Reliable Delivery with Unreliable Deliverers

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Dedicated to the memory of the late the Honourable Lois Hole, former Lieutenant Governor of Alberta.

The Problem

We have to send a secret file to our embassy in an unfriendly country. We can use only our sixteen agents who can take sensitive material across borders. The former head of our secret agency had indicated that three of the agents are suspected of being double agents. The double agents will deliver what they are carrying to the enemy instead. If our embassy does not get the entire file, or if the enemy gets hold of the entire file, it will mean big trouble for us. Unfortunately, we were not told which agents might be unreliable. The former head did not want to prejudice us against any individuals without concrete evidence that they were, indeed, double agents.

Our Thinking

Clearly, we must divide the file into at least four parts. If it is divided into only three parts and there are indeed three double agents, each of them may be carrying a different part, and they will among them deliver the entire file to the enemy. Also, we must have at least four copies of each part. If we have only three copies and there are indeed three double agents, each of them may be carrying the same part, and our embassy will not be able to get a complete copy of the file. On the other hand, more than four copies will only make things easier for the enemy. As long as no agent carries duplicate parts, four copies are enough.

Suppose we divide each of the four copies of the file into four parts—A, B, C and D. We may have agents 001, 002, 003 and 004 carry A; 005, 006, 007 and 008 carry B; 009, 010, 011 and 012 carry C; and 013, 014, 015 and 016 carry D. This will guarantee that our embassy will get a complete copy of the file and that the enemy cannot. Unfortunately, we discover that agent 016 has been eliminated, and we are down to fifteen agents. Even more unfortunately, we may still have three double agents.

Suppose we still divide each copy of the file into four parts. This is no longer enough. A key result in the explanation is the Mean Value Principle, which states that in a finite set of real numbers, there is at least one not greater than the average and there is at least one not less than the average.

If we have sixteen parts altogether and only fifteen agents, the average number of parts each carries is greater than one. Some agent must carry not less than the average number of parts. This means that this agent must carry at least two parts, say A and B. Now a second agent must carry C and a third agent must carry D. If these three are double agents, the enemy will have a complete copy of the file. Later, we will make use of the Mean Value Principle without mentioning it explicitly.

With fifteen agents, will be it enough to divide the file into five parts? It turns out that it is enough even if 015 is unavailable. Let the five parts be A, B, C, D and E. We may have 001 and 002 carry A and B; 003 and 004 carry C and A; 005 and 006 carry B and C; 007, 008, 009 and 010 carry D; and 011, 012, 013 and 014 carry E. In order for the enemy to get all of A, B and C, two of 001 to 006 must be double agents. In order for the enemy to get D, one of 007 to 010 must be a double agent. In order for the enemy to get E, one of 011 to 014 must be a double agent. Thus three double agents will not be enough for the enemy.

So we successfully send the secret file to our embassy. However, we are worried that more agents may be eliminated. It is better for us to work out well in advance what we would do in each scenario.

Scenario 1: Agents 014 and 015 have also been eliminated.

We may divide the file into six parts, even if 013 is unavailable. Let the parts be A, B, C, D, E and F.

We may have 001 and 002 carry B and C, 003 and 004 carry C and A, 005 and 006 carry A and B, 007 and 008 carry F and D, 009 and 010 carry D and E, and 011 and 012 carry E and F. In order for the enemy to get all of A, B and C, two of 001 to 006 must be double agents. In order for the enemy to get all of D, E and F, two of 007 to 012 must be double agents. Thus three double agents will not be enough.

We now prove that six parts are necessary. Suppose we have only five parts, say A, B, C, D and E. If someone carries three parts, say A, B and C, someone else must carry D and someone E. These three will have a complete file among them. Hence no agent carries three or more parts. We may assume that someone carries A and B. If someone else carries two of the remaining parts, say C and D, a third agent must carry E, and the three of them will have a complete file among them. It follows that four of the other twelve agents carry C, four carry D and four carry E. Now someone must carry the second copy of A, say an agent carrying C. Then no agent can carry two of B, D and E. Hence the other three copies of B must be carried by those who are carrying C. Now we have someone carrying B and C, and whoever carries the third copy of A will also carry D or E. We will have three agents who have a complete file among them.

Scenario 2: Agents 012 and 013 have also been eliminated.

We may divide the file into eight parts. Let the parts be A, B, C, D, E, F, G and H. We may have 001 carry A, B, C and D; 002 carry A, B, C and E; 003 carry A, B, D and E; 004 carry A, C, D and E; 005 carry B, C, D and E; 006 and 007 carry H and F; 008 and 009 carry F and G; and 010 and 011 carry G and H. In order for the enemy to get all of A, B, C, D and E, two of 001 to 005 must be double agents. In order for the enemy to get all of F, G and H, two of 006 to 011 must be double agents. Thus three double agents will not be enough.

We now prove that eight parts are necessary. Suppose we have only seven parts and someone carries at least four parts. Then someone else must carry at least two of the remaining three parts. Along with someone who carries the remaining part, these three agents among them carry all seven parts, and the enemy can obtain a complete copy of the file. Henceforth, we assume that everyone carries at most three parts. At the same time, no two carry between them six different parts. At least six agents must carry three parts each, and it is not possible for every two of them to carry two common parts. Hence we may assume that agent 001 carries A, B and C, while 002 carries A, D and E. Now no agent can carry both F and G. Hence, we may assume that F is carried by 003 to 006, while G is carried by 007 to 010. Now one of these eight agents must carry B, and we may assume that 003 does. Then none of 007 to 010 can carry C. If any of them carry B, then none of 003 to 006 can carry C either, and we will not have enough agents carrying C. Hence none of 007 to 010 can carry B. So each of them carries D or E in addition to G, but none can carry both D and E. Similarly, each of 003 to 006 carries B or C in addition to F, but not both B and C, and none of them can carry D or E. However, this means that 011 must carry B, C, D and E. This is a contradiction.

Scenario 3: Agent 011 has also been eliminated.

We may divide the file into ten parts. Let the parts be A, B, C, D, E, F, G, H, I and J. We may have agent 001 carry A, B, C and D; 002 carry A, B, C and E; 003 carry A, B, D and E; 004 carry A, C, D and E; 005 carry B, C, D and E; 006 carry F, G, H and I; 007 carry F, G, H and J; 008 carry F, G, I and J; 009 carry F, H, I and J; and 010 carry G, H, I and J. In order for the enemy to get all of A, B, C, D and E, two of 001 to 005 must be double agents. In order for the enemy to get all of F, G, H, I and J, two of 006 to 010 must be double agents. Thus, three double agents will not be enough. We postpone the proof that ten parts are necessary to Appendix A.

Scenario 4: Agent 010 has also been eliminated.

We may divide the file into twelve parts. Let the parts be A, B, C, D, E, F, G, H, I, J, K and L. Let agent 001 carry A, E, F, G, H and I; 002 carry B, D, F, G, H and J; 003 carry C, D, E, G, I and J; 004 carry B, C, E, F and K; 005 carry A, C, D, F and K; 006 carry A, B, D, E and L; 007 carry A, B, C, G and L; 008 carry H, I, J, K and L; and 009 carry H, I, J, K and L. In order for the enemy to get all of A, B, C, D, E, F and G, all three double agents must be among 001 to 007. In order for the enemy to get all of H, I and J, two of 001 to 003 must be double agents. In order for the enemy to get K, one of 004 and 005 must be a double agent. In order for the enemy to get L, one of 006 and 007 must be a double agent. Thus three double agents will not be enough. We postpone the proof that twelve parts are necessary to Appendix B.

Scenario 5: Agent 009 has also been eliminated.

We may divide the file into fourteen parts. Let the parts be A, B, C, D, E, F, G, *A, B, C, D, E, F* and *G*. Let agent 001 carry A, B, C, D, E, F and G; 002 carry A, B, C, *D, E, F* and *G*; 003 carry A, D, E, *B, C, F* and *G*; 004 carry A, F, G, *B, C, D* and *E*; 005 carry B, D, F, *A, C, E* and *G*; 006 carry B, E, G, *A, C, D* and *F*; 007 carry C, D, G, *A, B, E* and *F*; and 008 carry C, E, F, *A, B, D* and *G*. In order for the enemy to get all of *A, B, C, D, E, F* and *G*, all three double agents must be among 002 to 008. In order for the enemy to get all of A, B, C, D, E, F and G, all three double agents must carry a common part, say A. Then none of them carry *A*. This is a contradiction.

We now prove that fourteen parts are necessary. Suppose we have only thirteen parts. Since there are fifty-two copies of these parts combined, someone must carry at least seven parts. Consider the other seven agents and six parts. Since there are twenty-four copies of these parts combined, someone must carry at least four parts. Consider now the remaining six agents and two parts. Someone must carry both of them. These three agents among them carry all thirteen parts, and the enemy can obtain a complete file.

Scenario 6: Agent 008 has also been eliminated.

Suppose 001, 002 and 003 are double agents. Then there must be a part which none of them carry. Let it be A. Since there are four copies of A, each of 004, 005, 006 and 007 must carry one. Suppose 001, 002 and 004 are double agents. Then there must also be a part which none of them carry. Moreover, this cannot be A since 004 is carrying it. Let it be B. Then each of 003, 005, 006 and 007 carries B. So we need a part for every triple of agents. Since the number of triples of agents is thirty-five, dividing the file into thirty-five parts is both necessary and sufficient.

Scenario 7: Agent 007 has also been eliminated.

As it turns out, we will be helpless without James Bond. If we are down to six agents, and three of them may still be double agents, then it is mission impossible. This is because three of the six are on our side while the other three are on the other side. Whatever we can deliver with three agents, the other side can get with three double agents. Whatever the other side cannot get with three double agents, we cannot deliver with three agents. This is an incidence of the important notion of *symmetry*.

Appendix A

Suppose we have ten agents and only nine parts A, B, C, D, E, F, G, H and I. It is routine to verify that any scheme in which one of the agents carries at least five parts will not work. Henceforth, we assume that each agent carries at most four parts. We may assume that agent 001 carries A, B, C and D.

Since there are twenty copies of E, F, G, H and I, and only nine other agents, we may assume that 002 carries E, F and G. Nobody can then carry both H and I. Hence we may assume that 003 to 006 carry H and 007 to 010 carry I. Since there are nine other copies of E, F and G, 003 must carry at least two of them, say E and F. This means that the other three copies of G must be carried by 004 to 006. None of these three can carry E or F, and none of 007 to 010 can carry both E and F. Hence 007 and 008 carry E and 009 and 010 carry F. We have arrived at the following situation.

001	002	003	004	005	006	007	008	009	010
A	E	E	G	G	G	E	E	F	F
B	F	F	H	H	H	I	I	I	I
C	G	H							
D									

We still have to add three copies of each of A, B, C and D. Someone other than agent 001 must also carry two of them. We consider two subcases.

Case 1: 007 carries A and B. Neither 009 nor 010 can carry both C and D. Hence one of 002 to 006 must carry C. Suppose it is 002. Then nobody can carry both D and H, so that the other three copies of D are carried by 008 to 010. Then none of 004 to 006 can carry C, so that the last two copies of C are carried by 003 and 008. None of 004 to 006 can carry both A and B. Hence 009 carries A, but 004 must carry B. Now 004, 008 and 009 carry all the parts among them. It follows that neither 002 nor 003 carries C, so 004 does. Then nobody can carry both D and F, so the other three copies of D are carried by 005, 006 and 008. Neither 005 nor 006 can carry C, so 009 must. Now 004, 007 and 009 carry all the parts among them.

Case 2: None of 007 to 010 carries two of A, B, C and D. This means that each of 004 to 006 carries two of them while each of 002, 003 and 007 to 010 carries one. We may assume that 004 carries A and B. Suppose 002 carries C. Then nobody can carry both D and I, so that the other three copies of D are carried by 003, 005 and 006. By the same reasoning, the other two copies of C are carried by 005 and 006. Let 007 carry A. Then either 009 or 010 must carry B. This agent and 004 and 007 will carry all the parts among them. It follows that neither 002 nor 003 can carry either C or D. Now neither 005 nor 006 can carry both C and D, as otherwise 002 and 003 cannot carry A or B either. Hence each of 005 to 010 carries one of C and D. Now one of 007 and 008 must carry a different part (C or D) from either of 009 and 010. Along with 004, we again have three agents who carry all the parts among them.

Appendix B

Suppose we have nine agents and only eleven parts (A, B, C, D, E, F, G, H, I, J and K). It is routine to verify that any scheme in which one of the agents carries at least six parts will not work. Henceforth, we assume that each agent carries at most five parts. We may assume that 001 carry A, B, C, D and E.

Suppose another agent carries four of the other six parts. Consider the remaining seven agents and two parts. Someone must carry both parts. These three agents among them carry all eleven parts, and the enemy can obtain a complete file. Thus we may assume that everyone except 001 carries exactly three of F, G, H, I, J and K. We may assume that 002 carries A, B, F, G and H. Then each of 003 to 009 carries exactly three of C, D, E, I, J and K. However, none of them can carry all of I, J and K.

Suppose some agent carries none of I, J and K. Then he must carry all of C, D, E, F, G and H, which is too many. Hence, every agent carries at least one of I, J and K. Since there are twelve copies and seven other agents, we may assume that 003 carries exactly one of I, J or K—say I. Then he must also carry two of C, D and E and two of F, G and H—say D, E, G and H. Then each of 004 to 009 carries exactly three of A, B, C, F, J and K. Moreover, none of them can carry J and K plus C or F.

There are eight copies of J and K, and only six agents left. Therefore, at least two of them, say 004 and 005, must carry both J and K. Each must then carry one of A and B and one of D and E, as well as one of G and H. Now we still have nine copies of C, F and I, but only four agents left. Hence at least one of them, say 006, must carry all of C, F and I. He must still carry either J or K, but nothing else.

In order for 004, 005 and 006 not to have all the parts among them, 004 and 005 must carry a third common part other than J and K. However, this means that one of the other seven agents must carry four parts not carried by 004. This is a contradiction.

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