7}x+\frac{1}{5}(-\frac{845}{73})=23(\frac{73}{73}) \Rightarrow \frac{1}{7}x-\frac{169}{73}=\frac{1679}{73} \Rightarrow x=\frac{7(1679+169)}{73} \quad [x=\frac{12936}{73}]$$

$$\text{Check } A:\ \frac{1}{11}(\frac{12936}{73})-\frac{1}{13}(-\frac{845}{73})y=17 \Rightarrow \frac{1176}{73}+\frac{65}{73}y=17 \Rightarrow \frac{1241}{73}=17 \quad 17=17$$

$$\text{Check } B:\ \frac{1}{7}(\frac{12936}{73})+\frac{1}{5}(-\frac{845}{73})y=23 \Rightarrow \frac{1848}{73}-\frac{169}{73}y=23 \Rightarrow \frac{1679}{73}=23 \quad 23=23$$

Linear Inequalities

Linear inequalities can be reformulated by keeping the abstract portion of the inequality unmodified. Add the equivalent of zeros to obtain the matching terms, and then multiply by the equivalent of one to obtain a matching coefficient. If the coefficient of the variable term is negative, the sense of the inequality must be reversed at the final step.

LI1) Visual solution, variable with positive coefficient:

$$-3\le 2x+4\le 10 \Rightarrow 4-4-3\le 2x+4\le 10+4-4 \Rightarrow -7+4\le 2x+4\le 6+4$$

$$\frac{2}{2}(-7)+4\le 2x+4\le \frac{2}{2}(6)+4 \Rightarrow 2(\frac{-7}{2})+4\le 2x+4\le 2(3)+4$$

Range of the variable: $(-3.5)\le x\le 3$ from one-to-one matching.

LI2) Visual solution, variable with negative coefficient:

$$-3\le -2x+4\le 10 \Rightarrow 4-4-3\le -2x+4\le 10+4-4 \Rightarrow -7+4\le -2x+4\le 6+4$$

$$\frac{-2}{-2}(-7)+4\le 2x+4\le \frac{-2}{-2}(6)+4 \Rightarrow -2(\frac{-7}{-2})+4\le -2x+4\le -2(-3)+4$$

Reverse sense of the inequality. Range: $(3.5)\ge x\ge -3$ from one-to-one matching.

Rational Equations

Rational equations are particularly difficult to comprehend precisely because they contain abstract fractions. The visual solution generally keeps the more difficult side of the equation unmodified and rewrites the less difficult side to obtain symmetry. In some cases partial symmetry is created, followed by division. Division generally produces remainders on both sides of an equation. The remainders must be equated repeatedly to reduce the original rational equation to a linear form.

R1) Visual solution: create partial symmetry to avoid the binomial multiplications.

$$\frac{x+3}{x+5}=\frac{2x-7}{2x+11} \Rightarrow \frac{x+5-2}{x+5}=\frac{2x+11-18}{2x+11} \Rightarrow 1+\frac{-2}{x+5}=1+\frac{-18}{2x+11} \Rightarrow \frac{-2}{x+5}(\frac{9}{9})=\frac{-18}{2x+11}$$

$9x+45=2x+11 \Leftarrow$ Formulate and add a symmetrical equation

$-7x=-7(\frac{34}{-7}) \Rightarrow x=-\frac{34}{7}$ from one-to-one matching

$2x+45=2x+45 \quad \Leftarrow$ Symmetry verifies that $x=-\frac{34}{7}$; no check is needed.

R2) Visual solution: rational equation that contains multiple fractions

$$\frac{x+5}{2}+\frac{1}{2}=2x-\frac{x-3}{8}$$

$$2x-\frac{x-3}{8}-2x+\frac{x-3}{8}+\frac{x+5+1}{2}=2x-\frac{x-3}{8} \quad \Leftarrow \text{ LHS: add } 0=\text{RHS}-\text{RHS}$$

$$2x-\frac{x-3}{8}-2x(\frac{8}{8})+\frac{x-3}{8}+\frac{x+6}{2}(\frac{4}{4})=2x-\frac{x-3}{8}$$

$$2x-\frac{x-3}{8}+\frac{-16x}{8}+\frac{x-3}{8}+\frac{4x+24}{8}=2x-\frac{x-3}{8}$$

$$2x-\frac{x-3}{8}+\frac{-11x+21}{8}=2x-\frac{x-3}{8}$$

Unmatched term (numerator) = zero; reformulate to contain an inverse coefficient.

$-11x+21=0 \Rightarrow -11x+11(\frac{21}{11})=0 \quad x=\frac{21}{11}$ from one - to - one matching.

LHS expression check: is the original expression equivalent to the final expression?

There is no need to use $x=\frac{21}{11}$ for the LHS expression check; $x=0$ will do.

Original expression: $\frac{5}{2}+\frac{1}{2}=\frac{6}{2}=3$ Final expression: $-\frac{-3}{8}+\frac{21}{8}=\frac{+3+21}{8}=\frac{24}{8}=3$

R3) Visual solution: rational expression equal to an abstract fraction

$$\frac{x-4}{3x}+\frac{x-8}{5x}=\frac{-16}{x} \quad \Rightarrow \quad \frac{x-4}{3x}+\frac{x-8}{5x}=\frac{-16}{x}(\frac{15}{15})$$

$$\frac{5}{5}(\frac{x-4}{3x})+\frac{3}{3}(\frac{x-8}{5x})=\frac{5}{5}(\frac{x-4}{3x})+\frac{3}{3}(\frac{x-8}{5x})+\frac{-240-5(x-4)-3(x-8)}{15x}$$

Unmatched term (numerator) = zero: $-240-5x+20-3x+24=0$

$-196-8x=0 \Rightarrow +8(\frac{-196}{+8})-8x=0 \Rightarrow x=-\frac{49}{2}$ from one-to-one matching

This unique value produces symmetry: $\frac{5}{5}(\frac{x-4}{3x})+\frac{3}{3}(\frac{x-8}{5x})=\frac{5}{5}(\frac{x-4}{3x})+\frac{3}{3}(\frac{x-8}{5x})$

in the final equation and thereby verifies that the solution is $x=-\frac{49}{2}$. Check the modified

expression: does final RHS = original RHS? Do not use $x=-\frac{49}{2}$ to check, $x=1$ will do.

$$\frac{5(1-4)+3(1-8)}{15(1)}+\frac{-240-5(1-4)-3(1-8)}{15(1)}=\frac{-16}{1} \quad \Rightarrow \quad \frac{-240}{15(1)}=\frac{-16}{1} \quad \Rightarrow \quad \frac{-16}{1}=\frac{-16}{1}$$

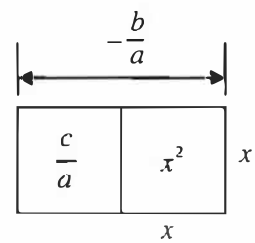
Note: in R2 and R3, the axiomatic method (multiply through by 8 and $15x$, and so on) requires 26 and 24 arithmetic operations respectively, including the check, versus 11 and 9 for the visual method.

Quadratic Equations

In quadratic equations, completing the square is currently in vogue as the only general solution of these equations. A geometric–numerical method, formulated at the dawn of the mathematical sciences, is also a general solution. A quadratic equation can be represented by a geometric diagram, which, in turn, leads to a simple numerical solution.

Generalized Geometric–Numerical Method

$$f(x)=ax^2+bx+c=0 \quad \Rightarrow \quad x^2+\frac{c}{a}=-\frac{b}{a}x \quad \Rightarrow$$



The diagram is symmetrical when the equation is a trinomial square, that is, area $\frac{c}{a}$ = area x^2.

The redundant roots are : $x=[-\frac{b}{2a},-\frac{b}{2a}]$. Assuming that every quadratic equation has these roots would produce an error, $|e^2|\geq 0$. The deviation from a trinomial square can be computed.

$$e^2=-x^2-\frac{b}{a}x-\frac{c}{a} \quad \Rightarrow \quad e^2=-(-\frac{b}{2a})^2-\frac{b}{a}(-\frac{b}{2a})-\frac{c}{a} \quad \Rightarrow \quad e^2=-\frac{b^2}{4a^2}+(\frac{b^2}{2a^2})-\frac{c}{a}$$

$$e^2=\frac{b^2}{4a^2}-\frac{c}{a} \quad \Rightarrow \quad e^2=\frac{(\frac{b}{2})^2}{a^2}-\frac{ac}{a^2} \quad \Rightarrow \quad e^2=\frac{(\frac{b}{2})^2-ac}{a^2} \qquad f(-\frac{b}{2a})=\frac{-e^2}{a}$$

Roots of a quadratic equation are $x=\frac{-\frac{b}{2}}{a}\pm\frac{\sqrt{e^2}}{a}$ and the vertex is $v=(-\frac{b}{2a},\frac{-e^2}{a})$.

Redefine e^2 as the numerator $(\frac{b}{2})^2-ac$, and express as a determinant to facilitate computation.

Example : $f(x)=3x^2-18x+10 \qquad e^2=\begin{vmatrix}-\frac{b}{2} & a \\ c & -\frac{b}{2}\end{vmatrix}=\begin{vmatrix}-\frac{-18}{2} & 3 \\ 10 & -\frac{-18}{2}\end{vmatrix}=\begin{vmatrix}9 & 3 \\ 10 & 9\end{vmatrix}=[81]-[30]=51$

Root: $x=\frac{9}{3}\pm\frac{\sqrt{51}}{3} \Rightarrow x=3\pm\frac{\sqrt{51}}{3}$ Vertex: $v=(3,\frac{-51}{3}) \Rightarrow v=(3,-17)$

The geometric–numerical method requires six arithmetic operations to compute the roots and vertex. Considering that three coordinates can be found quickly using only arithmetic, the method lends itself to graphing quadratic equations. In this example, the quadratic formula requires nine arithmetic operations to compute the roots alone. The geometric–numerical method also calls into question the need to learn the factoring procedures to solve quadratic equations.

General Quadratic Equation: $f(x)=ax^2+bx+c=0$

Symmetrical form: $ax(x+\frac{b}{a})+c=a[-\frac{b}{2a}\pm\frac{\sqrt{e^2}}{a}]([-\frac{b}{2a}\pm\frac{\sqrt{e^2}}{a}]+\frac{b}{a})+c$

Factored form: $(x+\frac{b}{2a}+\frac{\sqrt{e^2}}{a})(x+\frac{b}{2a}-\frac{\sqrt{e^2}}{a})=0$

Geometric–numerical solution of quadratic equations

$$e^2 = \begin{vmatrix} -\frac{b}{2} & a \\ c & -\frac{b}{2} \end{vmatrix}, \quad \text{Roots: } x = \frac{-\frac{b}{2}}{a} \pm \frac{\sqrt{e^2}}{a}, \quad \text{Vertex: } v = (\frac{-\frac{b}{2}}{a}, \frac{-e^2}{a})$$

Trinomial square deviation: e^2 determines the nature of the solutions.

Q 1) $9x^2 - 24x + 16 = 0 \qquad e^2 = \begin{vmatrix} 12 & 9 \\ 16 & 12 \end{vmatrix} = [144] - [144] \qquad e^2 = 0$

e^2 is zero;equation is a trinomial square. Roots: $x = \frac{4}{3} \pm 0$ Vertex: $v = (\frac{4}{3}, 0)$

Q 2) $2x^2 - 6x - 7 = 0 \qquad e^2 = \begin{vmatrix} 3 & 2 \\ -7 & 3 \end{vmatrix} = [9] - [-14] \qquad e^2 = 23$

e^2 is positive; roots are irrational. Roots: $x = \frac{3}{2} \pm \frac{\sqrt{23}}{2}$ Vertex: $v = (\frac{3}{2}, \frac{-23}{2})$

Q 3) $-3x^2 - x + 2 = 0 \quad e^2 = \begin{vmatrix} \frac{1}{2} & -3 \\ 2 & \frac{1}{2} \end{vmatrix} = [\frac{1}{4}] - [-6] \qquad e^2 = \frac{4(6)+1}{4} \qquad e^2 = \frac{25}{4}$

e^2 is a positive square; roots are rational. Roots: $x = -\frac{1}{6} \pm \frac{5}{6}$ Vertex: $v = (-\frac{1}{6}, \frac{25}{12})$

Q 4) $3x^2 - 6x + 13 = 0 \qquad e^2 = \begin{vmatrix} 3 & 3 \\ 13 & 3 \end{vmatrix} = [9] - [39] \qquad e^2 = -30$

e^2 is negative; roots are complex. Roots: $x = 1 \pm \frac{\sqrt{-30}}{3}$ Vertex: $v = (1, 10)$

or Roots: $x = 1 \pm \frac{\sqrt{30}}{3} i$

Note: the second level of abstraction, $i = \sqrt{-1}$, can be avoided by instructing the students that the square of a square root with a negative radicand is a negative number, for example, $(\sqrt{-30})^2 = -30$.

Exponential Equations

Exponential equations with variables or fractions as exponents can be reformulated to obtain symmetry by using the change-of-base formula in logarithms. Symmetrical exponential equations eliminate the need to take the logarithm of both sides of an equation.

E1) Visual solution: an equation containing a variable exponent

$$24 = 5(2)^x \quad \Rightarrow \quad 5(\frac{24}{5}) = 5(2)^x \quad \Rightarrow \quad 5(4.8) = 5(2)^x$$

$$5(2)^{\frac{\log(4.8)}{\log(2)}} = 5(2)^x \qquad 5(2)^{\frac{\mathrm{Ln}(4.8)}{\mathrm{Ln}(2)}} = 5(2)^x$$

$x = \frac{\mathrm{Log}(4.8)}{\mathrm{Log}(2)}$ or $x = \frac{\mathrm{Ln}(4.8)}{\mathrm{Ln}(2)}$ from one-to-one matching.

E2) Visual solution: an equation containing an irrational exponent

$$3x^{\pi} - 7 = 4 \quad \Rightarrow \quad 3x^{\pi} - 7 = 3(\frac{4+7}{3}) - 7 \quad \Rightarrow \quad 3x^{\pi} - 7 = 3x^{\frac{\mathrm{Ln}(\frac{11}{3})}{\mathrm{Ln}(x)}} - 7$$

$\pi = \dfrac{\mathrm{Ln}(\frac{11}{3})}{\mathrm{Ln}(x)}$ from one-to-one matching. $\mathrm{Ln}(x) = \dfrac{\mathrm{Ln}(\frac{11}{3})}{\pi} \quad \Rightarrow \quad x = 1.51221$

E3) Visual solution: an equation containing a variable exponent and base

$y = x^{x} \quad \Rightarrow \quad x^{\frac{\ln y}{\ln x}} = x^{x} \quad \Rightarrow \quad \dfrac{\ln y}{\ln x} = x$ from one-to-one matching

$\ln y = x \ln x \quad \Rightarrow$ limit as $x \Rightarrow 0$; $\ln y = 0 \quad \Rightarrow \quad y = 1 \quad \Rightarrow \quad \therefore 0^{0} = 1$

Note that this statement or a similar one appears in all of the algrebra textbooks:

For any real number a, $a \neq 0$, $a^{0} = 1$.

Anynonzero number raised to the zero power is 1.

This is the eternal textbook error as shown in example E3. While the error is minor, the statement is false and does not belong in an algebra textbook.

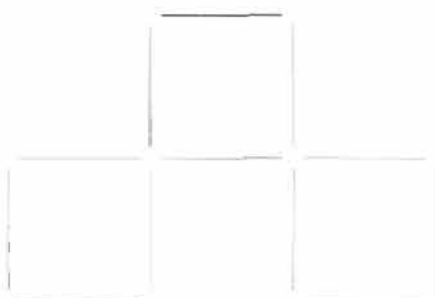
Conclusion

There is substantial experimental evidence that all human abilities are normally distributed. The centre point mean of a normal distribution implies that one-half of the student population will be below average in mathematical ability. Every student, to some degree, has a measure of mathematical ability that appears to depend on the degree to which one accumulates the mental representation of mathematical objects whose properties are reproducible. Intuition is the faculty by which one can consider or examine the mathematical objects that are stored in a mental set of neurons. When contemplating a problem, internal mathematical intuition is the faculty of browsing in one's neuron library until a new insight or connection between the objects is found. When contemplating a diagram that has a quantitative connotation, visual mathematical intuition is the faculty that enables the brain to discover the mathematical truth revealed in the diagram. The axiomatic method, the basis of which is the concept of formal proof (assumption–deduction–conclusion), has reigned too long as the exclusive basis of mathematics pedagogy, to the detriment of those students who lack the intuition necessary to comprehend the concept. Because visual learning is the dominant mode, visual mathematical intuition is also the dominant mode. This is why teaching the axiomatic and visual methods in parallel, both in the classroom and in the textbooks, would create a more powerful pedagogy that more closely corresponds to the mathematical abilities of the typical class in algebra. Such pedagogy will demonstrate that the visual method is the inverse of the axiomatic method in the sense that both achieve the unique solution, one through an axiomatic reformulation to translational symmetry and the other through an axiomatic reduction to reflective symmetry. Moreover, the visual method may be the only means available to rescue those students whose circumstances place them at high risk of failure in abstract mathematics.

Bill Mandras is a retired teacher and aerospace engineer. He has extensive experience as a college and university professor. His current interests include mathematics education and quantum physics. At age 75, he is delighted with our decision to publish his first manuscript in delta-K *and hopes that the ideas presented here will accrue to the benefit of thousands of students in the Alberta school system over time.*

A Page of Problems

Gladys Sterenberg



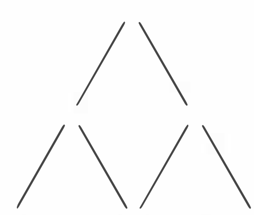
Move three toothpicks to form
five squares.

Move two toothpicks to form
four squares.

Remove four toothpicks to form
one square.

Remove four toothpicks to form
two squares.

Remove two toothpicks to form
two squares.



Remove three toothpicks to form one triangle.

Remove three toothpicks to form two triangles.

Remove four toothpicks to form two triangles.

Remove two toothpicks to form two triangles.

Use six toothpicks to form four
congruent triangles

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