A Grade 4 Adventure with Multiplication on the Chinese Abacus

Jerry Ameis

An arithmetic algorithm or procedure is a series of steps performed to obtain an answer to an arithmetic task. Currently, there is a shift towards teaching early- and middle-years students the reasoning involved in an algorithm. The National Council of Teachers of Mathematics (NCTM 2000) supports this shift: "Fluency refers to having efficient, accurate, and generalizable methods (algorithms) for computing that are based on well-understood properties and number relationships" (p 144). Why does the algorithm produce a correct result? What properties are involved?

Past practice has tended to teach multidigit multiplication in a manner that did not involve student understanding but, rather, student mimicry. The distributive property was seldom developed as the critical justification of the algorithm. This article investigates the experiences of five Grade 4 students learning a functional sense of the distributive property that was then applied to multidigit multiplication. The students extended their understanding by working with the Chinese abacus (*suan pan*) and by subsequently teaching early-years preservice teachers those concepts using the abacus. This article focuses on one aspect of the experience concerning the initial development of the distributive property, and the students' experiences teaching early-years preservice teachers multiplication on the Chinese abacus.

I had been working with a number of students on Saturday mornings, over a period of years, as a long-term research project that explored how children learn mathematics and what mathematics they can learn. These students had been having some difficulties learning mathematics in their school settings, and the Saturday sessions supplemented their school experiences. The students were now in Grade 4. During Saturday sessions, the students learned mathematics in a problem-solving climate of learning, where sense-making and meaningful contexts were important aspects of instruction.

Developing the Distributive Property

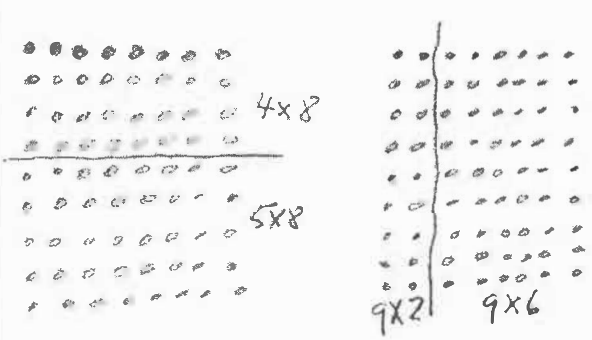
The students were presented with the problem: "Grade 4 students are about 9 years old. Suppose you are exactly 9 years old. How many days old are you?" I decided to ignore leap years in this instance, as I did not want to muddy the waters with an additional feature of the problem. Past experience with these students had indicated that it was not a wise teaching strategy to include matters that were not critical to the situation. The students were asked to represent the problem with a multiplication number sentence. With some guidance that mainly involved the number of days in a year, the students responded "9×365." The students were asked to obtain the answer by using repeated addition (with pencil and paper) or place value materials. The students chose to work with place value materials. Their thinking involved making 9 groups of 365 with the materials, trading as needed, and then writing in symbolic form the answer represented by the materials. Four of the five answers were correct. Some of the students had made 9 groups of 300, 9 groups of 60, and 9 groups of 5 in the course of their work with the place value materials. This was an indication that students recognized that the 3 in 365 was in the hundreds place, the 6 was in the tens place, and the 5 was in the ones place, and that multiplication involved groups. However, the students' ability to make groups of 9 using place value does not necessarily indicate that they understood or were consciously aware of the distributive property. That property involves understanding that one or both numbers of a product can be split, with the subproducts added to yield the final result. For example, 8×23 can be thought of as $(3 + 5) \times (15 + 8)$. This particular splitting of 8 and 23 results in the four subproducts $3 \times 15 + 3 \times 8 + 5 \times 15 + 5 \times 8$. Completing the task leads to $45 + 24 + 75 + 40 = 184$. While a different splitting of 8 and 23 would result in different subproducts, the final result would still be 184.

When I asked the students why they made groups of 9 in the way they did, their responses indicated an understanding of place value and multiplication only as a grouping concept (eg, "There are 9 groups of 365 and there are 3 hundreds in 365, so I built the 3 hundreds 9 times by putting down 9 piles of 3 flats").

A discussion followed about the amount of work needed to obtain the answer when using repeated addition or place value materials. I suggested there might be a short cut for multiplying large numbers using pencil and paper and that, before learning it, they first needed to understand an important mathematical idea.

Students were presented with the multiplication task 9 × 8. Most of them knew the answer automatically. They were asked to confirm it by making a dot diagram, splitting the number of rows with a horizontal line, representing each part of the cut using multiplication, and obtaining the answer from the subproducts (see Figure 1). They accomplished the task with some assistance. We discussed various ways of splitting 9 (the number of rows). Students were then asked to split the number of columns with a vertical line, to represent each part of the cut in a multiplication way, and to obtain the answer to 9 × 8 from the parts (see Figure 1). We discussed various ways of splitting 8 (the number of columns).

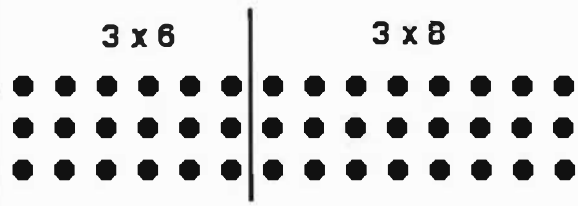
Figure 1
9 × 8 as 4 × 8 + 5 × 8 and as 9 × 2 and 9 × 6



Students were beginning to understand the distributive property in a functional sense. They realized that splitting one number of a product in whatever way they wanted allowed them to think in terms of subproducts that were added to obtain the final result. It was time to develop the strategy of splitting a number using place value concepts because it led to less cumbersome arithmetic.

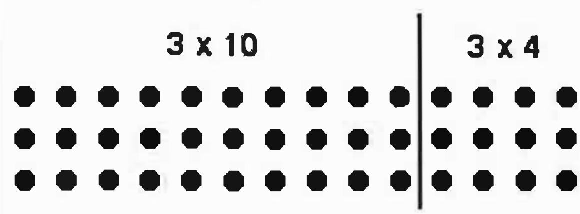
The students were asked to make up a 1-digit × 2-digit task, with the 2-digit number between 10 and 20 and the 1-digit multiplier between 2 and 5. The 1-digit number had to be different from the ones digit of the 2-digit number (eg, 2 × 12 was not allowed). They selected 3 × 14 for the task. They were asked to represent 3 × 14 using rows and columns, to split up the long side of the array using a vertical cut anywhere they chose, and to obtain the answer to 3 × 14 from the two parts (see Figure 2).

Figure 2
3 × 14 as 3 × 6 + 3 × 8



We discussed whether the smaller multiplications were easy to do and whether there might be a better way to split 14. The students were asked to split 14 using place-value thinking for 14 (see Figure 3). We discussed whether place value spitting made the smaller multiplications easier. Those who were adept with multiplying by 10s realized the advantage. One student was not adept and did not readily see the benefit of thinking of 14 as 10 + 4.

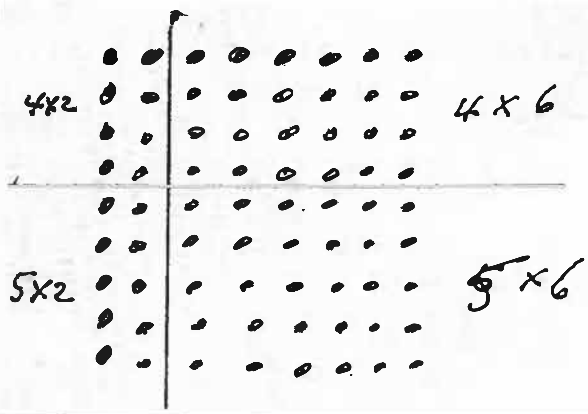
Figure 3
3 × 14 as 3 ×10 + 3 × 4



The students were asked to obtain answers to two more 1-digit × 2-digit multiplications by splitting the 2-digit number into tens and ones (I provided dot diagrams for this). They completed the tasks successfully. I felt it was time to move toward multi-digit multiplication but decided that the students' understanding of the distributive property had to be extended.

We revisited 9 × 8. Students were asked if it might be possible to cut both the 9 and the 8. One student suggested cutting across and down and showed the thinking using a diagram (see Figure 4). The students were asked how the diagram could be used to obtain the answer to 9 × 8. With minimal assistance they realized that each subproduct needed to be found and the results added.

Figure 4
9 × 8 as 4 × 2 + 4 × 6 + 5 × 2 + 5 × 6



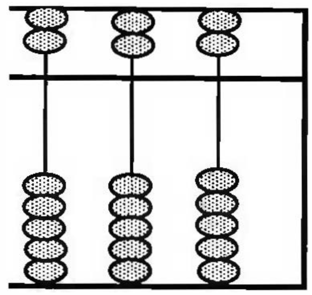
The students were asked to do two more 1-digit × 1-digit multiplication tasks by splitting the dot diagram across and down. We discussed the thinking involved. I asked students to explain in their own words what one could do when multiplying two numbers. A sample explanation follows. It indicates that the students were ready for 2-digit × 2-digit multiplication: "When you multiply two numbers like 7 × 6, you can break up each number and multiply the pieces. Then you add what you get."

Working with the Chinese Abacus

The Chinese abacus was used to strengthen the students' understanding of the functional sense of the distributive property and to extend their understanding of the corresponding multiplication algorithm. In my experiences working with students, the abacus is a powerful motivational device for engaging arithmetic. The abacus brings authenticity to the table. The need to use mental arithmetic strategies (eg, thinking of 17 + 8 as 17 + 10 – 2) arises naturally when working with the abacus. Mental arithmetic is required and is sometimes made more complex by the need to trade between columns (eg, 10 ones for 1 ten) and/or the need to trade within a column (eg, 1 five-bead for 5 one-beads).

The Chinese abacus has seven beads in each column (see Figure 6). The rightmost column represents the ones place, the next column the tens place, and so on. The two upper beads each represent a count of 5 and the five lower beads each represent a count of 1. A horizontal divider separates the two types of beads. A bead counts *only* when it touches the horizontal divider (akin to the yin and yang meeting each other).

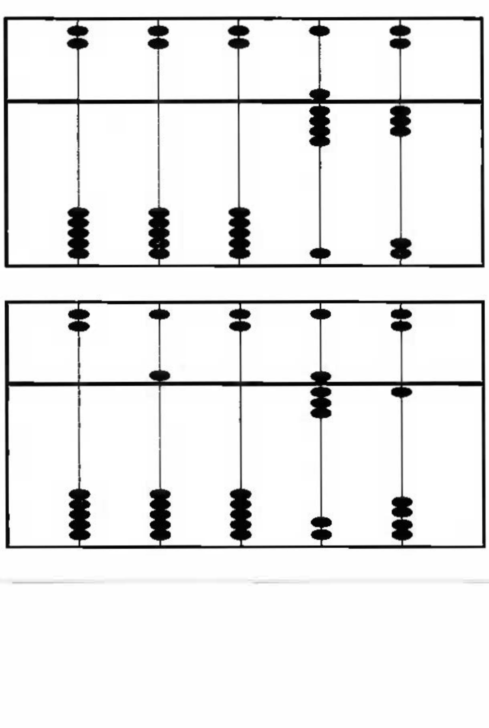
Figure 6
Three columns of a Chinese abacus with 7 beads in each column



The seven-bead structure makes it possible to represent a count of 15 (2 × 5 + 5 × 1) in each column. An advantage of this structure is that it allows for a variety of strategies for doing addition and subtraction, the core operations on the abacus. Multiplication involves the distributive property, with subproducts added as a running total.

The students could already represent numbers on the Chinese abacus (see Figure 7) and do addition on it. These understandings had been developed in conjunction with place value and multi-digit addition.

Figure 7
Representing 93 and 5081 on the Chinese abacus (a bead counts only when it touches the horizontal divider)



The students did not know how to use the abacus to multiply. This ability was developed over two Saturdays. The main difficulty for students was not the distributive property. Rather, it was doing the subproduct multiplication (eg, 7 × 20) mentally. Students who had a great degree of fluency with the basic facts of multiplication and multiplying by 10s had far less difficulty retaining the multiplication result in their mind and then adding it onto the existing number representation on the abacus.

Teaching Preservice Teachers to Use the Chinese Abacus

After the students could use the abacus to multiply reasonably well, we co-constructed two lesson plans that the students would use to teach preservice teachers. The first plan was for developing the distributive property and the 2-digit × 2-digit multiplication algorithm. The second plan was for developing representation, addition and multiplication on the abacus. Both plans involved the same approach that I had used when teaching the Grade 4 students those concepts. The students rehearsed the plans by using them to teach me. During the rehearsal I asked questions that a novice learner might ask. This provided the students with more practice, deepened their understanding and prepared them for questions from the preservice teachers.

During the next week, the students acted as instructors for two early-years mathematics methods classes. One purpose for doing this was to alter the preservice teachers' beliefs about the mathematical capabilities of elementary school students. The preservice teachers tended to see these students through the lens of their own mathematics anxiety and ability. This limited their perception of the mathematics that elementary students might be able to learn.

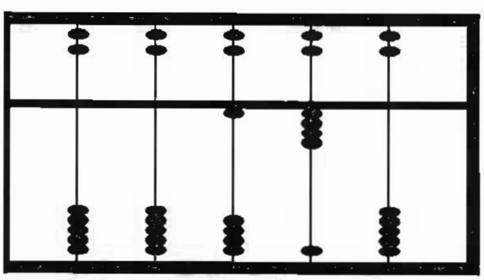
The first lesson involved developing a functional sense of the distributive property and a multiplication algorithm for which the distributive property is transparent. Needless to say, the mind shift involved from the traditional "magical" algorithm learned long ago by the preservice teachers generated feelings ranging from frustration to wonderment in them. The preservice teachers asked a number of questions during the first lesson. A sample question and Grade 4 student response follows:

> *Question*: When multiplying 5 × 13, can you split up the 5 instead of the 13?
>
> *Response*: Yes, but the answer will be harder to get.

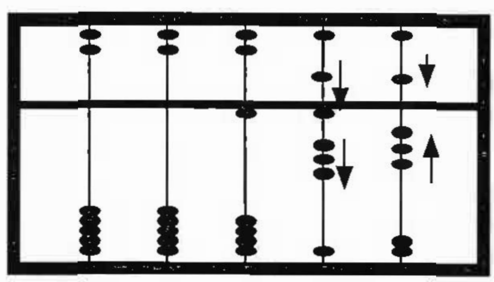
The final part of the second lesson developed multiplication on the abacus. Multiplication involves

Figure 8

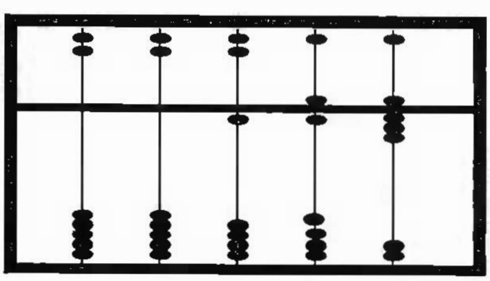
Multiplying 7 × 24 by thinking 7 × 20 + 7 × 4



Recording 140 (7 x 20)



Adding 28 (7 x 4) by adding 8 in the ones column and adding 50 and removing 30 in the tens column



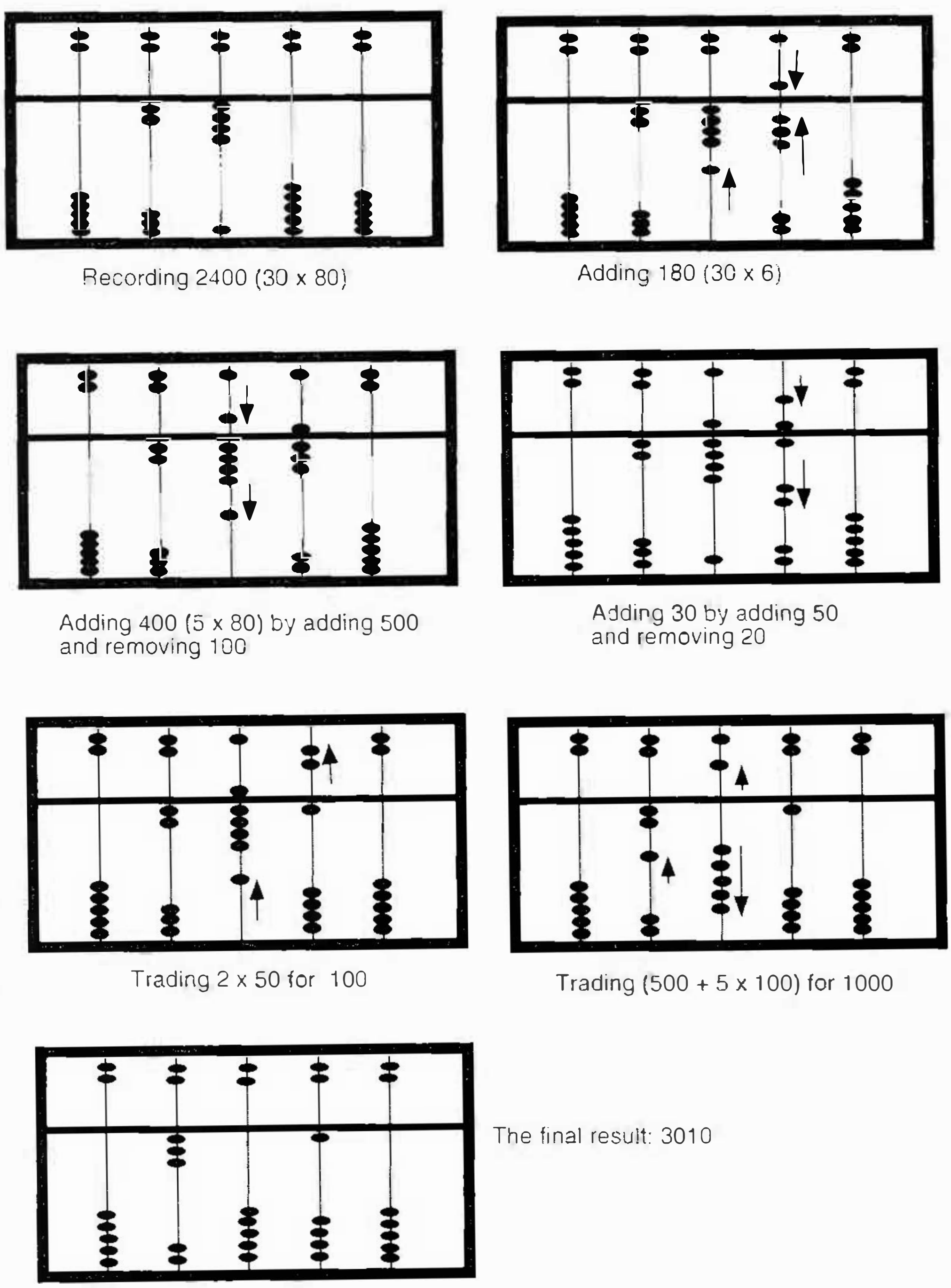
The final result: 168

adding subproducts using a running total. The subproducts are calculated mentally, not on the abacus. The process requires an understanding of the distributive property and the mental arithmetic strategies and concepts central to addition.

The first multiplication task presented was 3 × 12. Most of the preservice teachers completed it by adding 12 three times. The inadequacy of this approach was made evident by the next task: 45 × 67. Many of them realized that the distributive property was

Figure 9

Multiplying 35 × 86 by doing 30 × 80 + 30 × 6 + 5 x 80 + 5 x 6



needed to make this task manageable. Once this realization occurred, the lesson continued by presenting tasks progressively from 1-digit × 2-digit to 2-digit × 2-digit multiplication. One of the 1-digit × 2-digit tasks was 7 × 24 (see Figure 8).

The lesson concluded with 2-digit × 2-digit tasks. Figure 9 provides an example of 35 × 86. The thinking involved is 30 × 80 + 30 x 6 + 5 × 80 + 5 × 6.

Typically, the largest subproduct (eg, 30 × 80) is recorded first on the abacus because doing so tends to simplify the addition of subsequent subproducts and because of the way Chinese numerals are written and decoded (the largest position is at the top of the vertically written numeral). The preservice teachers tended to record the smallest subproduct first because that is how the traditional paper and pencil algorithm they had learned works (ones × ones, ones × tens, and so on). The Grade 4 students were far more comfortable with working with the largest place value position first because of their experiences doing addition with base 10 materials and with the abacus that had occurred during Saturday sessions with me. It was interesting to observe the Grade 4 students' impatience with the preservice teachers in relation to this. For example, one student made the following remark to a preservice teacher who insisted on recording the ones × ones partial product first and who was having difficulty seeing that it could be done differently: "Why do you want to do things the hard way? What is so special about doing 5 × 8 first?"

To be truly "Chinese" when using the abacus means that the calculation and memory of subproducts takes place solely in the mind, with subproduct results continuously added to the running total. Results are not written down on paper. The abacus *is* the paper, so to speak. If authenticity is expected when working with the abacus, that is another reason that the abacus can serve as a powerful device for stimulating and developing mental arithmetic. In effect, the Grade 4 students asked the preservice teachers to be Chinese in spirit when using the abacus. Those preservice teachers who had multiplicative fluency (basic multiplication facts and multiplying by 10s) were most comfortable with this. Those who used cognitive bypasses (such as skip counting) to obtain subproducts experienced frustration and had difficulty completing multiplication tasks successfully, even though they understood the distributive property.

The two lessons went well. The Grade 4 students presented the tasks clearly and facilitated learning capably. Especially notable was the confident manner in which they responded to questions. They felt proud about teaching mathematics to adults. The preservice teachers learned a functional sense of the distributive property (that I later revisited using bracket notation) and an understanding of why the multiplication algorithm that they had been taught in a rote manner when they were elementary students worked. The experience contributed significantly to reshaping their perceptions of the mathematical capabilities of elementary school students and their ability to concentrate for a long period of time. It also provided the preservice teachers with an exposure to a possible teaching strategy, namely, having a few students teach the other students in the class, where the teacher provides assistance with the matter to be taught and the teaching plan.

References

National Council of Teachers of Mathematics (NCTM). 2000. *Principles and Standards for School Mathematics.* Reston, Va: NCTM.

Note

The website Chinese Abacus, at www.mandarintools.com/abacus.html, provides further detail.

Jerry Ameis is an associate professor of mathematics education at the Faculty of Education, University of Winnipeg, where he teaches mathematics and mathematics curriculum and instruction courses to K–8 preservice teachers. His research interests concern how children learn mathematics and the kind of mathematics they can learn. A recent research project involved learning mathematics in the ArtSmarts and Learning Through the Arts programs. His view of teaching and learning borrows from the fundamental notions of a complex chaotic system where predictability is suspect and often not possible. Translated into educational research and practice, this means that we must beware of our fondness for using narrow lenses to view phenomena of teaching and learning and our penchant for adopting bandwagon approaches to pedagogy.