

The Graphing Calculator: A Valuable Tool in Mathematics Education

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The purpose of this letter is to respond to the article entitled "One Way to Avoid Using a Graphing Calculator," by Dr Indy Lagu in the December 2007 issue of *delta-K*.

It is unfortunate that the diagram on the front cover of the *delta-K* that illustrates Dr Lagu's article is mislabelled. It also appears that Dr Lagu's original article was slightly edited. The distance a should be the distance from the *level* of the observer's eye (that is, from the vertex of the right angle) to the top of the statue (as Dr Lagu had written in the original version of his article), rather than the distance from the observer's eye to the top of the statue (the hypotenuse), as was printed. Similarly, the distance b should be the distance from the *level* of the observer's eye to the bottom of the statue. He gives the angular height of the statue (that is, the angle subtended at the observer's eye by the statue) as a function of θ as $\theta = \tan^{-1}\left(\frac{a}{x}\right) - \tan^{-1}\left(\frac{b}{x}\right)$, where x is the horizontal distance from the observer's eye to the statue. We are required to determine x that will maximize θ —that is, where do we stand in order that the statue will appear to be as large as possible? Calculus and derivatives are employed to determine that $x = \sqrt{ab}$, and Dr Lagu points out the very nice connection that x is the geometric mean of a and b .

Dr Lagu goes on to write; "A lower-tech approach (mathematically and intellectually) is to plug the equation into a graphing calculator and eyeball the maximum." He states that "I do not mind the first approach, but I abhor the graphing calculator approach." To our memories, it has always been the case that that postsecondary institutions in Alberta generally do not allow graphing calculators, especially in Calculus I classes. We have always wondered why, and Dr Lagu's statement has prompted us to again ask the question. We fail to see why a mathematics

department would favour the use of programs such as Maple or Mathematica (even though they are excellent programs) over the graphing calculator in at least some lab classes associated with Calculus I courses. At mathematics conferences, we have attended sessions at which the presenter explored the concept of calculus reform. We have taught advanced placement (AP) calculus and international baccalaureate (IB) Math 31 classes for many years, and both of these heavily involve the use of the graphing calculator. In fact, as of 2008, the IB mathematics standard level exam has both a calculator and non-calculator section; AP calculus exams have had this structure for many years. We consider it to be a problem that some high school teachers have told students *not* to use their calculators. When asked why, a statement such as "The university doesn't allow them," is often given. We maintain that such an attitude is very short-sighted.

The most recent revision of the common curriculum framework for the Western and Northern Canada Protocol (WNCP) Grades 10–12 mathematics (January 2008), states that "Technology contributes to a learning environment in which the curiosity of students can lead to rich mathematical discoveries at all grade levels. The use of technology should not replace mathematical understanding. Instead, technology should be used as one of a variety of approaches and tools for creating mathematical understanding" (p 9).

We heartily agree with the position taken in the above statement; we maintain that there is interplay between the use of technology and mathematical understanding, especially at the high-school level. We certainly do not advocate the use of technology to bypass or shortcut mathematical process. At the Grade 10 level, simply realizing that a function may have a maximum ($y = -x^2$), a minimum ($y = x^2$) or

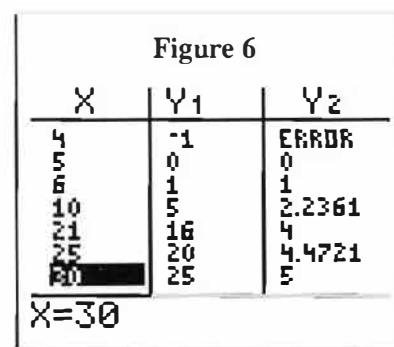
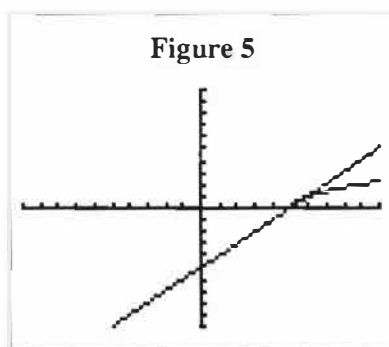
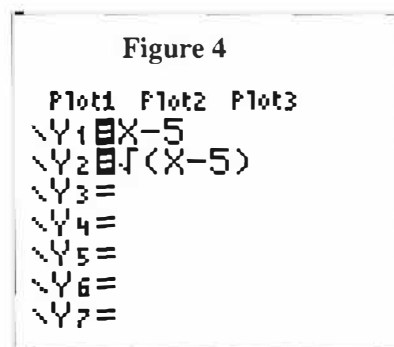
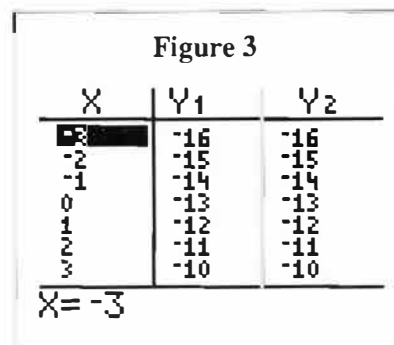
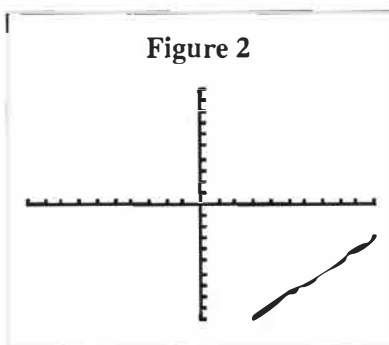
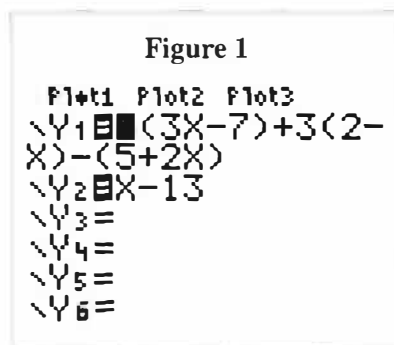
both $(y = x + \frac{1}{x})$ may be sufficient to give students a glimpse of their mathematical future. Yes, we use (and perhaps even misuse) the calculator as a tool, but in the right hands it can be used to take students where they have not gone before, and to pique their interest in mathematics.

For example, students in Mathematics 10P are routinely asked to simplify polynomial expressions such as $2(3x-7)+3(2-x)-(5+2x)$ by using the distributive property and then adding like terms to (hopefully) obtain the simplification $x-13$. Mathematically, what is to be gained by looking for the answer in “the back of the book”? Rather, students may enter the original expression in Y_1 and the simplification in Y_2 (Figure 1); pressing ZOOM6 will graph both relations (Figure 2) in the viewing window $x_{\min} = -10, x_{\max} = 10, x_{\text{scf}} = 1, y_{\min} = -10, y_{\max} = 10, y_{\text{scf}} = 1$.

Students quickly come to understand that if they see a single line (Figure 2), then their simplification is probably correct. Even more important, they come to realize that if, for example, the constant term was too small or large because of arithmetic error, there could be a second line that is beyond the viewing window. Examination of the table of values (Figure 3) (TBL SET: Tbl Start = -3, ΔTbl=1, Indpnt: AUTO, Depend: AUTO) shows that both the given relation and its simplification return the same value of the dependent variable for a given value of the independent variable—and yes, using such terminology is important to the beginning mathematics student.

Discussions of these techniques quickly involve mathematical topics such as domain and range, viewing window, and relation and function. Think of all the mathematics that the beginning algebra student has been exposed to if the use of the graphing calculator has been encouraged! In the days before the invention of the graphing calculator, it was difficult and often extremely tedious to make mathematics meaningful to many students, particularly those students who were weak or unmotivated in mathematics. Any concept is far easier to visualize when one already understands a particular mathematical concept, but this is not the case for the student who is studying new material. What is wrong with providing students who cannot already understand with a means of visualizing an image that we, as teachers of mathematics, already possess in our minds? Our experience in teaching mathematics has given us an aptitude for explanation, but sometimes even that is not enough. The adage “A picture is worth a thousand words” has never been truer than in the study of mathematics, and the graphing calculator and the new SMARTBoards are particularly advantageous.

In a later course (and this will be a specific outcome in Math 30-1) when radicals are the topic, one can easily define a radical function such as $y = g(x) = \sqrt{x-5}$ (Figure 4). It is instructive to graph both $y = f(x) = x-5$ and $y = g(x) = \sqrt{x-5}$ on the same axes (Figure 5). Notice that the graphs of functions f and g both exist (that is, they have a common domain) only when $x-5 \geq 0$.



Here, then, we have $x-5 \geq 0$ if and only if $x \geq 5$. On the calculator, we could use TBL SET: (ignore TblStart and Δ Tbl) Independ ASK, Depend AUTO. Students will quickly relate the domain of $y = g(x) = \sqrt{x-5}$ to the table of values shown (Figure 6); they will quickly realize that if they wish to avoid "ERROR" in the column of values for Y_2 , then they must input a value for x that is at least 5. Mathematics is all about making connections, and

we certainly may deal with $f(x) = \sqrt{ax+b} = \sqrt{a\left(x+\frac{b}{a}\right)}$ as either a transformation of $y = \sqrt{x}$ or a composition of $y = g(x) = \sqrt{x}$ with $y = h(x) = ax+b$ to obtain $y = f(x) = g(h(x)) = \sqrt{h(x)} = \sqrt{ax+b}$.

As an extension of the foregoing, one could easily ask a class to determine the set of values for x so that the function $f(x) = \sqrt{ax+b}$ will generate real numbers. If you include a statement such as "Your answer must be in terms of a and b ," then any advantage attributed to the graphing calculator is removed from the question.

This same concept is easily extended to $y = \sqrt{ax^2+bx+c}$ and so on, since, generally, the function $y = \sqrt{f(x)}$ will generate real numbers for output if and only if $f(x) \geq 0$.

In the unit on relations and functions in Math 10P, students could be asked to write a function for the sum of a number and its reciprocal. Hopefully, students will contribute something of the form $h(x) = x + \frac{1}{x}$. Here again, a discussion of domain may ensue, and perhaps even a sketch of the graph *without* using the calculator; students will surely know by now that the graph of $f(x) = x$ is linear and, if nothing else, they may be guided towards sketching the graph of $g(x) = \frac{1}{x}$. If both f and g are sketched on the same plane, it is obvious that there are points of intersection at (1,1) and also at (-1,-1). (Can anyone name every real number that is its own reciprocal?)

Students may even be able to identify the minimum that occurs at (1,2) and the maximum that occurs at (-1,-2) on the graph of $h(x) = x + \frac{1}{x}$ by using addition of ordinates. In support of these techniques, students

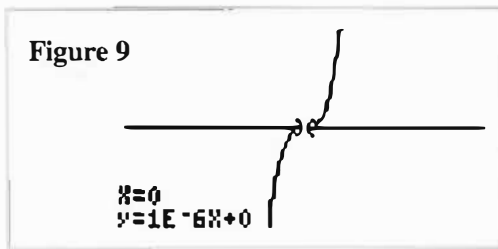
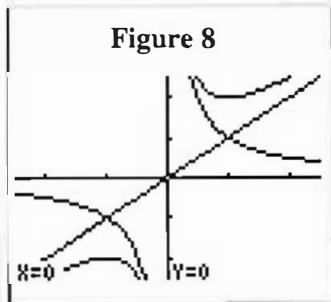
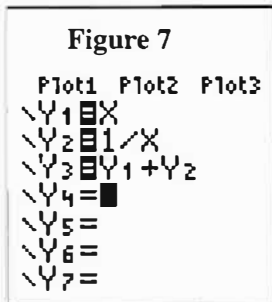
may produce the graph of $y = x + \frac{1}{x}$ on their calculator (ZOOM6, then ZOOM2) (Figure 7 and Figure 8).

The mathematics is plentiful here! Discussions with respect to addition of functions to produce a new function (if $f(x) = x$ and $g(x) = \frac{1}{x}$, then $h(x) = f(x) + g(x) = x + \frac{1}{x}$), domain (function h is defined only where f and g are both defined), range, maximum and minimum, and even limits are all possible here. In the calculus classroom, students may do any or all of the above, but then, of course, the techniques of differentiation and first or second derivative tests for maxima or minima will take us to the critical points of the function. Prior to the introduction of calculus, Dr Lagu's technique of reasoning that $y = x + \frac{1}{x}$ is minimized (or maximized, depending on the domain in which we are working) when $x = \frac{1}{x}$, which solves to give $x = \pm 1$, may be employed.

Now we must sort out what occurs where and, at the non-calculus level, that involves the definition of maximum of a function: a function f has a local (or relative) maximum at $x = c$ if $f(c) \geq f(x)$ when x is near c . A similar definition exists for the local minimum of a function.

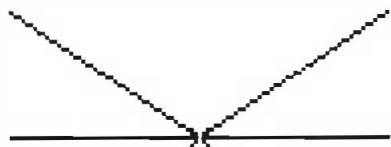
In the beginning calculus class, students will undoubtedly study Fermat's Theorem, which states that "If f has a local maximum or minimum at $x = c$, and if the derivative, $f'(c)$ exists, then $f'(c) = 0$ " (Stewart 2003, 282). Fermat's Theorem merely says that if we are at a point on the graph of a function where either a local maximum or minimum exists, and if the slope of the tangent can be expressed as a real number, then the value of that real number is 0. That is, the tangent line must be horizontal.

Students almost always assume that the converse is also true; that is, if the slope of the tangent is 0 at a particular point, then a maximum or minimum must occur there. One way to dispel such thoughts is to produce the graph of the function $y = x^3$ (Figure 9), but first, turn the coordinate axes OFF (2nd ZOOM (FORMAT) and down arrow to Axes Off ENTER). The tangent at $x = 0$ is easily drawn by using 2nd PRGM (Draw 5 ENTER). Clearly, no maximum exists at $x = 0$ for the graph of $y = x^3$, even though the slope of the tangent line is 0. The calculator gives the equation of the tangent as $y = (-1 \times 10^{-6})x + 0$.



A second counter-example to the converse of Fermat's Theorem is provided using the function $g(x)=|x|$, and this example shows a definite weakness in the capability of the graphing calculator.

Figure 10



The graph of $y = g(x) = |x|$, axes OFF, and the "tangent" drawn to the graph at the origin with the equation $y = 0x + 0$

The function $y = f(x) = |x|$ is expressed piecewise as $y = f(x) = \begin{cases} -1 \cdot x & x < 0 \\ x & x \geq 0 \end{cases}$. The derivative of this function is $y' = g'(x) = \begin{cases} -1, & x < 0 \\ 1, & x > 0 \end{cases}$. The derivative does NOT exist at $x = 0$, since $(\lim_{x \rightarrow 0^-} f'(x) = -1) \neq (\lim_{x \rightarrow 0^+} f'(x) = 1)$.

This is an instance that always generates much discussion and centres on the definition of *derivative*. One of the forms of this definition that many texts use is $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{(x+h) - x} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$; this definition contains a quotient that is sometimes referred to as the *difference quotient*.

The calculator, on the other hand, uses a definition that involves the limit of what is called the *symmetric difference quotient*, which is written algebraically as $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x-h)}{(x+h) - (x-h)} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x-h)}{2h}$. Using this definition and $y = f(x) = |x|$, we obtain $f'(x) = \lim_{h \rightarrow 0} \frac{|x+h| - |x-h|}{2h} = \lim_{h \rightarrow 0} \frac{(x+h) - (x-h)}{2h} = \lim_{h \rightarrow 0} \frac{x}{h}$.

The value of h will default to 0.001 (the calculator's ability to take limits is "limited"), so it essentially evaluates the last expression as $\frac{2(0)}{2(0.001)} = 0$.

Geometrically, the calculator approximates the slope of the tangent at $x = 0$ by using a sequence of secants where the endpoints of each secant define a closed interval that is (horizontally) symmetric across the origin. Since each of those secants has a slope of 0, it should be no surprise that the calculator reports the slope of the tangent at the origin as 0. Now, ask your students if they can find other functions for which the calculator will give incorrect values for slopes of tangents, and get out of the way!

There are countless examples showing that graphing calculators can help students develop a stronger understanding of the concept of limits, which is a difficult topic for many students.

Consider the function $f(x) = \frac{x^2 - 4}{x - 2}$ and, more specifically, $\lim_{x \rightarrow 2} f(x)$. Hopefully, the student will realize that attempting to evaluate $f(2)$ will produce the indeterminate form $\frac{0}{0}$. The graph of $y = f(x)$ is shown (Figure 11)

using the window settings: $x[-1, 5, 1]$ $y[-2, 6, 1]$. With an appropriate window setting, students can clearly see the "hole" in the graph (a point or removable discontinuity) at $x = 2$. Discussions will certainly centre on the difference between this function and the simplified function $g(x) = x + 2$. This can help the student understand that for both $f(x)$ and $g(x)$, the limiting value as x approaches 2 is 4, regardless of whether or not $f(2)$ exists. The table of values from the calculator (Figure 12), where $Y_1 = \frac{x^2 - 4}{x - 2}$ and $Y_2 = x + 2$, also aids in that discussion.

Now define the function $g(x) = \frac{1}{f(x)} = \frac{x - 2}{x^2 - 4}$ and consider $\lim_{x \rightarrow 2} \frac{x - 2}{x^2 - 4}$ to enhance students' understanding of the difference between a point (removable) discontinuity and an infinite (nonremovable) discontinuity.

A second limit example involves the very interesting function $f(x) = x \cdot \sin\left(\frac{1}{x}\right)$ and more specifically the $\lim_{x \rightarrow 0} \left(x \sin\left(\frac{1}{x}\right) \right)$. If we let $x = 0$, we get $f(x) = 0 \cdot \sin \infty$,

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P1ot1 P1ot2 P1ot3
\Y1 (X^2-4)/(X-2)
\Y2 X+2
\Y3 =
\Y4 =
\Y5 =
\Y6 =
    
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Figure 11

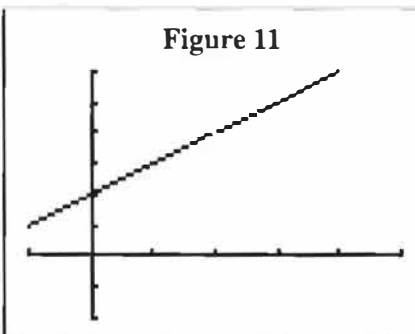


Figure 12

X	Y1	Y2
0	ERR	ERR
1	ERR	ERR
2	ERR	ERR
3	ERR	ERR
4	ERR	ERR
5	ERR	ERR
6	ERR	ERR
X=0		

which is indeterminate. Algebraically, we could get help from L'Hospital's Rule, but this is not part of the high school curriculum. Let's see what the graphing calculator can do. When $f(x)$ is graphed on the window $x[-\pi, \pi, 0]$ $y[-0.5, 0.5, 0]$ (Figure 13), the student may be led to believe that the limit value does not exist, as the graph shows somewhat chaotic behaviour near zero.

When regraphed using a Zoombox window around the origin (Figure 14), the graph appears to be more predictable near the origin; it now appears that the limiting value is zero. A table of values using the Ask feature on the Independent variable seems to confirm this. Algebraically, it is also possible to relate this limit to the more common function $y = \frac{\sin \theta}{\theta}$, since

$$\lim_{x \rightarrow 0} \left(x \sin \left(\frac{1}{x} \right) \right) = \lim_{x \rightarrow 0} \left(\frac{\sin \left(\frac{1}{x} \right)}{\frac{1}{x}} \right) = \lim_{\frac{1}{x} \rightarrow \infty} \left(\frac{\sin \left(\frac{1}{x} \right)}{\frac{1}{x}} \right) = \lim_{\theta \rightarrow 0} \left(\frac{\sin \theta}{\theta} \right) = 0,$$

where $\theta = \frac{1}{x}$.

In our minds, this development is useful not just in providing a visual way to see limit values, but also in teaching students to be careful with the calculator window in making judgments about functions and associated limits.

Our third and final limit examples are $\lim_{x \rightarrow \infty} \frac{x}{\sqrt{x^2 - x + 1}}$. Intuitively, we should realize that as $x \rightarrow \infty$, the

radicand in the denominator behaves like x^2 , and the denominator will therefore behave as x . Since the quotient is now essentially $\frac{x}{x}$, then $\lim_{x \rightarrow \infty} \frac{x}{\sqrt{x^2 - x + 1}} = 1$.

Similarly, $\lim_{x \rightarrow -\infty} \frac{x}{\sqrt{x^2 - x + 1}} = -1$, since the principal root

in the denominator ensures that the quotient will always be negative when $x < 0$.

The graph of $f(x) = \frac{x}{\sqrt{x^2 - x + 1}}$, $x \in \mathbb{R}$ (Figure 15)

with window settings $x[-20, 20, 5]$, $y[-2.2, 1]$ illustrates that 1 is the limiting value as $x \rightarrow +\infty$, and -1 is the limiting value as $x \rightarrow -\infty$.

Students are surprised to see a function having not one, but two, horizontal asymptotes. Students' past experience with rational functions—a rational function never intersects its (vertical) asymptotes—will likely cause them to be surprised that this function *does* intersect one of its horizontal asymptotes. They also find it surprising that both limits are approached from above.

Algebraically, the limits in question are easily determined:

$$\lim_{x \rightarrow \infty} \frac{x}{\sqrt{x^2 - x + 1}} = \lim_{x \rightarrow \infty} \sqrt{\frac{x^2}{x^2 - x + 1}} \cdot \frac{1}{x} = \lim_{x \rightarrow \infty} \sqrt{\frac{1}{1 - \frac{1}{x} + \frac{1}{x^2}}} = 1,$$

since $x = \sqrt{x^2}$ whenever $x \geq 0$.

Figure 13

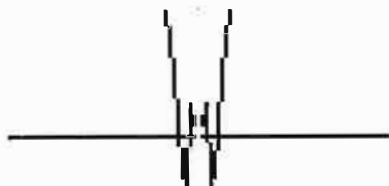
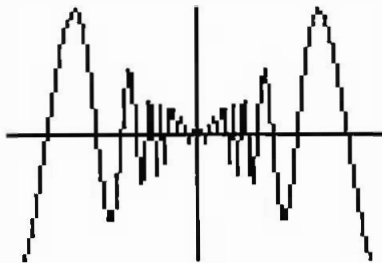


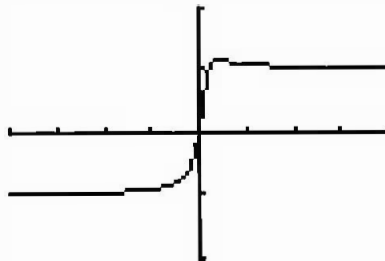
Figure 14



X	Y1
0	ERR:
.01	-.0051
1E-4	-3E-5
1E-6	-3E-7
-.01	-.0051
-1E-4	-3E-5
-1E-6	-3E-7

X = -1E-6

Figure 15



X	Y1
1	1
100	1.005
1000	1.0005
1E6	1.000005
-1	-.5774
-10	-.9492
-1000	-.9995

Y1 = 1.000005

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Plot1 Plot2 Plot3
\Y1 X/√(X²-X+1)
\Y2 =
\Y3 =
\Y4 =
\Y5 =
\Y6 =
\Y7 =

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Similarly, $x = \sqrt{x^2} = |x| = -1 \cdot x$ whenever $x < 0$, so that

$$\lim_{x \rightarrow -\infty} \frac{|x|}{\sqrt{x^2 - x + 1}} = \lim_{x \rightarrow -\infty} \frac{-1 \cdot x}{\sqrt{x^2 - x + 1}} = -1 \cdot \lim_{x \rightarrow -\infty} \frac{x}{\sqrt{x^2 - x + 1}} = -1 \cdot 1 = -1$$

In conclusion, we believe that the examples shown provide strong evidence that the use of graphing calculators enriches the high school mathematics curriculum by providing meaningful connections among all aspects of mathematics. We applaud Alberta Education for continuing to mandate the appropriate use of technology in high school mathematics and science classrooms in Alberta. We encourage teachers at all levels to make use of the many features of graphing technology to enhance mathematics education for their students so that they may consider mathematics problems not only algebraically, but also numerically and graphically.

Reference

Stewart, J. 2003. *Calculus: Early Transcendentals*. 5th ed. Florence, Ky: Thomson Brooks/Cole.

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