delta-k

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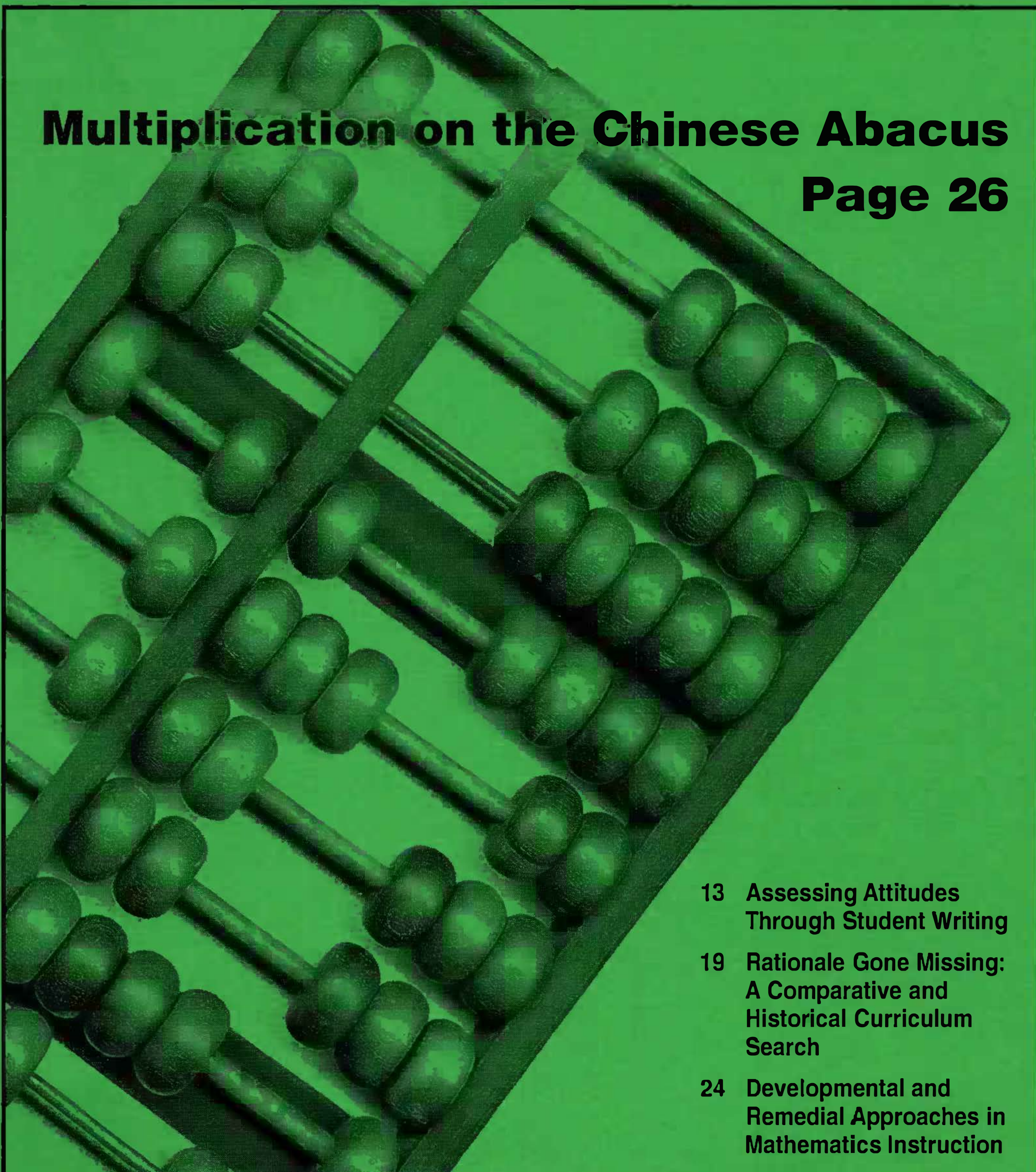
December 2008



Multiplication on the Chinese Abacus
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Guidelines for Manuscripts

delta-K is a professional journal for mathematics teachers in Alberta. It is published twice a year to

- promote the professional development of mathematics educators, and
- stimulate thinking, explore new ideas, and offer various viewpoints.

Submissions are requested that have a classroom as well as a scholarly focus. They may include

- personal explorations of significant classroom experiences;
- descriptions of innovative classroom and school practices;
- reviews or evaluations of instructional and curricular methods, programs or materials;
- discussions of trends, issues or policies;
- a specific focus on technology in the classroom; or
- a focus on the curriculum, professional and assessment standards of the NCTM.

Suggestions for Writers

1. *delta-K* is a refereed journal. Manuscripts submitted to *delta-K* should be original material. Articles currently under consideration by other journals will not be reviewed.
2. If a manuscript is accepted for publication, its author(s) will agree to transfer copyright to the Mathematics Council of the Alberta Teachers' Association for the republication, representation and distribution of the original and derivative material.
3. All manuscripts should be typewritten, double-spaced and properly referenced. All pages should be numbered.
4. The author's name and full address should be provided on a separate page. If an article has more than one author, the contact author must be clearly identified. Authors should avoid all other references that may reveal their identities to the reviewers.
5. All manuscripts should be submitted electronically, using Microsoft Word format.
6. Pictures or illustrations should be clearly labelled and placed where you want them to appear in the article. A caption and photo credit should accompany each photograph.
7. References should be formatted consistently using *The Chicago Manual of Style*'s author-date system.
8. If any student sample work is included, please provide a release letter from the student's parent/guardian allowing publication in the journal.
9. Articles are normally 8–10 pages in length.
10. Letters to the editor or reviews of curriculum materials are welcome.
11. Send manuscripts and inquiries to the editor: Gladys Sterenberg, 195 Sheep River Cove, Okotoks, AB, T1S 2L4; email gladyss@ualberta.ca

MCATA Mission Statement

Providing leadership to encourage the continuing enhancement of teaching, learning and understanding mathematics.

delta-k

Volume 46, Number 1 December 2008

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EDITORIAL

This summer, I had the pleasure of teaching graduate students enrolled in a master's program centred on elementary mathematics and science. Many of our conversations focused on the changes in the new curriculum for mathematics. Change is a recurring theme in conversations I have with teachers throughout the province: What is different about this new program of studies? What do these new things mean for me in the classroom? Are they just fads? How can I teach to encourage a deeper understanding of mathematics?

delta-K is dedicated to the exploration of change. In response to questions generated by teachers this summer, and with the help of Karen Virag, supervising editor at the ATA, I investigated why our journal is named *delta-K*. I share the following excerpt with you:

> In December 1970, a major change was initiated, and the intent was explained in the *Newsletter*. Due to the nature of the articles being printed, the publication had grown to be more than a newsletter; it had developed into a professional journal. Suggestions for a name for the new journal were requested from members. The *Mathematics Council Newsletter* was renamed *delta-K* in May 1971. The chosen name represents *delta* (Δ), the fourth letter in the Greek alphabet used in mathematics to represent an increment or increase. *K* is for knowledge: knowledge of mathematics, knowledge of teaching mathematics and knowledge of new methods and developments in our discipline. (Worth and Jorgenson 1995, 37)

Indeed, we are in an era of change and this issue of *delta-K* presents thought-provoking articles on what this change might mean for mathematics teachers in Alberta. As you catch your breath during this festive season, take time to reflect on the legacy we bring to this significant endeavour.

Gladys Sterenberg

Reference

Worth, J, and A Jorgenson. 1995. *Thirty-Four Years and Counting: The History of the Mathematics Council of the Alberta Teachers' Association*. Edmonton, Alta: Alberta Teachers' Association.

From the President's Pen

We are almost halfway through another school year. For many of us, this has been an exciting year as we try to wrap our minds around the new curriculum. With the mandatory implementation of a new program of studies for kindergarten and Grades 1, 4 and 7, and the optional implementation of Grades 2, 5 and 8, mathematics teachers across the province have been very busy. There are many chances to confer with colleagues and many professional development opportunities. The new curriculum ties into many of the current AISI projects. Whether your district is focusing on differentiation, inquiry, assessment, character education or other priorities, you should be able to see some correlation to the new curriculum.

As teachers, we are used to change and have learned to embrace it. Often it is rejuvenating to try something new. I thought it would be a few more years until I would be implementing the new curriculum, but I changed positions this summer and have had the pleasure of teaching Math 7 this year. Watching students attach personal meaning to the mathematics that we are teaching is inspiring. Seeing a renewed confidence grow in my students makes me remember why I decided many years ago to go into education. Whether it is new curriculum, new grade levels or new methodology, we do our best to make the transition seamless.

This year, the MCATA executive has continued to work with Alberta Education representatives and regional consortia to help smooth the implementation process for teachers and students. We enjoyed seeing many of you in Jasper, in October, and hope that you keep in touch with the colleagues that you connected with there. The success of our mathematics program in Alberta can be attributed to the dedicated professionals who know how to work together so well.

As Christmas break approaches, we are sometimes overwhelmed by how much we have to do and get discouraged with the preholiday lack of energy. Our students are ready for a break and are losing some of their beginning-of-year excitement. Approach the holiday with some selfishness! Recharge your batteries guilt free. You have worked hard since September (or earlier) and deserve some time away from the job. Have a wonderful holiday and a very happy new year.

Sharon Gach

MCATA Annual Conference

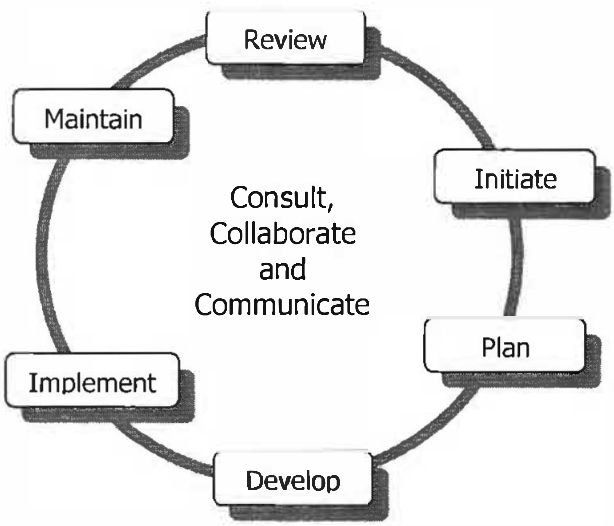
Mathematics teachers wanting to explore new depths and reach new heights joined with their colleagues in Jasper last October for the annual MCATA conference. Curious about the new program of studies and challenged to teach for deep understanding, teachers attended sessions where they were able to learn about and discuss how to integrate visualization in their lessons, encourage appropriate practice of skills, make connections among mathematics topics and promote problem solving in their lessons. Thank you to all members of the conference committee, especially the new council members who took a job on for the first time. Thank you also to Janis for coaching.

Please mark your calendars for the 2009 conference, to be held in Edmonton from October 23–25. Next year's conference will continue to address the challenges that teachers face integrating the new K–12 program of studies for mathematics.

Elaine Simmt

The Right Angle: Report from Alberta Education

The Alberta mathematics curriculum is entering new phases of the curriculum development, approval and implementation cycle.



K–9 Mathematics

The revised mathematics kindergarten to Grade 9 program of studies is available on the Alberta Education website, at www.education.gov.ab.ca/k_12/curriculum/bySubject/math/Kto9Math.pdf. Provincial implementation of the kindergarten and Grades 1, 4 and 7 program began in September 2008. English-language and French-language resources are available to support provincial implementation. Optional implementation of the Grades 2, 5 and 8 program also began in September 2008. English-language resources are available to support these grades, and French-language resources will be available in the spring of 2009 in preparation for provincial implementation in September 2009.

Alberta Education again offered a summer institute, in English and French, at the Lister Centre, in Edmonton, from July 8–10, 2008. The summer institute focused on kindergarten and Grades 1, 2, 4, 5, 7 and 8. The institute featured several speakers, including Dr Shaun Murphy, from the University of Saskatchewan, and Dr Lynn McGarvey, from the University of Alberta. The institute was offered at no cost to teachers. It is anticipated that there will be a third summer institute next year, focusing on Grades 2, 3, 5, 6, 8 and 9.

10–12 Mathematics

Development of the Alberta senior high school mathematics program has been completed, and the program will soon enter the implementation phase.

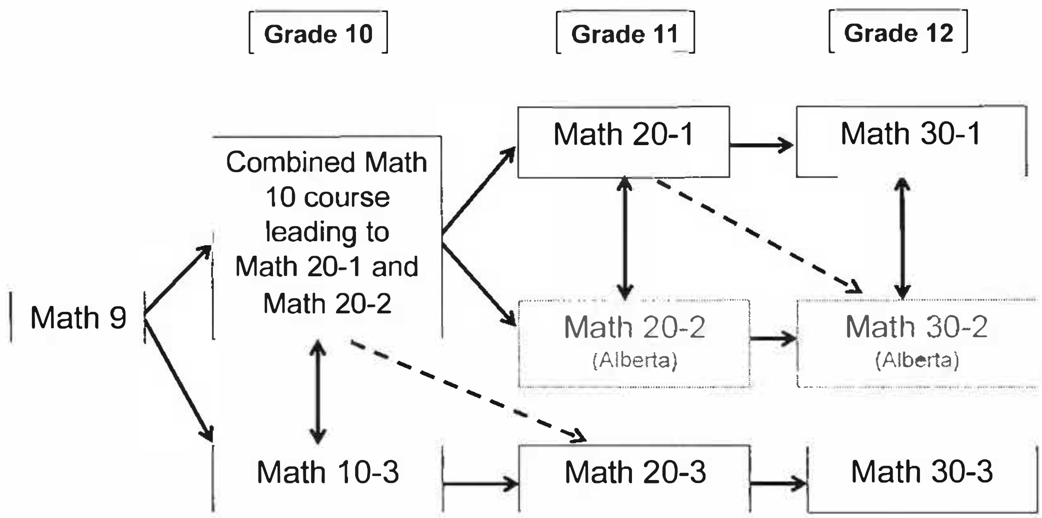
The Common Curriculum Framework for Grades 10–12 Mathematics: Western and Northern Canadian Protocol is posted in English on the WNCP website at www.wncp.ca, and in French at www.ponc.ca. The Alberta mathematics Grades 10–12 program of studies, which was derived from the *Common Curriculum Framework,* is posted on the Alberta Education website at http://education.alberta.ca/teachers/core/math/programs.aspx.

Meetings with postsecondary institutions continue. All postsecondary institutions have been asked to provide the Alberta Council on Admissions and Transfer (ACAT) with listings of programs and prerequisites well before implementation of the revised senior high school mathematics program of studies. The University of Alberta has indicated that it will accept Mathematics 30-2 as a prerequisite for Stat 141 and Math 153, thereby increasing postsecondary options, such as nursing, for students choosing Mathematics 30-2. Decisions on the acceptance of the

Alberta Mathematics Programs: Implementation Time Line

	2007	2008	2009	2010	2011	2012
Optional	K, 1, 4, 7	2, 5, 8	3, 6, 9			
Provincial		K, 1, 4, 7	2, 5, 8	3, 6, 9, 10	11	12

Structure for Alberta Senior High School Mathematics



mathematics course sequences are anticipated well before the first graduates of the revised program enter postsecondary education.

Note that the structure allows for transfer between the -1 and -2 course sequences at all levels. Should a student decide to switch course sequences, only one additional mathematics course would be necessary to switch to the -1 sequence, and no extra courses would be necessary if switching from Mathematics 20-1 to Mathematics 30-2.

Mathematics 30-1 will be the prerequisite for Mathematics 31, which remains as part of the maintenance phase.

Kathy McCabe

The Graphing Calculator: A Valuable Tool in Mathematics Education

Len Bonifacio and Darryl Smith

The purpose of this letter is to respond to the article entitled "One Way to Avoid Using a Graphing Calculator," by Dr Indy Lagu in the December 2007 issue of *delta-K*.

It is unfortunate that the diagram on the front cover of the *delta-K* that illustrates Dr Lagu's article is mislabelled. It also appears that Dr Lagu's original article was slightly edited. The distance a should be the distance from the *level* of the observer's eye (that is, from the vertex of the right angle) to the top of the statue (as Dr Lagu had written in the original version of his article), rather than the distance from the observer's eye to the top of the statue (the hypotenuse), as was printed. Similarly, the distance b should be the distance from the *level* of the observer's eye to the bottom of the statue. He gives the angular height of the statue (that is, the angle subtended at the observer's eye by the statue) as a function of θ as $\theta = \tan^{-1}\left(\frac{a}{x}\right) - \tan^{-1}\left(\frac{b}{x}\right)$, where x is the horizontal distance from the observer's eye to the statue. We are required to determine x that will maximize θ—that is, where do we stand in order that the statue will appear to be as large as possible? Calculus and derivatives are employed to determine that $x = \sqrt{ab}$, and Dr Lagu points out the very nice connection that x is the geometric mean of a and b.

Dr Lagu goes on to write; "A lower-tech approach (mathematically and intellectually) is to plug the equation into a graphing calculator and eyeball the maximum." He states that "I do not mind the first approach, but I abhor the graphing calculator approach." To our memories, it has always been the case that that postsecondary institutions in Alberta generally do not allow graphing calculators, especially in Calculus I classes. We have always wondered why, and Dr Lagu's statement has prompted us to again ask the question. We fail to see why a mathematics department would favour the use of programs such as Maple or Mathematica (even though they are excellent programs) over the graphing calculator in at least some lab classes associated with Calculus I courses. At mathematics conferences, we have attended sessions at which the presenter explored the concept of calculus reform. We have taught advanced placement (AP) calculus and international baccalaureate (IB) Math 31 classes for many years, and both of these heavily involve the use of the graphing calculator. In fact, as of 2008, the IB mathematics standard level exam has both a calculator and non-calculator section; AP calculus exams have had this structure for many years. We consider it to be a problem that some high school teachers have told students *not* to use their calculators. When asked why, a statement such as "The university doesn't allow them," is often given. We maintain that such an attitude is very short-sighted.

The most recent revision of the common curriculum framework for the Western and Northern Canada Protocol (WNCP) Grades 10–12 mathematics (January 2008), states that "Technology contributes to a learning environment in which the curiosity of students can lead to rich mathematical discoveries at all grade levels. The use of technology should not replace mathematical understanding. Instead, technology should be used as one of a variety of approaches and tools for creating mathematical understanding" (p 9).

We heartily agree with the position taken in the above statement; we maintain that there is interplay between the use of technology and mathematical understanding, especially at the high-school level. We certainly do not advocate the use of technology to bypass or shortcut mathematical process. At the Grade 10 level, simply realizing that a function may have a maximum ($y = -x^2$), a minimum ($y = x^2$) or both ($y = x + \frac{1}{x}$) may be sufficient to give students a glimpse of their mathematical future. Yes, we use (and perhaps even misuse) the calculator as a tool, but in the right hands it can be used to take students where they have not gone before, and to pique their interest in mathematics.

For example, students in Mathematics 10P are routinely asked to simplify polynomial expressions such as $2(3x-7)+3(2-x)-(5+2x)$ by using the distributive property and then adding like terms to (hopefully) obtain the simplification $x-13$. Mathematically, what is to be gained by looking for the answer in "the back of the book"? Rather, students may enter the original expression in Y_1 and the simplification in Y_2 (Figure 1); pressing ZOOM6 will graph both relations (Figure 2) in the viewing window $x_{min} = -10, x_{max} = 10, x_{scl} = 1, y_{min} = -10, y_{max} = 10, y_{scl} = 1$.

Students quickly come to understand that if they see a single line (Figure 2), then their simplification is probably correct. Even more important, they come to realize that if, for example, the constant term was too small or large because of arithmetic error, there could be a second line that is beyond the viewing window. Examination of the table of values (Figure 3) (TBL SET: Tbl Start = -3, ΔTbl = 1, Indpnt: AUTO, Depend: AUTO) shows that both the given relation and its simplification return the same value of the dependent variable for a given value of the independent variable—and yes, using such terminology is important to the beginning mathematics student. Discussions of these techniques quickly involve mathematical topics such as domain and range, viewing window, and relation and function. Think of all the mathematics that the beginning algebra student has been exposed to if the use of the graphing calculator has been encouraged! In the days before the invention of the graphing calculator, it was difficult and often extremely tedious to make mathematics meaningful to many students, particularly those students who were weak or unmotivated in mathematics. Any concept is far easier to visualize when one already understands a particular mathematical concept, but this is not the case for the student who is studying new material. What is wrong with providing students who cannot already understand with a means of visualizing an image that we, as teachers of mathematics, already possess in our minds? Our experience in teaching mathematics has given us an aptitude for explanation, but sometimes even that is not enough. The adage "A picture is worth a thousand words" has never been truer than in the study of mathematics, and the graphing calculator and the new SMARTBoards are particularly advantageous.

In a later course (and this will be a specific outcome in Math 30-1) when radicals are the topic, one can easily define a radical function such as $y = g(x) = \sqrt{x-5}$ (Figure 4). It is instructive to graph both $y = f(x) = x-5$ and $y = g(x) = \sqrt{x-5}$ on the same axes (Figure 5). Notice that the graphs of functions *f* and *g* *both* exist (that is, they have a common domain) only when $x - 5 \geq 0$.

Figure 1

```
Plot1 Plot2 Plot3
\Y1=(3X-7)+3(2-X)-(5+2X)
\Y2=X-13
\Y3=
\Y4=
\Y5=
\Y6=
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Figure 2

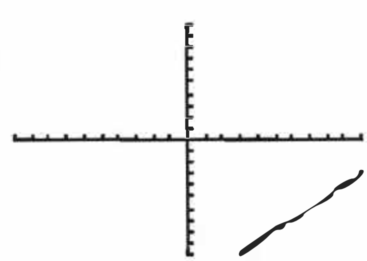

Figure 3

X	Y1	Y2
-3	-16	-16
-2	-15	-15
-1	-14	-14
0	-13	-13
1	-12	-12
2	-11	-11
3	-10	-10

X= -3

Figure 4

```
Plot1 Plot2 Plot3
\Y1=X-5
\Y2=√(X-5)
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
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Figure 5

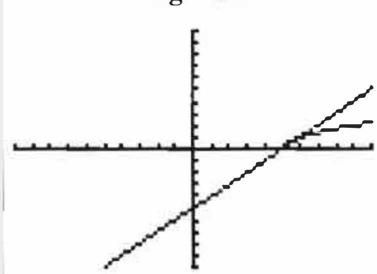


Figure 6

X	Y1	Y2
4	-1	ERROR
5	0	0
6	1	1
10	5	2.2361
21	16	4
25	20	4.4721
30	25	5

X=30

Here, then, we have $x-5 \geq 0$ if and only if $x \geq 5$. On the calculator, we could use TBL SET: (ignore TblStart and ΔTbl) Independ ASK, Depend AUTO. Students will quickly relate the domain of $y = g(x) = \sqrt{x-5}$ to the table of values shown (Figure 6); they will quickly realize that if they wish to avoid "ERROR" in the column of values for Y_2, then they must input a value for x that is at least 5. Mathematics is all about making connections, and we certainly may deal with $f(x) = \sqrt{ax+b} = \sqrt{a\left(x+\frac{b}{a}\right)}$ as either a transformation of $y = \sqrt{x}$ or a composition of $y = g(x) = \sqrt{x}$ with $y = h(x) = ax+b$ to obtain $y = f(x) = g(h(x)) = \sqrt{h(x)} = \sqrt{ax+b}$.

As an extension of the foregoing, one could easily ask a class to determine the set of values for x so that the function $f(x) = \sqrt{ax+b}$ will generate real numbers. If you include a statement such as "Your answer must be in terms of a and b," then any advantage attributed to the graphing calculator is removed from the question.

This same concept is easily extended to $y = \sqrt{ax^2+bx+c}$ and so on, since, generally, the function $y = \sqrt{f(x)}$ will generate real numbers for output if and only if $f(x) \geq 0$.

In the unit on relations and functions in Math 10P, students could be asked to write a function for the sum of a number and its reciprocal. Hopefully, students will contribute something of the form $h(x) = x + \frac{1}{x}$. Here again, a discussion of domain may ensue, and perhaps even a sketch of the graph *without* using the calculator; students will surely know by now that the graph of $f(x) = x$ is linear and, if nothing else, they may be guided towards sketching the graph of $g(x) = \frac{1}{x}$. If both f and g are sketched on the same plane, it is obvious that there are points of intersection at $(1,1)$ and also at $(-1,-1)$. (Can anyone name every real number that is its own reciprocal?)

Students may even be able to identify the minimum that occurs at $(1,2)$ and the maximum that occurs at $(-1,-2)$ on the graph of $h(x) = x + \frac{1}{x}$ by using addition of ordinates. In support of these techniques, students may produce the graph of $y = x + \frac{1}{x}$ on their calculator (ZOOM6, then ZOOM2) (Figure 7 and Figure 8).

Figure 7

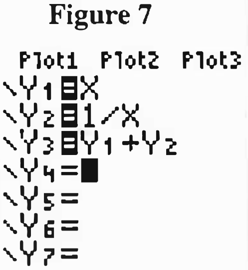
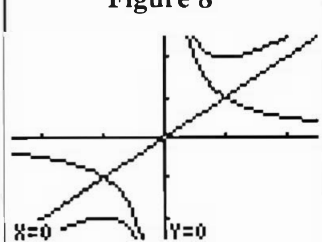


Figure 8

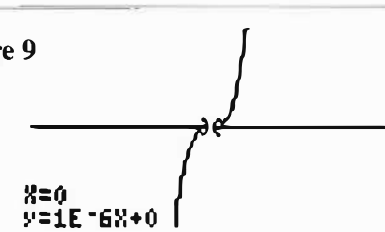


The mathematics is plentiful here! Discussions with respect to addition of functions to produce a new function (if $f(x) = x$ and $g(x) = \frac{1}{x}$, then $h(x) = f(x) + g(x) = x + \frac{1}{x}$), domain (function h is defined only where f and g are *both* defined), range, maximum and minimum, and even limits are all possible here. In the calculus classroom, students may do any or all of the above, but then, of course, the techniques of differentiation and first or second derivative tests for maxima or minima will take us to the critical points of the function. Prior to the introduction of calculus, Dr Lagu's technique of reasoning that $y = x + \frac{1}{x}$ is minimized (or maximized, depending on the domain in which we are working) when $x = \frac{1}{x}$, which solves to give $x = \pm 1$, may be employed. Now we must sort out what occurs where and, at the non-calculus level, that involves the definition of maximum of a function: a function f has a local (or relative) maximum at $x = c$ if $f(c) \geq f(x)$ when x is near c. A similar definition exists for the local minimum of a function.

In the beginning calculus class, students will undoubtedly study Fermat's Theorem, which states that "If f has a local maximum or minimum at $x = c$, and if the derivative, $f'(c)$ exists, then $f'(c) = 0$" (Stewart 2003, 282). Fermat's Theorem merely says that if we are at a point on the graph of a function where either a local maximum or minimum exists, and if the slope of the tangent can be expressed as a real number, then the value of that real number is 0. That is, the tangent line must be horizontal.

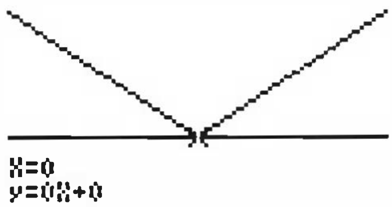
Students almost always assume that the converse is also true; that is, if the slope of the tangent is 0 at a particular point, then a maximum or minimum must occur there. One way to dispel such thoughts is to produce the graph of the function $y = x^3$ (Figure 9), but first, turn the coordinate axes OFF (2nd ZOOM (FORMAT) and down arrow to Axes Off ENTER). The tangent at $x = 0$ is easily drawn by using 2nd PRGM (Draw 5 ENTER). Clearly, no maximum exists at $x = 0$ for the graph of $y = x^3$, even though the slope of the tangent line is 0. The calculator gives the equation of the tangent as $y = (-1 \times 10^{-6})x + 0$.

Figure 9



A second counter-example to the converse of Fermat's Theorem is provided using the function $g(x) = |x|$, and this example shows a definite weakness in the capability of the graphing calculator.

Figure 10



The graph of $y = g(x) = |x|$, axes OFF, and the "tangent" drawn to the graph at the origin with the equation $y = 0x + 0$

The function $y = f(x) = |x|$ is expressed piecewise as $y = f(x) = \begin{cases} -1 \cdot x, x < 0 \\ x, x \geq 0 \end{cases}$. The derivative of this function is $y' = g'(x) = \begin{cases} -1, x < 0 \\ 1, x > 0 \end{cases}$. The derivative does NOT exist at $x = 0$, since $\left(\lim_{x \to 0^-} f'(x) = -1\right) \neq \left(\lim_{x \to 0^+} f'(x) = 1\right)$.

This is an instance that always generates much discussion and centres on the definition of *derivative*. One of the forms of this definition that many texts use is $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{(x+h) - h} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$; this definition contains a quotient that is sometimes referred to as the *difference quotient*.

The calculator, on the other hand, uses a definition that involves the limit of what is called the ***symmetric** difference quotient*, which is written algebraically as $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x-h)}{(x+h) - (x-h)} = \lim_{h \to 0} \frac{f(x+h) - f(x-h)}{2h}$. Using this definition and $y = f(x) = |x|$, we obtain $f'(x) = \lim_{h \to 0} \frac{|x+h| - |x-h|}{2h} = \lim_{h \to 0} \frac{(x+h) - (x-h)}{2h} = \lim_{h \to 0} \frac{x}{h}$.

The value of h will default to 0.001 (the calculator's ability to take limits is "limited"), so it essentially evaluates the last expression as $\frac{2(0)}{2(0.001)} = 0$. Geometrically, the calculator approximates the slope of the tangent at $x = 0$ by using a sequence of secants where the endpoints of each secant define a closed interval that is (horizontally) symmetric across the origin. Since each of those secants has a slope of 0, it should be no surprise that the calculator reports the slope of the tangent at the origin as 0. Now, ask your students if they can find other functions for which the calculator will give incorrect values for slopes of tangents, and get out of the way!

There are countless examples showing that graphing calculators can help students develop a stronger understanding of the concept of limits, which is a difficult topic for many students.

Consider the function $f(x) = \frac{x^2 - 4}{x - 2}$ and, more specifically, $\lim_{x \to 2} f(x)$. Hopefully, the student will realize that attempting to evaluate $f(2)$ will produce the indeterminate form $\frac{0}{0}$. The graph of $y = f(x)$ is shown (Figure 11) using the window settings: $x[-1, 5, 1]$ $y[-2, 6, 1]$. With an appropriate window setting, students can clearly see the "hole" in the graph (a point or removable discontinuity) at $x = 2$. Discussions will certainly centre on the difference between this function and the simplified function $g(x) = x + 2$. This can help the student understand that for both $f(x)$ and $g(x)$, the limiting value as x approaches 2 is 4, regardless of whether or not $f(2)$ exists. The table of values from the calculator (Figure 12), where $Y_1 = \frac{x^2 - 4}{x - 2}$ and $Y_2 = x + 2$, also aids in that discussion.

Now define the function $g(x) = \frac{1}{f(x)} = \frac{x - 2}{x^2 - 4}$ and consider $\lim_{x \to -2} \frac{x - 2}{x^2 - 4}$ to enhance students' understanding of the difference between a point (removable) discontinuity and an infinite (nonremovable) discontinuity.

A second limit example involves the very interesting function $f(x) = x \cdot \sin\left(\frac{1}{x}\right)$ and more specifically the $\lim_{x \to 0}\left(x \sin\left(\frac{1}{x}\right)\right)$. If we let $x = 0$, we get $f(x) = 0 \cdot \sin \infty$,

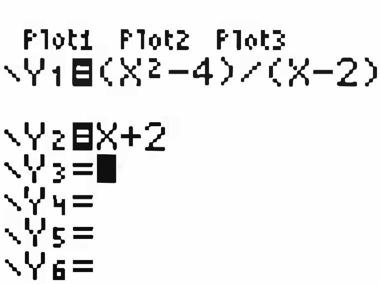


Figure 11

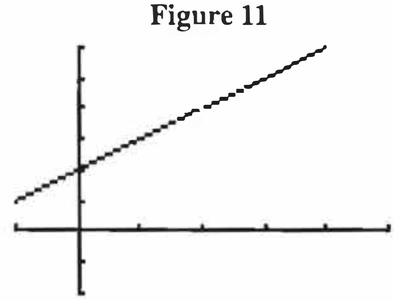
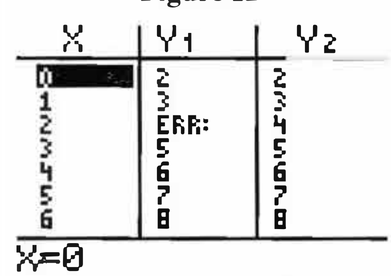


Figure 12



which is indeterminate. Algebraically, we could get help from L'Hospital's Rule, but this is not part of the high school curriculum. Let's see what the graphing calculator can do. When $f(x)$ is graphed on the window $x[-\pi,\pi,0]$ $y[-0.5,0.5,0]$ (Figure 13), the student may be led to believe that the limit value does not exist, as the graph shows somewhat chaotic behaviour near zero.

When regraphed using a Zoombox window around the origin (Figure 14), the graph appears to be more predictable near the origin; it now appears that the limiting value is zero. A table of values using the Ask feature on the Independent variable seems to confirm this. Algebraically, it is also possible to relate this limit to the more common function $y=\frac{\sin\theta}{\theta}$, since

$$\lim_{x\to 0}\left(x\sin\left(\frac{1}{x}\right)\right)=\lim_{x\to 0}\left(\frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}}\right)=\lim_{\frac{1}{x}\to\infty}\left(\frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}}\right)=\lim_{\theta\to\infty}\left(\frac{\sin\theta}{\theta}\right)=0,$$

where $\theta=\frac{1}{x}$.

In our minds, this development is useful not just in providing a visual way to see limit values, but also in teaching students to be careful with the calculator window in making judgments about functions and associated limits.

Our third and final limit examples are $\lim_{x\to\pm\infty}\frac{x}{\sqrt{x^2-x+1}}$. Intuitively, we should realize that as $x\to\infty$, the radicand in the denominator behaves like x^2, and the denominator will therefore behave as x. Since the quotient is now essentially $\frac{x}{x}$, then $\lim_{x\to\infty}\frac{x}{\sqrt{x^2-x+1}}=1$. Similarly, $\lim_{x\to-\infty}\frac{x}{\sqrt{x^2-x+1}}=-1$, since the principal root in the denominator ensures that the quotient will always be negative when $x<0$.

The graph of $f(x)=\frac{x}{\sqrt{x^2-x+1}}, x\in\Re$ (Figure 15) with window settings x [-20, 20, 5], y [-2,2,1] illustrates that 1 is the limiting value as $x\to+\infty$, and -1 is the limiting value as $x\to-\infty$.

Students are surprised to see a function having not one, but two, horizontal asymptotes. Students' past experience with rational functions—a rational function never intersects its (vertical) asymptotes—will likely cause them to be surprised that this function *does* intersect one of its horizontal asymptotes. They also find it surprising that both limits are approached from above.

Algebraically, the limits in question are easily determined:

$$\lim_{x\to\infty}\frac{x}{\sqrt{x^2-x+1}}=\lim_{x\to\infty}\sqrt{\frac{x^2}{x^2-x+1}\cdot\frac{\frac{1}{x^2}}{\frac{1}{x^2}}}=\lim_{x\to\infty}\sqrt{\frac{1}{1-\frac{1}{x}+\frac{1}{x^2}}}=1,$$

since $x=\sqrt{x^2}$ whenever $x\geq 0$.

Figure 13

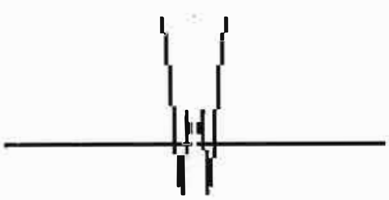
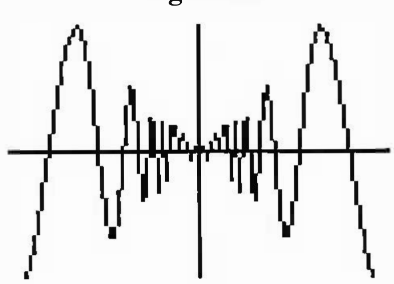


Figure 14



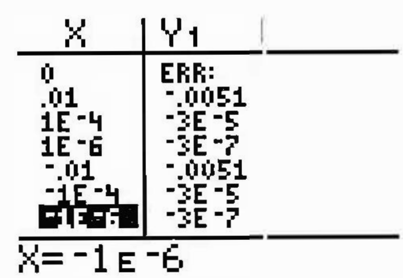
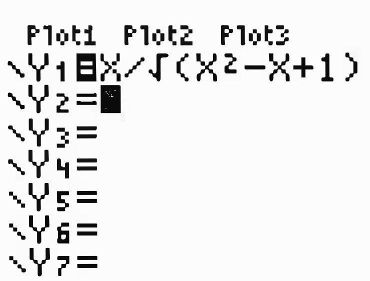
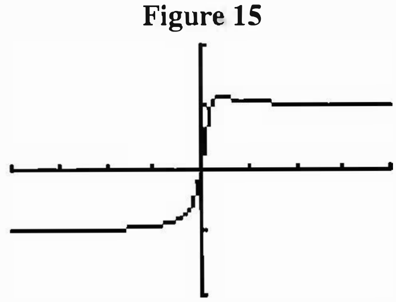
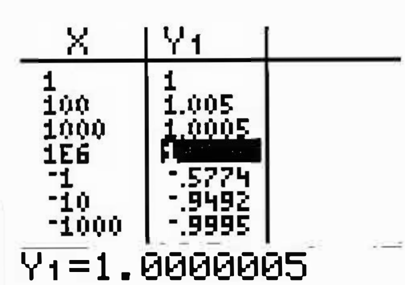


Figure 15







Similarly, $x = \sqrt{x^2} = |x| = -1 \cdot x$ whenever $x < 0$, so that

$$\lim_{x\to-\infty} \frac{|x|}{\sqrt{x^2-x+1}} = \lim_{x\to-\infty} \frac{-1 \cdot x}{\sqrt{x^2-x+1}} = -1 \cdot \lim_{x\to-\infty} \frac{x}{\sqrt{x^2-x+1}} = -1 \cdot 1 = -1$$

In conclusion, we believe that the examples shown provide strong evidence that the use of graphing calculators enriches the high school mathematics curriculum by providing meaningful connections among all aspects of mathematics. We applaud Alberta Education for continuing to mandate the appropriate use of technology in high school mathematics and science classrooms in Alberta. We encourage teachers at all levels to make use of the many features of graphing technology to enhance mathematics education for their students so that they may consider mathematics problems not only algebraically, but also numerically and graphically.

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Assessing Attitudes Through Student Writing

Lorelei Gibeau and Katherine Willson

Attitude and motivation have a profound impact on mathematics teaching and learning. Students who view mathematics as useful, worthwhile and sensible (Kilpatrick, Swafford and Findell 2001) are more likely to invest the cognitive energy required to become truly proficient mathematicians.

Recent reforms in mathematics education clearly acknowledge the importance of the affective domain in mathematics teaching and learning. The development of positive attitudes towards mathematics is now incorporated into many curriculum standards documents, such as the National Council of Teachers of Mathematics *Principles and Standards for School Mathematics* (2000) and the *Alberta Program of Studies for K–9 Mathematics* (2006). However, many children still develop negative attitudes towards mathematics, often at a very young age. Research shows that these negative attitudes are learned (Middleton and Spanias 2002) and that teaching practices greatly influence student motivation (Kazemi and Stipek 2002).

How can teachers create classroom environments characterized by enthusiasm for mathematics and high levels of engagement in challenging tasks? This issue was addressed in a year-long professional development project that involved eleven Grade 3 and Grade 4 teachers from schools representing a wide range of socioeconomic backgrounds. The Collaborative Project in Mathematics Professional Development focused on assessing student attitudes and mathematical thinking through writing. Teachers met once a month to share student work samples and discuss topics such as using open-ended tasks, asking better questions to promote student thinking and setting criteria for a good mathematics task. Student attitudes and motivation were the main topic of the September and June sessions.

September Attitude Tasks

In September, teachers read the article "Writing in Mathematics: Assess the Affective Domain" (McKay, Willson and Wolodko 2000). This article highlights the benefits of using writing tasks, including open-ended journal prompts and mathematical metaphors, to assess student attitudes and perceptions of mathematics. Teachers then created their own tasks to assess student attitudes towards mathematics. These included

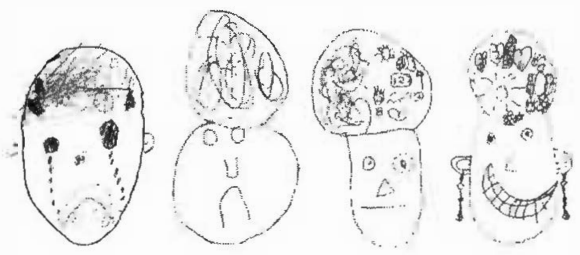
- Journal prompts
  - In your own words, explain what math means to you.
  - What do you like/not like about math?
  - When I do math, I feel …
  - What does math mean to you?
  - What are some words or symbols that come to your mind when you think of the word *math*?
  - Math is …
- Math art posters
  - A study of colour and contrast in art became the vehicle for students to create vivid posters that expressed their feelings toward math.
- Word webs
  - Grade 3 students created word webs using the prompt *Math is …*; and
- Math heads
  - One of the project teachers asked a Grade 1 colleague to assess student math attitudes. The teacher asked her students to draw "math heads" (Figure 1) that described what came to mind when they thought of mathematics (in their brain) and how they felt about mathematics (on their face).

Student Attitudes Towards Mathematics

A wide variety of student attitudes towards mathematics was evident in the writing samples collected in September. For example, the "math head" samples showed that children can have strong feelings about mathematics at a very young age.

Figure 1

Grade One "Math Heads"



One Grade 3 teacher using the prompt *What is math?* found very little negativity in her class towards mathematics. The teacher was pleased to find that there seemed to be a general consensus among students regarding the importance of mathematics in the real world.

In contrast, many students from the other classes in the project showed more negative attitudes towards mathematics. One Grade 3 student's metaphor in response to the stem *Math makes me feel like ...* was particularly powerful (Figure 2)—the student likened mathematics to running through a door without turning the handle.

Teachers discovered that positive attitudes toward mathematics are not always correlated with ability. Some children identified as strong math students by their teachers displayed negative attitudes towards the subject. For example, a Grade 4 teacher shared an art poster created by a child identified as the best math student in the class. The student drew a dark image of a boy spiralling down into a black hole and wrote words such as *boring*, *pathetic*, *death*, *scary* and *confusing* to describe his feelings toward mathematics. In contrast, a student in the same class, described by the teacher as a struggling math learner, drew a brightly coloured poster with images of smiling children.

Student Conceptions of Mathematics

Samples across the different Grade 3 and 4 classes showed that the children's conception of mathematics was closely tied to counting strategies and the four arithmetic operations. Noting a "mixed feeling tone" in the class, one of the project teachers stated that most students in her Grade 3 class viewed mathematics as consisting of number awareness and the operations, although the time unit they were working on at the time appeared in some student writing samples.

The teacher also noted that many students were apprehensive about the upcoming year, because they had heard about the challenges of the Grade 3 mathematics program, especially multiplication and division.

Open-ended journal prompts also revealed that many students' perceptions of mathematics emphasized getting the correct answer and completing drill exercises. Finding the correct answer was often linked to positive feelings. "Sometimes I feel good if I get them rite [right]" (Figure 2). This sentiment was also expressed by another student in the same class, who wrote, "When I do math I feel if I'm having truble [trouble] I don't feel good but if I know the anser [answer] I feel good that I know the anser." Figure 3 illustrates the conception of math as drill and practice. Other students' writing and drawing clearly supported this narrow view of mathematics learning.

Figure 2

A powerful negative image of mathematics by a grade three student

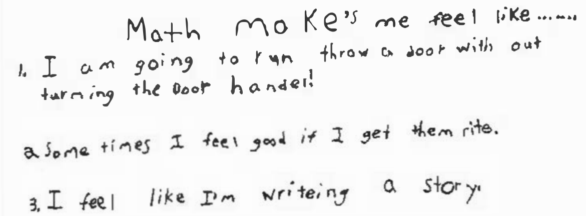
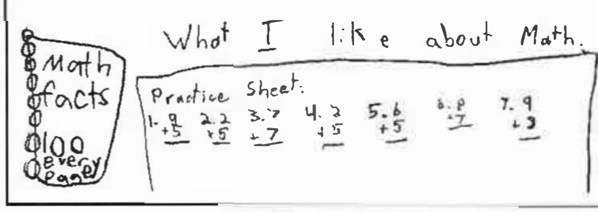


Figure 3

Conceptions of math as drill and practice



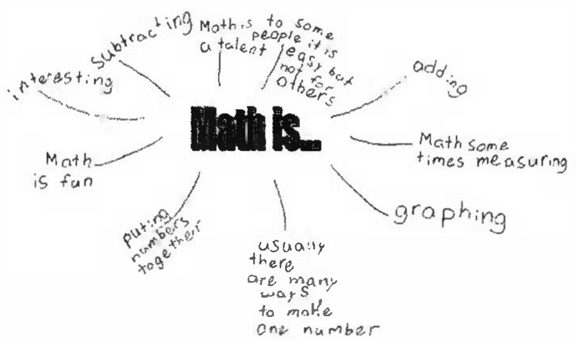
Another Grade 3 teacher who used the prompt *When I do math, I feel...* found that her students had similar concerns about correct answers. Her students also viewed mathematics as basic fact drills. She was dismayed to find that her students also equated success in math with getting the answer quickly and viewed incorrect answers as failures.

Student Beliefs About Mathematical Ability

Student writing samples also revealed the common belief that success in mathematics can be attributed to innate ability rather than effort. Grade 3 students created word webs centred on the prompt *Math is ...* to describe mathematics. As shown in Figure 4, many students had developed the belief that being good at math is more a personality trait than a goal for all learners. This student wrote "Math is a talent" and "to some people it is easy but not for others." Studies show that students who "conceive of ability as *fluid*, or subject to improvement through effort ... are better achievers than students who believe that mathematical ability is *fixed*." (Middleton and Spanias 2002, 10).

Figure 4

Students' perception that mathematical ability is innate or fixed



Incorporating Good Mathematical Tasks and Math Journals

Early in the project the teachers shared student writing samples they had collected in their classrooms. These student samples helped teachers develop their own criteria for a good mathematical task—one that provided a) student engagement, b) a range of possible solutions and/or solution strategies, c) access to significant mathematics, d) a meaningful context that supports student thinking and e) developmental appropriateness.

Project teachers experimented throughout the year with alternative forms of oral and written assessment that elicited student thinking. Through professional readings and collegial dialogue, teachers focused on incorporating more student writing in mathematics. This was achieved mainly through the use of open-ended math tasks and the use of math journals.

Several of the teachers had used mathematics journals, at least occasionally, in their classrooms, while some had never used a math journal before. As the project progressed, many teachers incorporated journalling as a regular part of their mathematics lessons.

In June, student writing samples were collected to help teachers assess the impact of the project on student attitudes and motivation. Teachers wondered if their focus on open-ended tasks and journalling throughout the year would affect student attitudes and conceptions of mathematics.

June Attitude Tasks

Most teachers used writing activities similar to those they used at the beginning of the project to reassess student attitudes. A few teachers experimented with a different activity that they felt would give them a more comprehensive picture. Attitude tasks included

- Journal prompts
  - Write a letter to Grade 3 students to tell them what to expect in Grade 4 math.
  - Write about your personal feelings towards mathematics.
  - Compare your thoughts and feeling about mathematics now as compared to the beginning of the year.
  - How do you feel about math since using the math journals?
- Math art posters
  - Create a poster to advertise what Grade 3 math is all about.

  and
- Word webs
  - students re-created word webs using the prompt *Math is ....*

Student Attitudes Toward Mathematics

In general, teachers stated that more positive attitudes were evident in the June writing samples. Students often expressed a sense of pride in their accomplishments. Many children wrote about personal learning gains or overcoming challenges throughout the year. This reinforces findings in motivation research that indicate students develop more positive attitudes if they interpret their successes and failures as being within their own control (Middleton and Spanias 2002).

Another notable change was a shift from judging mathematics as good or bad based on how easy it is to valuing mathematics as a subject worthy of effort. In September, negative attitudes were often accompanied by comments that mathematics is "hard." At the end of the year, many children stated that they liked mathematics despite the challenges it poses.

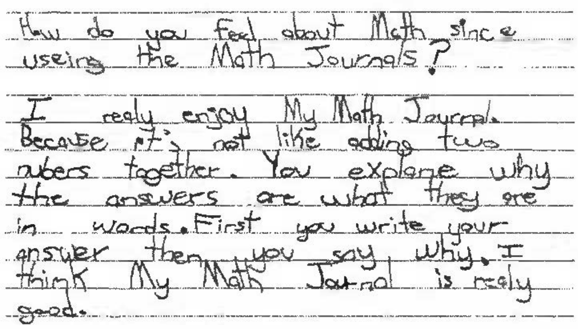
The June writing samples also revealed that journalling had a positive impact on student attitudes.

One teacher asked her Grade 3 students to write specifically about their experience with mathematics journals (Figure 5). Students stated not only that they liked working in their journals, but that they saw them as a valuable learning tool. As one teacher stated at the end of the project, "Incorporating journals introduced a new perspective on math; eg, rather than just worksheets/textbooks, the students could share ideas individually, in pairs or in large groups. Quiet students spoke up, where normally they wouldn't say anything."

All teachers agreed that the project had a positive impact on their students—especially on their attitudes towards mathematics and their conception of what is means to "do math." Many teachers attributed this also to their own enthusiasm for learning and trying out new ideas and approaches.

Figure 5

Student reflections on math journals



How do you feel about Math since useing the Math Journals?

I realy enjoy My Math Journal. Because it's not like adding two nubers together. You explane why the answers are what they are in words. First you write your answer then you say why. I think My Math Journal is realy good.

Student Conceptions of Mathematics

Various student samples collected in June showed that students held a much broader conception of mathematics than they had held the previous fall. Although many students wrote about addition, subtraction, multiplication and division, they also identified graphing, symmetry, measurement, probability and geometric shapes as part of mathematics.

Using mathematical tasks set in real-life contexts also seemed to have influenced students. As shown in Figure 6, students wrote about how mathematics was important and useful in the real world. Other comments from student writing samples included statements such as "Math is pretty much all around us" and "I will always know that math is important." Teachers confirmed that making connections between mathematics and real-life situations was key to improving student attitudes, as illustrated by the following quotes.

- "Connecting math to a story or mind map or project relating to life application was the real motivator."
- "Talking math was great. Students talked about the application of math and how important it is in the real world. Students were able to see the value of math."

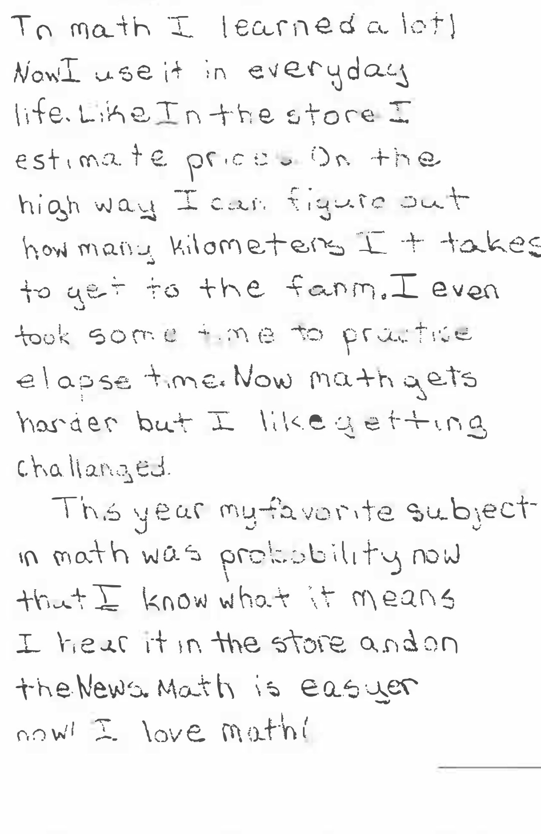
In general, teachers felt that students emphasized right answers much less in their year-end journal reflections. Instead, students began to write about using strategies and sharing their thinking. Below are some examples of student comments illustrating their conceptions of math at the end of the year:

- "Math gets your brain started."
- "Math is a way that helps you learn."
- "I think of math like a great thing because it has thinking things."
- "Math is ... thinking."
- "Math is ... strategies."

One Grade 4 student, whose task was to write a letter to the upcoming Grade 3 class telling them what to expect in Grade 4 math, was particularly encouraging. The student began his letter by writing "In Grade 4 math you will have tons of fun. There are so

Figure 6

Math is useful in the real world



In math I learned a lot! Now I use it in everyday life. Like In the store I estimate prices. On the high way I can figure out how many kilometers I t takes to get to the farm. I even took some time to practice elapse time. Now math gets harder but I like getting challanged.

This year my favorite subject in math was probability now that I know what it means I hear it in the store and on the News. Math is easyer now! I love math!

many different ways to learn math besides worksheets, quizzes and tables. When you learn math in fun ways, it makes you want to learn."

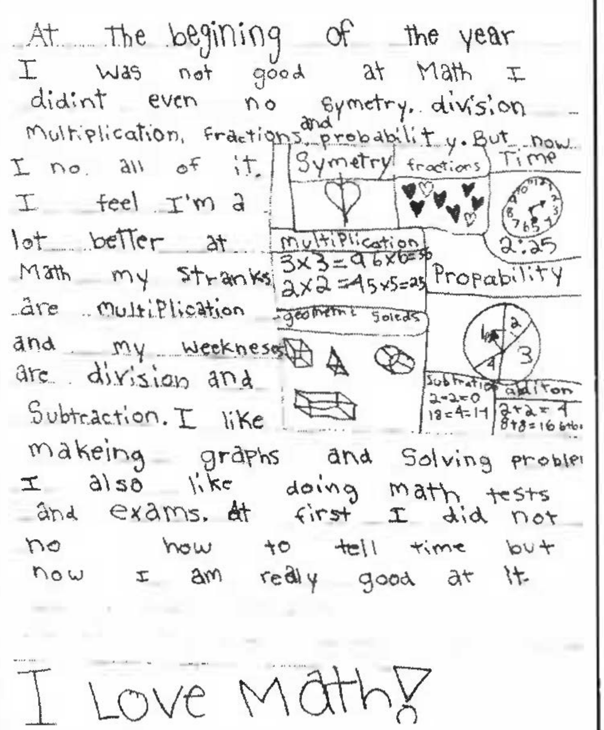
Teachers felt that as they selected better mathematics tasks and incorporated writing into daily instruction, students began to recognize that mathematics could go beyond memorizing procedures and completing practice exercises.

Student Beliefs About Mathematical Ability

In June, none of the students identified themselves as "bad at math." Only a very few stated that they were "good at math" in general. Instead, students simply identified personal areas of strength or weakness. Many stated that subtraction and division, for example, were things they had to work on. As illustrated in one Grade 3 student's journal entry (Figure 7), students became more reflective about their own learning. In June, students' self-assessment samples seemed to indicate that they believed that mathematics was within their reach if they tried. This was an important shift in student thinking from what the teachers had observed in September.

Figure 7

Students reflect on their learning



Conclusion

Examples of year-end reflections from the teachers sum up the influence of the year-long project on their students:

- My students, I believe for the first time in their "mathematical lives," really recognized and felt the freedom to be able to approach mathematical tasks knowing that there are always a variety of ways to solve a problem or represent a situation. I also am grateful for the positive attitude that all my students demonstrated towards themselves as math learners.
- Kids became more relaxed about trying things that reflected their understanding rather than "what I think the teacher wants."
- I think my students have more tools now in math and a broader understanding of what math is.
- I believe my students have gained a deeper understanding of math and are better suited to handle future learning.
- Students in my class were previously taught math skills/concepts in a traditional manner. The students became more confident in their mathematical abilities and showed great enthusiasm when using these innovative and exciting ideas.

The project illustrated that a focus on student writing and open-ended, engaging real-life mathematical tasks can increase student motivation and improve conceptions of mathematics. It is important that teachers develop an awareness of student attitudes as early as possible, because children's dispositions towards mathematics develop at a young age and are often difficult to change.

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Rationale Gone Missing: A Comparative and Historical Curriculum Search

Lynn McGarvey

Whether it is appropriate or not, there is an implicit hierarchy in the subject areas taught in public schools. Educational institutions are notoriously underfunded and have limited resources. Educators, particularly those in nonacademic subjects such as fine arts or physical education, need to simultaneously defend the importance of their subject areas and actively lobby for their fair share of time, money, equipment and personnel. As a mathematics educator, I often feel I am letting my subject area down. I usually leave my soapbox tucked neatly under my desk and keep quiet. Yet, even in my silence there is an assumed value and importance given to school mathematics by educators, parents, policy makers and most citizens. I have listened patiently to people explain how horrible they are in math but in the next breath say how important mathematics is for such things as everyday calculations, postsecondary institution admission, future job prospects and future income possibilities. Not satisfied with any of these reasons, I examined the front matter of the *Alberta K–9 Mathematics Program of Studies* (Alberta Education 2007)[1] in search of a rationale for teaching mathematics. Perhaps it would provide me with a thoughtful reason to dust off my soapbox.

As I read, I was bemused by the fact that the introduction included the heading, "Purpose of the Document," but the purpose of the content or of the entire subject was not clear. I read through the program of studies, asking the question, "Why is mathematics important for children?" Nothing in the early pages answered the question. Instead, the document provides goals for students such as, "to prepare students to use mathematics confidently to solve problems" (p 4). But what are the problems to be solved? If the main goal is to solve the problems that mathematics educators give students in the course of learning mathematics, then this goal is a self-referencing and self-serving. The goals chosen must be based on a rationale of some sort, but I couldn't find it.

I continued to sift through the document asking myself, "Why is mathematics important?" I found one sentence under "Nature of Mathematics" that was at least a partial response: "Mathematics is one way of trying to understand, interpret and describe our world" (p 10). This statement could form the basis for a rationale. It is, in fact, similar to the rationale provided in the science program of studies (1996) which appears under the explicit heading, "Rationale." It states: "Learning about science provides a framework for students to understand and interpret the world around them" (p 1). The statement in the mathematics program of studies is very similar, but it was buried on page 10 and no further explanation is given. Nowhere in the document is an explicit rationale provided.

On one hand, I was relieved that no attempt was made to pinpoint a rationale. Mathematics educators certainly do not agree on a singular, appropriate reason to teach mathematics—especially to all children (eg, Davis 2001; Huckstep 2000; Noddings 1994). If a rationale had been written, it surely would have been criticized by several stakeholders. On the other hand, I was dismayed. With no clearly defined vision for why mathematics is important, what assumptions are hidden beneath the choice of goals for students, content inclusion and exclusion, authorized resources, and so on?

Perhaps an explicit rationale in a program of studies is passé. I continued to think about my colleagues in other subject areas and their ability to put forth a united front regarding the purpose of their subject areas and their insistence that their subject is a vital component of schooling—at least in elementary schools. Out of curiosity I opened the Programs of Study (Core Curriculum) site from Alberta Education.

[1] All references to programs of studies are to programs produced by the Department of Education (1914–68), Alberta Education (1975–85; 2000–08) and Alberta Learning (1997).

Because I am a former physical education teacher, I opened the *K–12 Physical Education Program of Studies* (2000b) to see if it still included a rationale. It does. In fact, in contrast to the mathematics curriculum, the physical education document starts with "**Program Rationale and Philosophy**" in big bold letters. I asked the question, "Why is physical education important for children?" Part way through the first page I read:

> Physical activity is vital to all aspects of normal growth and development, and the benefits are widely recognized. Students do not develop automatically the requisite knowledge, skills and attitudes that lead to active, healthy lifestyles. Such learning should begin in childhood. Schools and teachers can be prime facilitators in providing opportunities for the development of the desire for lifelong participation in physical activity. (p 1)

Although we can critically dissect this statement and find faults, unlike the mathematics curriculum, the physical education document places a strong rationale for the subject up front. Not only does the document provide a justification for physical education, it also addresses the vital role that schools and teachers can (or should) play.

I also opened the programs of study for other marginalized subjects, including drama (1985b), art (1985a), music (1989) and health (2002). Each one started with "Program Rationale and Philosophy." Other academic subject areas begin the same way. Science, as I mentioned, starts with an explicit rationale. Social studies (2005), the program of studies that has been most recently revised, also provides extensive information on rationale, a program vision, a definition and the role of social studies in schools.

The remaining core subject, English language arts, arguably tops the hierarchy of subjects, particularly in elementary schools. As I had done with other subjects, I opened the document and started searching with the question, "Why is learning language arts important?" The word *rationale* is not used in the *K–9 English Language Arts Program of Studies* (2000a). Instead, the document begins with "The Importance of Language" followed by the subheading, "The nature of language."

> Language is the basis of all communication and the primary instrument of thought. Composed of interrelated and rule-governed symbol systems, language is a social and uniquely human means of exploring and communicating meaning. As well as being a defining feature of culture, language is an unmistakable mark of personal identity and is essential for forming interpersonal relationships, extending experiences, reflecting on thought and action, and contributing to society. (p 1)

Although the subheadings "the nature of language" and "the nature of mathematics" are parallel, their treatment and placement in the two documents are not at all equal.

Regardless of where the subject area was on the perceived hierarchy, all current curriculum documents, except mathematics, provide a (more or less) persuasive response to the question of why that subject matter is important.

Why, then, is no rationale for mathematics education provided? Is the justification so obvious that inclusion is not needed? Is it simply an oversight? Or were the writers not able to reach an agreement? Has an explicit rationale for teaching mathematics to children ever existed? This last question prompted an examination of the 1996 program of studies, then the 1982 curriculum, and compelled me to look through the dusty historical collection of almost a century of curriculum guides for elementary mathematics.

A Century of Rationales for Mathematics in Schools

Beginning in 1914, most Alberta programs of study for elementary schools begin with a general introduction and aims or guiding principles for education in schools. The introduction is then followed by pages devoted to individual subject areas, usually with their own set of aims that are usually aligned with and reference the general goals of education previously stated. In arithmetic/mathematics, several common aims seemed to resurface throughout the nearly 100-year history that I reviewed. The following discussion highlights a few recurrent aims and assumptions pertaining to the importance of mathematics.

Perhaps a place to start is with the assumption that mathematics sits at or near the top of the hierarchy of school subjects. In 1918, the *Course of Studies for the Public Schools Grades I–VIII* includes "academic subjects" along with nature study, art, manual and household arts, physical training, hygiene and music. The teacher is advised to follow the outline in each subject, "but should give each pupil a thorough training in Reading, Writing, Spelling, Oral and Written Composition, and Arithmetic, as these subjects form the basis for future progress of education" (J T Ross, introductory note).

The supposed supremacy of literacy and numeracy has existed without change since the inception of public schooling. This viewpoint is expressed clearly in 1918, but similar views are expressed several times in the historical documents: "From the earliest times mathematics has occupied an honored place in the courses of study that have been pursued" (Department of Education 1924, 144). Also, "[f]rom time immemorial mathematics has played, and will continue to play, an important role in the history of man's existence and for this" and other reasons provided, "mathematics is considered to be one of the 'basics' of education" (1982, 2). In many respects these statements claim that mathematics is and has always been important in education, but without explicitly stating why. Why does mathematics occupy an "honored place in the courses of study"? The reason that mathematics is more important than hygiene or physical training, for example, is never explained or questioned.

Most Alberta curricula through the years do provide some form of rationale for why learning mathematics is important. Perhaps the most common reason stated is the utility of mathematics to solve everyday problems. However, there appeared to be a shift with respect to what and whose problems were important. The early years of public schooling emphasized "training" in and "mastery" of the four operations to solve "concrete problems" (1918, 7) and "problems ordinarily met within the activities of life" (1924, 144). Also, in Grade VIII (1924), the pinnacle of education for many students, the aim was training in "the genuine problems of life met within ordinary occupations of the community" (p 158). Throughout the years, mathematical skills are valued as a means to solve concrete problems, everyday problems and problems related to community occupations. However, I wonder why the problems related to mathematics are more important than the problems of, say, manual and household arts or hygiene?

Occasionally, the perspective on problem solving in mathematics shifted to a focus on problems arising from the children's present needs and interests. This trend was particularly evident in the "progressive" era of the 1930s, which emphasized a Deweyan perspective. That is, learning should be interesting and relevant for children "in the life they are living as boys and girls (1936, 4). A similar emphasis also appears in 1982: "The [mathematics] program should be focused on the child's world .... An awareness of some real-world applications of mathematics and some of the technological advances *which will directly affect the child's life*, should be imparted to the student" [italics added] (p 5).

The 2007 mathematics curriculum is sufficiently vague in its position on the utility of mathematics to solve problems. While students are expected to be confident problem solvers, the type of problems and their relevance are never specified.

Another common assumption is that learning mathematics is justified because it is required for future learning of mathematics. For example, "[o]f equal importance [to the growth and development of the child] is the aim of providing pupils with the background they will require for the study of mathematics in the later years of their school life (1962, 31; 1963, 24; 1968, 19). Even in Grade 1, "[t]he major objective of a primary number curriculum is to provide through enriching experiences a background of attitudes, appreciations, facts, and skills that will aid in the understanding of the formal Arithmetic of later grades" (1936, 89; 1940, 255). Rationalizing the learning of mathematics in the present for the learning of mathematics in the future is self-serving and rarely convincing.

Mathematics is also said to be important for future careers. In 1997, mathematics is justified because "a greater proficiency in using mathematics increases the opportunities available to individuals" (p 2). Although this rationale appears only once in the documents, it is one of the most common notions I hear from others who explain why they think mathematics is important. However, if "[w]e are teaching for a changing world" (1936, 89), how can we know what the opportunities will be available in a "rapidly advancing, technological society" (1997, 1)?

A final assumption appearing throughout the decades is the focus on the individual learner. In the first half of the last century, educational goals focused exclusively on individuals. Consider these statements from the *Programme of Studies for the Elementary School*, written in 1947:

> The ultimate goal of education is the happiness of the **individual**. Accordingly, the teacher's purpose is to assist each child in the class to unfold as fully as possible his unique potentialities. (1947, 10)

The arithmetic "point of view," in 1947, also focuses on individual learners, but I didn't see anything suggesting that the mathematics program was purposefully contributing to the children's happiness. The view that education should foster "the fullest development of each child's potentialities" occurred repeatedly (1963, 7; 1968, 4), particularly in mathematics.

In 1975, the educational goals shift radically to suggest that "education must provide opportunities to meet individual and societal needs" (p 1). However, the mathematics general objectives make little mention of society and continued to read "the aim of Elementary School Mathematics is to foster continuous and maximum development of each child's potentialities" (p 21). In fact, it emphasizes, "the growth of a mature individual who thinks and acts effectively, and who, as a result *may contribute to society*" [italics added] (p 21). The hedge on making a contribution to society seems particularly odd. In 1978, the educational goals also emphasize fulfilling "personal aspirations while making a positive contribution to society" (p 3), but the mathematics goals in that year focus exclusively on individuals and make no mention of society or social relationships.

Final Thoughts

My quest to locate a convincing rationale for teaching mathematics through almost a century of curriculum documents has been wholly unsuccessful. To my mind, learning mathematics for the purposes of future progress of education, for its utility, for later mathematics learning, for future careers and for individual growth are, at best, inadequate and, at worst, inappropriate. Why do we teach mathematics to children?

The current program of studies says that "Mathematics is one way of trying to understand, interpret and describe our world" (2007, 10). I mentioned previously that this is perhaps one part of a rationale, but it, too, is inadequate. The statement neglects the fact that humans who use mathematics play an essential role in *creating* our world—a mathematical world. Much like the "nature of language" in the English language arts curriculum (2000a), mathematics is also a "primary instrument of thought" and a "defining feature of culture" (p 1). How does mathematics shape our thinking? How has it defined our culture? What has mathematics allowed us to create? How has it constrained or perhaps disallowed possibilities for knowing, seeing, doing and being? Why is mathematics the only "academic" school subject that does not include a critical thinking component or address issues within the discipline? How does our uncritical stance on the value of mathematics prevent us from moving beyond individual development towards cultural implications of a mathematical world? Mathematics is important for children, but there are many more questions that need to be raised and discussed before I'll be comfortable dusting off and stepping on my soapbox.

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Developmental and Remedial Approaches in Mathematics Instruction

Marlow Ediger

There are two approaches in mathematics instruction, referred to as *developmental* and *remediation*, that are significant in instructional settings. These approaches need to be clarified and implemented to improve teaching and learning. A developmentally appropriate mathematics curriculum begins at the learner's current level of achievement; this point represents a starting place for instructional purposes. The chosen objectives for teaching and learning need to be challenging yet achievable.

Some students achieve at a slower rate than others in the same class. A student's failure to achieve requires the teacher to determine where the learner's understanding becomes incomplete. From that point on, new learning opportunities must be presented to the student.

A Developmental Mathematics Curriculum

A developmental mathematics program is carefully designed so that students' differences are adequately provided for in the classroom. Each student is assisted to achieve as optimally as possible. The objectives are challenging, but attainable. Balance among objectives is stressed in that knowledge, skills, and attitudinal ends of instruction are emphasized.

Knowledge objectives must

- stress salient structural ideas and supporting facts. There are key subject matter objectives that all students need to secure. Understanding basic facts, place value, regrouping and renaming, fractions and decimals, area, number systems, volume, and commutative, associative and distributive properties are important subjects that all learners in mathematics need to understand;
- contain higher levels of cognition, such as analytical and creative thinking;
- include problem solving situations, either contrived or real. Thus, important word problems may be written by the teacher as well as by students. Word problems might also come from basal series of textbooks. Actual experiences in number use need to be encouraged by the teacher. Students need to apply, both in school and in society, what has been learned;
- encourage project development with the use of pupil purpose, planning, doing and evaluating in its endeavours; and
- develop logical thinking in mathematics.

Skills outcomes for student achievement in mathematics should include

- accuracy in computation;
- neatness and legibility in written work;
- reading for comprehension and meaning;
- identification of unknown words through phonics, syllabication and/or context; and
- development and use of graphs, charts and tables.

Attitudinal outcomes must emphasize

- appreciating mathematics for its own sake, as well as for its practicality;
- completing assignments on time;
- asking for assistance when it is needed;
- assisting others in committee and small-group endeavours;
- volunteering to do additional work in mathematics; and
- working up to one's optimal ability level (National Council Teachers of Mathematics 1989).

Objectives provide goals for students to achieve. Teacher-written assessments may be formative if they assist students to achieve end-of-unit objectives, or if they result in students achieving No Child Left Behind (NCLB) mandated objectives. Teacher-written objectives are summative if the mathematics unit has been completed and the teacher is evaluating the quality of the unit to determine what changes, if any, should be made for future teaching of that unit. Thorough planning of mathematics units helps to ensure success in teaching, higher student achievement and learner attainment of relevant ends of instruction (Ediger and Rao 2003).

A Remedial Mathematics Curriculum

Remedial work is necessary when specific objectives have not been achieved by students. These deficiencies must be identified carefully. Quality ordering or sequencing of objectives is important. The student-learning sequence may have been interrupted due to the students' lack of attention in class, or the teacher may not have explained a mathematics process carefully. Whatever the cause, the teacher must now plan instruction to diagnose and remedy the involved difficulties. If, for example, a student did not understand regrouping in addition, then the teacher needs to plan and implement remedial instruction. Thus, the teacher may use a place-value chart to indicate the 1s, 10s, and 100s columns, with congruent slips of construction paper used to represent those columns. Regrouping must occur in addition if there are ten or more strips in any one column. Thus, ten strips in the 1s column may be replaced with one paper strip to represent one 10 in the 10s column. This may be explained clearly with the use of a deductive procedure. Alternatively, the teacher may raise vital questions leading students to the correct response through induction. By examining students' written work or oral answers, the teacher may assist in correcting student deficiencies. Thus, improved sequencing should be inherent in student achievement (Kennedy and Tripps 1991).

The mathematics teacher who follows selected guidelines in teaching, whether they are developmental or remedial, will facilitate student achievement of objectives. The following guidelines must be followed by mathematics teachers:

- Students need to be actively engaged in learning; interest is a powerful factor to consider in learning.
- Students need to attach meaning to ongoing learning in mathematics; understanding of subject matter is vital in the curriculum.
- Students must experience quality sequencing in ongoing lessons.
- Students should achieve adequate background learning to benefit from an ensuing lesson.
- Students must perceive purpose in learning. Thus, there are inherent reasons accepted by the learner for achieving objectives of instruction (Ediger 2005).

The qualified mathematics teacher is well prepared in subject-matter knowledge to teach students. The undergraduate and graduate programs taken at an accredited institution have prepared the teacher to be highly proficient in mathematics. An in-depth, demanding set of courses taken in mathematics has assisted the teacher to acquire vital facts, concepts and generalizations.

A second dimension of teaching mathematics is quality pedagogy. Courses in pedagogy taken at an accredited university must develop teacher efficiency in the instructional arena; teachers need to be well trained in planning daily lessons and units of instruction. The total university curriculum in teacher education needs to result in highly competent teaching in the public schools.

Teachers must continue growing and achieving in teaching mathematics. The beginning teacher needs quality assistance from a mentor who is interested in assisting the neophytes to teach well and treats them as equals in a democratic atmosphere. Inservice growth should be provided along the way. Activities such as workshops, courses taken at universities, independent studies, action research conducted in school and attendance at professional conferences should help develop the professional mathematics teacher. Observational visits and follow-up conferences with the mathematics supervisor in clinical settings should further enhance teaching skills and abilities (Cavanagh and Samuels 2006).

Closing

Mathematics teachers need to experience a quality program of teacher education at the undergraduate and graduate levels. The teacher of mathematics must possess confidence to teach well. Good teacher preparation, together with inservice education, will certainly assist in improving mathematics instruction. A good teacher teaches well so that learners might achieve their potential; when they don't, remedial work is necessary.

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A Grade 4 Adventure with Multiplication on the Chinese Abacus

Jerry Ameis

An arithmetic algorithm or procedure is a series of steps performed to obtain an answer to an arithmetic task. Currently, there is a shift towards teaching early- and middle-years students the reasoning involved in an algorithm. The National Council of Teachers of Mathematics (NCTM 2000) supports this shift: "Fluency refers to having efficient, accurate, and generalizable methods (algorithms) for computing that are based on well-understood properties and number relationships" (p 144). Why does the algorithm produce a correct result? What properties are involved?

Past practice has tended to teach multidigit multiplication in a manner that did not involve student understanding but, rather, student mimicry. The distributive property was seldom developed as the critical justification of the algorithm. This article investigates the experiences of five Grade 4 students learning a functional sense of the distributive property that was then applied to multidigit multiplication. The students extended their understanding by working with the Chinese abacus (*suan pan*) and by subsequently teaching early-years preservice teachers those concepts using the abacus. This article focuses on one aspect of the experience concerning the initial development of the distributive property, and the students' experiences teaching early-years preservice teachers multiplication on the Chinese abacus.

I had been working with a number of students on Saturday mornings, over a period of years, as a long-term research project that explored how children learn mathematics and what mathematics they can learn. These students had been having some difficulties learning mathematics in their school settings, and the Saturday sessions supplemented their school experiences. The students were now in Grade 4. During Saturday sessions, the students learned mathematics in a problem-solving climate of learning, where sense-making and meaningful contexts were important aspects of instruction.

Developing the Distributive Property

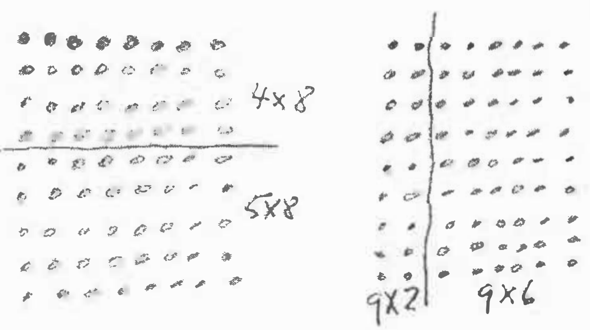
The students were presented with the problem: "Grade 4 students are about 9 years old. Suppose you are exactly 9 years old. How many days old are you?" I decided to ignore leap years in this instance, as I did not want to muddy the waters with an additional feature of the problem. Past experience with these students had indicated that it was not a wise teaching strategy to include matters that were not critical to the situation. The students were asked to represent the problem with a multiplication number sentence. With some guidance that mainly involved the number of days in a year, the students responded "9 × 365." The students were asked to obtain the answer by using repeated addition (with pencil and paper) or place value materials. The students chose to work with place value materials. Their thinking involved making 9 groups of 365 with the materials, trading as needed, and then writing in symbolic form the answer represented by the materials. Four of the five answers were correct. Some of the students had made 9 groups of 300, 9 groups of 60, and 9 groups of 5 in the course of their work with the place value materials. This was an indication that students recognized that the 3 in 365 was in the hundreds place, the 6 was in the tens place, and the 5 was in the ones place, and that multiplication involved groups. However, the students' ability to make groups of 9 using place value does not necessarily indicate that they understood or were consciously aware of the distributive property. That property involves understanding that one or both numbers of a product can be split, with the subproducts added to yield the final result. For example, 8×23 can be thought of as $(3 + 5) \times (15 + 8)$. This particular splitting of 8 and 23 results in the four subproducts $3 \times 15 + 3 \times 8 + 5 \times 15 + 5 \times 8$. Completing the task leads to $45 + 24 + 75 + 40 = 184$. While a different splitting of 8 and 23 would result in different subproducts, the final result would still be 184.

When I asked the students why they made groups of 9 in the way they did, their responses indicated an understanding of place value and multiplication only as a grouping concept (eg, "There are 9 groups of 365 and there are 3 hundreds in 365, so I built the 3 hundreds 9 times by putting down 9 piles of 3 flats").

A discussion followed about the amount of work needed to obtain the answer when using repeated addition or place value materials. I suggested there might be a short cut for multiplying large numbers using pencil and paper and that, before learning it, they first needed to understand an important mathematical idea.

Students were presented with the multiplication task 9 × 8. Most of them knew the answer automatically. They were asked to confirm it by making a dot diagram, splitting the number of rows with a horizontal line, representing each part of the cut using multiplication, and obtaining the answer from the subproducts (see Figure 1). They accomplished the task with some assistance. We discussed various ways of splitting 9 (the number of rows). Students were then asked to split the number of columns with a vertical line, to represent each part of the cut in a multiplication way, and to obtain the answer to 9 × 8 from the parts (see Figure 1). We discussed various ways of splitting 8 (the number of columns).

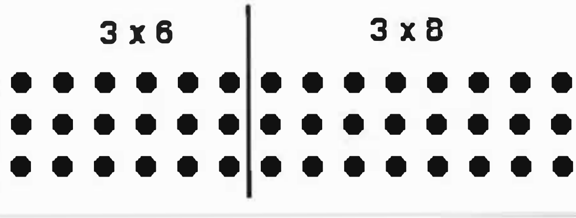
Figure 1
9 × 8 as 4 × 8 + 5 × 8 and as 9 × 2 and 9 × 6



Students were beginning to understand the distributive property in a functional sense. They realized that splitting one number of a product in whatever way they wanted allowed them to think in terms of subproducts that were added to obtain the final result. It was time to develop the strategy of splitting a number using place value concepts because it led to less cumbersome arithmetic.

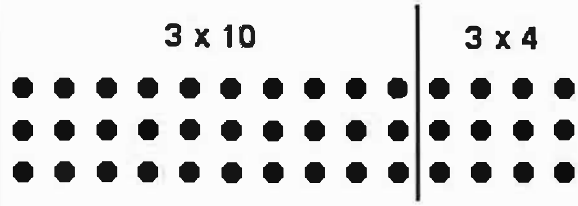
The students were asked to make up a 1-digit × 2-digit task, with the 2-digit number between 10 and 20 and the 1-digit multiplier between 2 and 5. The 1-digit number had to be different from the ones digit of the 2-digit number (eg, 2 × 12 was not allowed). They selected 3 × 14 for the task. They were asked to represent 3 × 14 using rows and columns, to split up the long side of the array using a vertical cut anywhere they chose, and to obtain the answer to 3 × 14 from the two parts (see Figure 2).

Figure 2
3 × 14 as 3 × 6 + 3 × 8



We discussed whether the smaller multiplications were easy to do and whether there might be a better way to split 14. The students were asked to split 14 using place-value thinking for 14 (see Figure 3). We discussed whether place value spitting made the smaller multiplications easier. Those who were adept with multiplying by 10s realized the advantage. One student was not adept and did not readily see the benefit of thinking of 14 as 10 + 4.

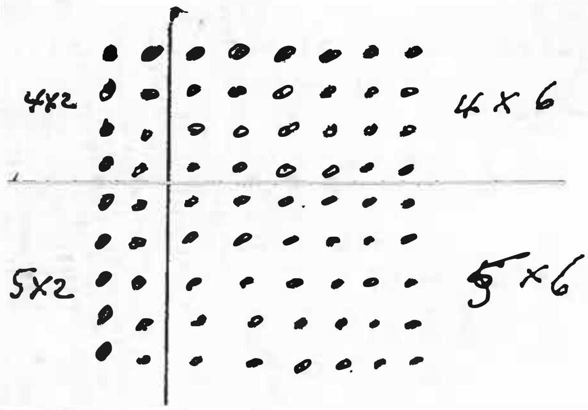
Figure 3
3 × 14 as 3 ×10 + 3 × 4



The students were asked to obtain answers to two more 1-digit × 2-digit multiplications by splitting the 2-digit number into tens and ones (I provided dot diagrams for this). They completed the tasks successfully. I felt it was time to move toward multi-digit multiplication but decided that the students' understanding of the distributive property had to be extended.

We revisited 9 × 8. Students were asked if it might be possible to cut both the 9 and the 8. One student suggested cutting across and down and showed the thinking using a diagram (see Figure 4). The students were asked how the diagram could be used to obtain the answer to 9 × 8. With minimal assistance they realized that each subproduct needed to be found and the results added.

Figure 4
9 × 8 as 4 × 2 + 4 × 6 + 5 × 2 + 5 × 6



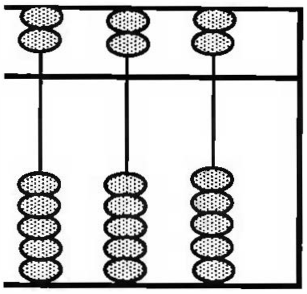
The students were asked to do two more 1-digit × 1-digit multiplication tasks by splitting the dot diagram across and down. We discussed the thinking involved. I asked students to explain in their own words what one could do when multiplying two numbers. A sample explanation follows. It indicates that the students were ready for 2-digit × 2-digit multiplication: "When you multiply two numbers like 7 × 6, you can break up each number and multiply the pieces. Then you add what you get."

Working with the Chinese Abacus

The Chinese abacus was used to strengthen the students' understanding of the functional sense of the distributive property and to extend their understanding of the corresponding multiplication algorithm. In my experiences working with students, the abacus is a powerful motivational device for engaging arithmetic. The abacus brings authenticity to the table. The need to use mental arithmetic strategies (eg, thinking of 17 + 8 as 17 + 10 – 2) arises naturally when working with the abacus. Mental arithmetic is required and is sometimes made more complex by the need to trade between columns (eg, 10 ones for 1 ten) and/or the need to trade within a column (eg, 1 five-bead for 5 one-beads).

The Chinese abacus has seven beads in each column (see Figure 6). The rightmost column represents the ones place, the next column the tens place, and so on. The two upper beads each represent a count of 5 and the five lower beads each represent a count of 1. A horizontal divider separates the two types of beads. A bead counts *only* when it touches the horizontal divider (akin to the yin and yang meeting each other).

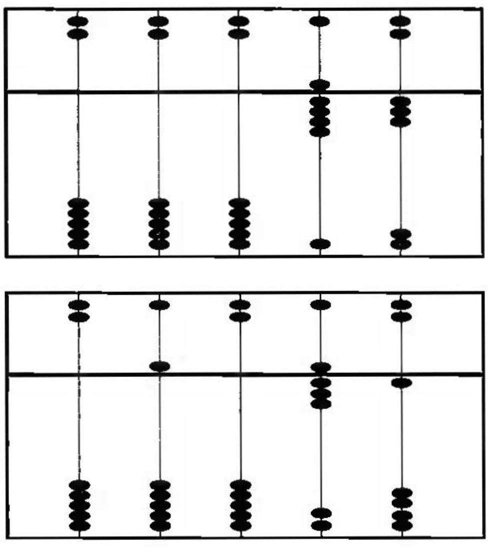
Figure 6
Three columns of a Chinese abacus with 7 beads in each column



The seven-bead structure makes it possible to represent a count of 15 (2 × 5 + 5 × 1) in each column. An advantage of this structure is that it allows for a variety of strategies for doing addition and subtraction, the core operations on the abacus. Multiplication involves the distributive property, with subproducts added as a running total.

The students could already represent numbers on the Chinese abacus (see Figure 7) and do addition on it. These understandings had been developed in conjunction with place value and multi-digit addition.

Figure 7
Representing 93 and 5081 on the Chinese abacus (a bead counts only when it touches the horizontal divider)



The students did not know how to use the abacus to multiply. This ability was developed over two Saturdays. The main difficulty for students was not the distributive property. Rather, it was doing the subproduct multiplication (eg, 7 × 20) mentally. Students who had a great degree of fluency with the basic facts of multiplication and multiplying by 10s had far less difficulty retaining the multiplication result in their mind and then adding it onto the existing number representation on the abacus.

Teaching Preservice Teachers to Use the Chinese Abacus

After the students could use the abacus to multiply reasonably well, we co-constructed two lesson plans that the students would use to teach preservice teachers. The first plan was for developing the distributive property and the 2-digit × 2-digit multiplication algorithm. The second plan was for developing representation, addition and multiplication on the abacus. Both plans involved the same approach that I had used when teaching the Grade 4 students those concepts. The students rehearsed the plans by using them to teach me. During the rehearsal I asked questions that a novice learner might ask. This provided the students with more practice, deepened their understanding and prepared them for questions from the preservice teachers.

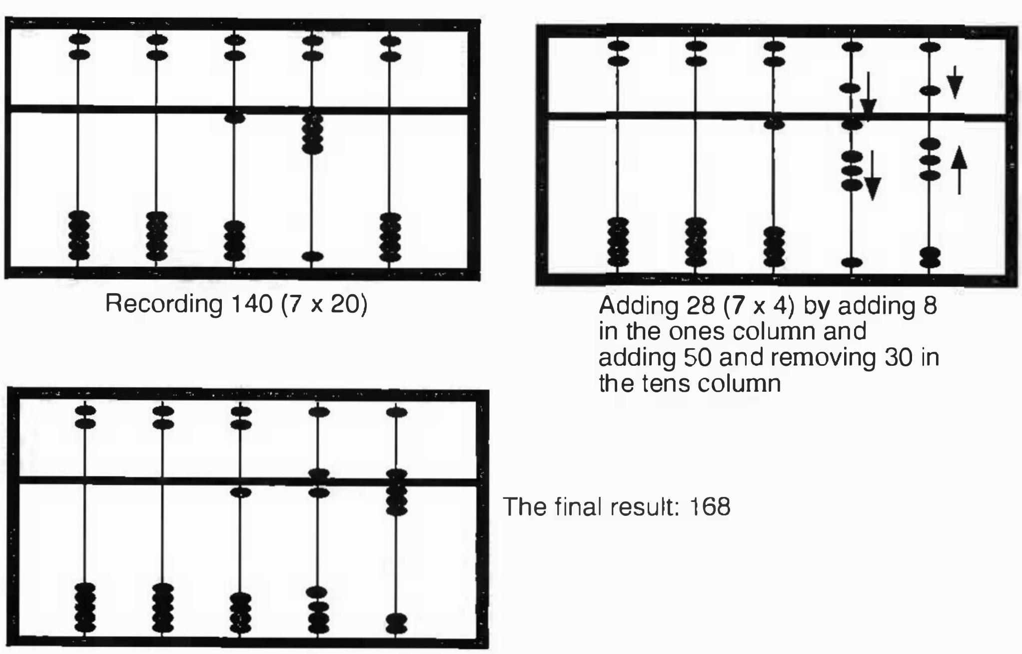
During the next week, the students acted as instructors for two early-years mathematics methods classes. One purpose for doing this was to alter the preservice teachers' beliefs about the mathematical capabilities of elementary school students. The preservice teachers tended to see these students through the lens of their own mathematics anxiety and ability. This limited their perception of the mathematics that elementary students might be able to learn.

The first lesson involved developing a functional sense of the distributive property and a multiplication algorithm for which the distributive property is transparent. Needless to say, the mind shift involved from the traditional "magical" algorithm learned long ago by the preservice teachers generated feelings ranging from frustration to wonderment in them. The preservice teachers asked a number of questions during the first lesson. A sample question and Grade 4 student response follows:

> *Question*: When multiplying 5 × 13, can you split up the 5 instead of the 13?
> *Response*: Yes, but the answer will be harder to get.

The final part of the second lesson developed multiplication on the abacus. Multiplication involves

Figure 8
Multiplying 7 × 24 by thinking 7 × 20 + 7 × 4

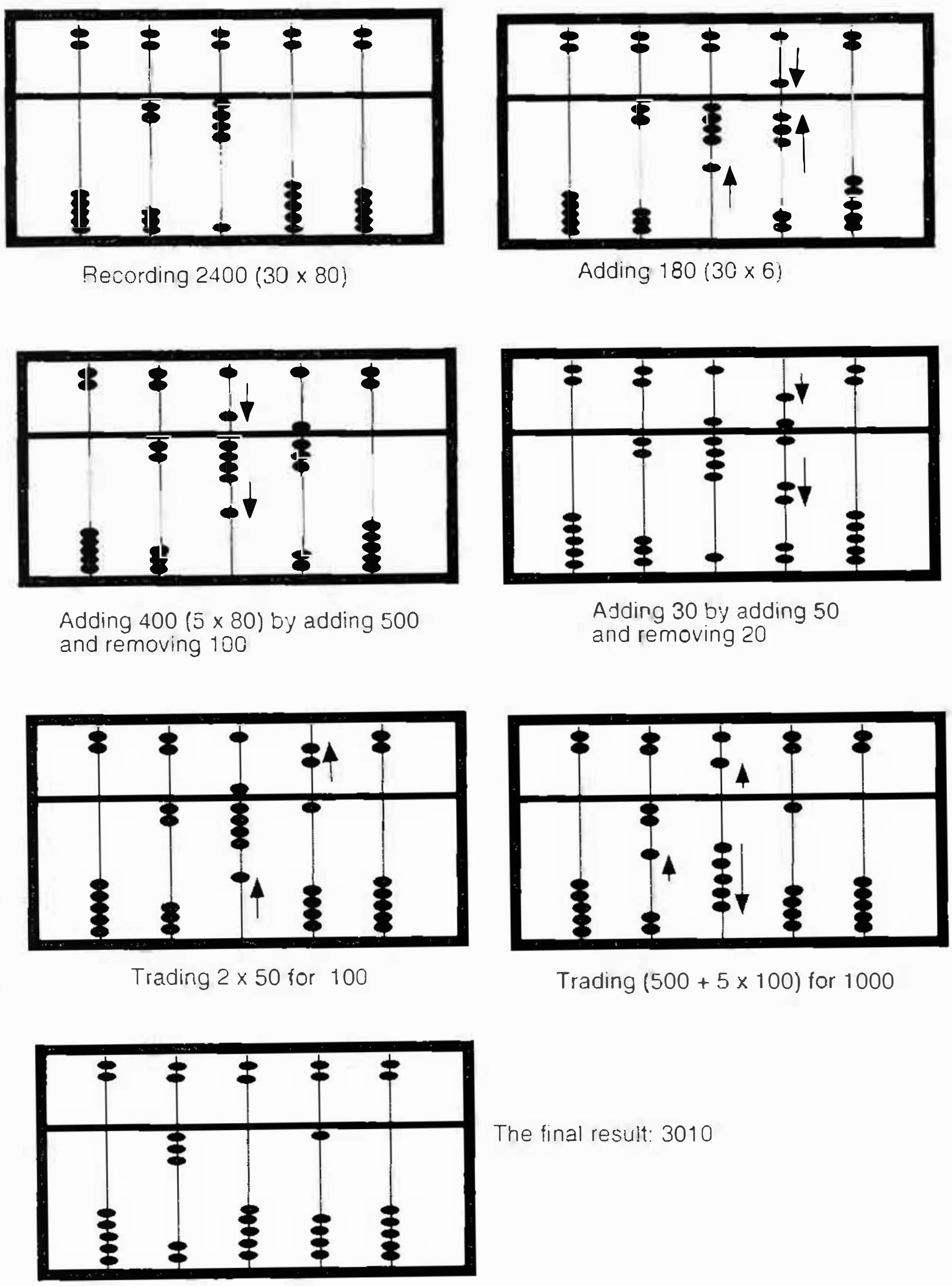


adding subproducts using a running total. The subproducts are calculated mentally, not on the abacus. The process requires an understanding of the distributive property and the mental arithmetic strategies and concepts central to addition.

The first multiplication task presented was 3 × 12. Most of the preservice teachers completed it by adding 12 three times. The inadequacy of this approach was made evident by the next task: 45 × 67. Many of them realized that the distributive property was

Figure 9

Multiplying 35 × 86 by doing 30 × 80 + 30 × 6 + 5 x 80 + 5 x 6



needed to make this task manageable. Once this realization occurred, the lesson continued by presenting tasks progressively from 1-digit × 2-digit to 2-digit × 2-digit multiplication. One of the 1-digit × 2-digit tasks was 7 × 24 (see Figure 8).

The lesson concluded with 2-digit × 2-digit tasks. Figure 9 provides an example of 35 × 86. The thinking involved is 30 × 80 + 30 x 6 + 5 × 80 + 5 × 6.

Typically, the largest subproduct (eg, 30 × 80) is recorded first on the abacus because doing so tends to simplify the addition of subsequent subproducts and because of the way Chinese numerals are written and decoded (the largest position is at the top of the vertically written numeral). The preservice teachers tended to record the smallest subproduct first because that is how the traditional paper and pencil algorithm they had learned works (ones × ones, ones × tens, and so on). The Grade 4 students were far more comfortable with working with the largest place value position first because of their experiences doing addition with base 10 materials and with the abacus that had occurred during Saturday sessions with me. It was interesting to observe the Grade 4 students' impatience with the preservice teachers in relation to this. For example, one student made the following remark to a preservice teacher who insisted on recording the ones × ones partial product first and who was having difficulty seeing that it could be done differently: "Why do you want to do things the hard way? What is so special about doing 5 × 8 first?"

To be truly "Chinese" when using the abacus means that the calculation and memory of subproducts takes place solely in the mind, with subproduct results continuously added to the running total. Results are not written down on paper. The abacus *is* the paper, so to speak. If authenticity is expected when working with the abacus, that is another reason that the abacus can serve as a powerful device for stimulating and developing mental arithmetic. In effect, the Grade 4 students asked the preservice teachers to be Chinese in spirit when using the abacus. Those preservice teachers who had multiplicative fluency (basic multiplication facts and multiplying by 10s) were most comfortable with this. Those who used cognitive bypasses (such as skip counting) to obtain subproducts experienced frustration and had difficulty completing multiplication tasks successfully, even though they understood the distributive property.

The two lessons went well. The Grade 4 students presented the tasks clearly and facilitated learning capably. Especially notable was the confident manner in which they responded to questions. They felt proud about teaching mathematics to adults. The preservice teachers learned a functional sense of the distributive property (that I later revisited using bracket notation) and an understanding of why the multiplication algorithm that they had been taught in a rote manner when they were elementary students worked. The experience contributed significantly to reshaping their perceptions of the mathematical capabilities of elementary school students and their ability to concentrate for a long period of time. It also provided the preservice teachers with an exposure to a possible teaching strategy, namely, having a few students teach the other students in the class, where the teacher provides assistance with the matter to be taught and the teaching plan.

References

National Council of Teachers of Mathematics (NCTM). 2000. *Principles and Standards for School Mathematics.* Reston, Va: NCTM.

Note

The website Chinese Abacus, at www.mandarintools.com/abacus.html, provides further detail.

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Reliable Delivery with Unreliable Deliverers

Sven Ti-lin Chou and Jason Wen-pen Liao

Dedicated to the memory of the late the Honourable Lois Hole, former Lieutenant Governor of Alberta.

The Problem

We have to send a secret file to our embassy in an unfriendly country. We can use only our sixteen agents who can take sensitive material across borders. The former head of our secret agency had indicated that three of the agents are suspected of being double agents. The double agents will deliver what they are carrying to the enemy instead. If our embassy does not get the entire file, or if the enemy gets hold of the entire file, it will mean big trouble for us. Unfortunately, we were not told which agents might be unreliable. The former head did not want to prejudice us against any individuals without concrete evidence that they were, indeed, double agents.

Our Thinking

Clearly, we must divide the file into at least four parts. If it is divided into only three parts and there are indeed three double agents, each of them may be carrying a different part, and they will among them deliver the entire file to the enemy. Also, we must have at least four copies of each part. If we have only three copies and there are indeed three double agents, each of them may be carrying the same part, and our embassy will not be able to get a complete copy of the file. On the other hand, more than four copies will only make things easier for the enemy. As long as no agent carries duplicate parts, four copies are enough.

Suppose we divide each of the four copies of the file into four parts—A, B, C and D. We may have agents 001, 002, 003 and 004 carry A; 005, 006, 007 and 008 carry B; 009, 010, 011 and 012 carry C; and 013, 014, 015 and 016 carry D. This will guarantee that our embassy will get a complete copy of the file and that the enemy cannot. Unfortunately, we discover that agent 016 has been eliminated, and we are down to fifteen agents. Even more unfortunately, we may still have three double agents.

Suppose we still divide each copy of the file into four parts. This is no longer enough. A key result in the explanation is the Mean Value Principle, which states that in a finite set of real numbers, there is at least one not greater than the average and there is at least one not less than the average.

If we have sixteen parts altogether and only fifteen agents, the average number of parts each carries is greater than one. Some agent must carry not less than the average number of parts. This means that this agent must carry at least two parts, say A and B. Now a second agent must carry C and a third agent must carry D. If these three are double agents, the enemy will have a complete copy of the file. Later, we will make use of the Mean Value Principle without mentioning it explicitly.

With fifteen agents, will be it enough to divide the file into five parts? It turns out that it is enough even if 015 is unavailable. Let the five parts be A, B, C, D and E. We may have 001 and 002 carry A and B; 003 and 004 carry C and A; 005 and 006 carry B and C; 007, 008, 009 and 010 carry D; and 011, 012, 013 and 014 carry E. In order for the enemy to get all of A, B and C, two of 001 to 006 must be double agents. In order for the enemy to get D, one of 007 to 010 must be a double agent. In order for the enemy to get E, one of 011 to 014 must be a double agent. Thus three double agents will not be enough for the enemy.

So we successfully send the secret file to our embassy. However, we are worried that more agents may be eliminated. It is better for us to work out well in advance what we would do in each scenario.

Scenario 1: Agents 014 and 015 have also been eliminated.

We may divide the file into six parts, even if 013 is unavailable. Let the parts be A, B, C, D, E and F.

We may have 001 and 002 carry B and C, 003 and 004 carry C and A, 005 and 006 carry A and B, 007 and 008 carry F and D, 009 and 010 carry D and E, and 011 and 012 carry E and F. In order for the enemy to get all of A, B and C, two of 001 to 006 must be double agents. In order for the enemy to get all of D, E and F, two of 007 to 012 must be double agents. Thus three double agents will not be enough.

We now prove that six parts are necessary. Suppose we have only five parts, say A, B, C, D and E. If someone carries three parts, say A, B and C, someone else must carry D and someone E. These three will have a complete file among them. Hence no agent carries three or more parts. We may assume that someone carries A and B. If someone else carries two of the remaining parts, say C and D, a third agent must carry E, and the three of them will have a complete file among them. It follows that four of the other twelve agents carry C, four carry D and four carry E. Now someone must carry the second copy of A, say an agent carrying C. Then no agent can carry two of B, D and E. Hence the other three copies of B must be carried by those who are carrying C. Now we have someone carrying B and C, and whoever carries the third copy of A will also carry D or E. We will have three agents who have a complete file among them.

Scenario 2: Agents 012 and 013 have also been eliminated.

We may divide the file into eight parts. Let the parts be A, B, C, D, E, F, G and H. We may have 001 carry A, B, C and D; 002 carry A, B, C and E; 003 carry A, B, D and E; 004 carry A, C, D and E; 005 carry B, C, D and E; 006 and 007 carry H and F; 008 and 009 carry F and G; and 010 and 011 carry G and H. In order for the enemy to get all of A, B, C, D and E, two of 001 to 005 must be double agents. In order for the enemy to get all of F, G and H, two of 006 to 011 must be double agents. Thus three double agents will not be enough.

We now prove that eight parts are necessary. Suppose we have only seven parts and someone carries at least four parts. Then someone else must carry at least two of the remaining three parts. Along with someone who carries the remaining part, these three agents among them carry all seven parts, and the enemy can obtain a complete copy of the file. Henceforth, we assume that everyone carries at most three parts. At the same time, no two carry between them six different parts. At least six agents must carry three parts each, and it is not possible for every two of them to carry two common parts. Hence we may assume that agent 001 carries A, B and C, while 002 carries A, D and E. Now no agent can carry both F and G. Hence, we may assume that F is carried by 003 to 006, while G is carried by 007 to 010. Now one of these eight agents must carry B, and we may assume that 003 does. Then none of 007 to 010 can carry C. If any of them carry B, then none of 003 to 006 can carry C either, and we will not have enough agents carrying C. Hence none of 007 to 010 can carry B. So each of them carries D or E in addition to G, but none can carry both D and E. Similarly, each of 003 to 006 carries B or C in addition to F, but not both B and C, and none of them can carry D or E. However, this means that 011 must carry B, C, D and E. This is a contradiction.

Scenario 3: Agent 011 has also been eliminated.

We may divide the file into ten parts. Let the parts be A, B, C, D, E, F, G, H, I and J. We may have agent 001 carry A, B, C and D; 002 carry A, B, C and E; 003 carry A, B, D and E; 004 carry A, C, D and E; 005 carry B, C, D and E; 006 carry F, G, H and I; 007 carry F, G, H and J; 008 carry F, G, I and J; 009 carry F, H, I and J; and 010 carry G, H, I and J. In order for the enemy to get all of A, B, C, D and E, two of 001 to 005 must be double agents. In order for the enemy to get all of F, G, H, I and J, two of 006 to 010 must be double agents. Thus, three double agents will not be enough. We postpone the proof that ten parts are necessary to Appendix A.

Scenario 4: Agent 010 has also been eliminated.

We may divide the file into twelve parts. Let the parts be A, B, C, D, E, F, G, H, I, J, K and L. Let agent 001 carry A, E, F, G, H and I; 002 carry B, D, F, G, H and J; 003 carry C, D, E, G, I and J; 004 carry B, C, E, F and K; 005 carry A, C, D, F and K; 006 carry A, B, D, E and L; 007 carry A, B, C, G and L; 008 carry H, I, J, K and L; and 009 carry H, I, J, K and L. In order for the enemy to get all of A, B, C, D, E, F and G, all three double agents must be among 001 to 007. In order for the enemy to get all of H, I and J, two of 001 to 003 must be double agents. In order for the enemy to get K, one of 004 and 005 must be a double agent. In order for the enemy to get L, one of 006 and 007 must be a double agent. Thus three double agents will not be enough. We postpone the proof that twelve parts are necessary to Appendix B.

Scenario 5: Agent 009 has also been eliminated.

We may divide the file into fourteen parts. Let the parts be A, B, C, D, E, F, G, *A, B, C, D, E, F* and *G*. Let agent 001 carry A, B, C, D, E, F and G; 002 carry A, B, C, *D, E, F* and *G*; 003 carry A, D, E, *B, C, F* and *G*; 004 carry A, F, G, *B, C, D* and *E*; 005 carry B, D, F, *A, C, E* and *G*; 006 carry B, E, G, *A, C, D* and *F*; 007 carry C, D, G, *A, B, E* and *F*; and 008 carry C, E, F, *A, B, D* and *G*. In order for the enemy to get all of *A, B, C, D, E, F* and *G*, all three double agents must be among 002 to 008. In order for the enemy to get all of A, B, C, D, E, F and G, all three double agents must carry a common part, say A. Then none of them carry *A*. This is a contradiction.

We now prove that fourteen parts are necessary. Suppose we have only thirteen parts. Since there are fifty-two copies of these parts combined, someone must carry at least seven parts. Consider the other seven agents and six parts. Since there are twenty-four copies of these parts combined, someone must carry at least four parts. Consider now the remaining six agents and two parts. Someone must carry both of them. These three agents among them carry all thirteen parts, and the enemy can obtain a complete file.

Scenario 6: Agent 008 has also been eliminated.

Suppose 001, 002 and 003 are double agents. Then there must be a part which none of them carry. Let it be A. Since there are four copies of A, each of 004, 005, 006 and 007 must carry one. Suppose 001, 002 and 004 are double agents. Then there must also be a part which none of them carry. Moreover, this cannot be A since 004 is carrying it. Let it be B. Then each of 003, 005, 006 and 007 carries B. So we need a part for every triple of agents. Since the number of triples of agents is thirty-five, dividing the file into thirty-five parts is both necessary and sufficient.

Scenario 7: Agent 007 has also been eliminated.

As it turns out, we will be helpless without James Bond. If we are down to six agents, and three of them may still be double agents, then it is mission impossible. This is because three of the six are on our side while the other three are on the other side. Whatever we can deliver with three agents, the other side can get with three double agents. Whatever the other side cannot get with three double agents, we cannot deliver with three agents. This is an incidence of the important notion of *symmetry*.

Appendix A

Suppose we have ten agents and only nine parts A, B, C, D, E, F, G, H and I. It is routine to verify that any scheme in which one of the agents carries at least five parts will not work. Henceforth, we assume that each agent carries at most four parts. We may assume that agent 001 carries A, B, C and D.

Since there are twenty copies of E, F, G, H and I, and only nine other agents, we may assume that 002 carries E, F and G. Nobody can then carry both H and I. Hence we may assume that 003 to 006 carry H and 007 to 010 carry I. Since there are nine other copies of E, F and G, 003 must carry at least two of them, say E and F. This means that the other three copies of G must be carried by 004 to 006. None of these three can carry E or F, and none of 007 to 010 can carry both E and F. Hence 007 and 008 carry E and 009 and 010 carry F. We have arrived at the following situation.

001	002	003	004	005	006	007	008	009	010
A	E	E	G	G	G	E	E	F	F
B	F	F	H	H	H	I	I	I	I
C	G	H							
D									

We still have to add three copies of each of A, B, C and D. Someone other than agent 001 must also carry two of them. We consider two subcases.

Case 1: 007 carries A and B. Neither 009 nor 010 can carry both C and D. Hence one of 002 to 006 must carry C. Suppose it is 002. Then nobody can carry both D and H, so that the other three copies of D are carried by 008 to 010. Then none of 004 to 006 can carry C, so that the last two copies of C are carried by 003 and 008. None of 004 to 006 can carry both A and B. Hence 009 carries A, but 004 must carry B. Now 004, 008 and 009 carry all the parts among them. It follows that neither 002 nor 003 carries C, so 004 does. Then nobody can carry both D and F, so the other three copies of D are carried by 005, 006 and 008. Neither 005 nor 006 can carry C, so 009 must. Now 004, 007 and 009 carry all the parts among them.

Case 2: None of 007 to 010 carries two of A, B, C and D. This means that each of 004 to 006 carries two of them while each of 002, 003 and 007 to 010 carries one. We may assume that 004 carries A and B. Suppose 002 carries C. Then nobody can carry both D and I, so that the other three copies of D are carried by 003, 005 and 006. By the same reasoning, the other two copies of C are carried by 005 and 006. Let 007 carry A. Then either 009 or 010 must carry B. This agent and 004 and 007 will carry all the parts among them. It follows that neither 002 nor 003 can carry either C or D. Now neither 005 nor 006 can carry both C and D, as otherwise 002 and 003 cannot carry A or B either. Hence each of 005 to 010 carries one of C and D. Now one of 007 and 008 must carry a different part (C or D) from either of 009 and 010. Along with 004, we again have three agents who carry all the parts among them.

Appendix B

Suppose we have nine agents and only eleven parts (A, B, C, D, E, F, G, H, I, J and K). It is routine to verify that any scheme in which one of the agents carries at least six parts will not work. Henceforth, we assume that each agent carries at most five parts. We may assume that 001 carry A, B, C, D and E.

Suppose another agent carries four of the other six parts. Consider the remaining seven agents and two parts. Someone must carry both parts. These three agents among them carry all eleven parts, and the enemy can obtain a complete file. Thus we may assume that everyone except 001 carries exactly three of F, G, H, I, J and K. We may assume that 002 carries A, B, F, G and H. Then each of 003 to 009 carries exactly three of C, D, E, I, J and K. However, none of them can carry all of I, J and K.

Suppose some agent carries none of I, J and K. Then he must carry all of C, D, E, F, G and H, which is too many. Hence, every agent carries at least one of I, J and K. Since there are twelve copies and seven other agents, we may assume that 003 carries exactly one of I, J or K—say I. Then he must also carry two of C, D and E and two of F, G and H—say D, E, G and H. Then each of 004 to 009 carries exactly three of A, B, C, F, J and K. Moreover, none of them can carry J and K plus C or F.

There are eight copies of J and K, and only six agents left. Therefore, at least two of them, say 004 and 005, must carry both J and K. Each must then carry one of A and B and one of D and E, as well as one of G and H. Now we still have nine copies of C, F and I, but only four agents left. Hence at least one of them, say 006, must carry all of C, F and I. He must still carry either J or K, but nothing else.

In order for 004, 005 and 006 not to have all the parts among them, 004 and 005 must carry a third common part other than J and K. However, this means that one of the other seven agents must carry four parts not carried by 004. This is a contradiction.

Acknowledgement

Our work extends the results in "A Space Interlude," by Gilbert Lee, Kenneth Ng and Philip Stein, published on pages 31–32 of *Mathematics for Gifted Students II*, a special edition of *delta-K*, volume 33, number 3, December 1996.

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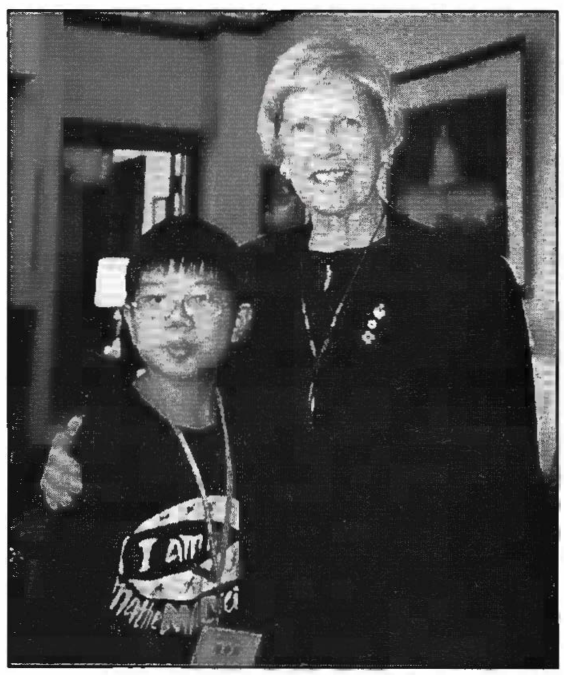


Boxes for Water: We Can Make a Difference

Nancy Espetveidt

This activity has excellent potential to incorporate aspects of all core subjects. Most important, though, it offers students the opportunity to start a project and follow through to a very tangible end: they will create a significant difference in the lives of others. The activity is based on the collaborative efforts of the group, thereby building communication skills and cohesion (especially if the activity is shared with other classrooms) as they learn together the potential of socially responsible citizenship. The students will immerse themselves in the project and will learn to appreciate the ways in which they use mathematics in their daily lives. This is an exceptional way to teach the concepts indicated because they are used authentically and meaningfully. Through this experience, the mathematical vocabulary will become a part of students' daily lives. There is great potential with this project—the sky is the limit! This article serves as an introduction and explains math processes the students will practise (possible crosscurricular connections are attached as Appendix A).

2007 Mathematics Curriculum, Grade 3

Strand: Number
General Outcome
Develop number sense
Specific Outcomes

8. Apply estimation strategies to predict sums and differences of two-digit numerals in a problem-solving context.
11. Demonstrate an understanding of multiplication to 5 × 5 by
    - creating and solving problems in context that involve multiplication and
    - relating multiplication to repeated addition.

Strand: Statistics and Probability (Data Analysis)
General Outcome
Collect, display and analyze data to answer questions
Specific Outcomes

1. Collect first-hand data and organize it using tally marks and charts to answer questions.
2. Construct, label and interpret bar graphs to solve problems.

—Alberta Learning 2007

Required Materials

- Chart paper and markers for running tally
- Calculators to help figure out year-long results
- Chart with jobs and names (to be decided each Monday by drawing popsicle sticks with students' names)
- Questions to get students thinking about the task
- Handout: How Many Boxes?
- Book: *One Well: The Story of Water on Earth* (Strauss 2007)

Overview

In this project, students collect juice boxes, clean them out and save them for recycling. The class will keep a running tally of the juice boxes recycled and the amount of money that has consequently been saved (each juice box is worth $0.05). The application is a daily task with weekly rotating duties for different class members. The goal of the application is to raise enough money to sponsor a potable water site through Ryan's Well Foundation (www.ryanswell.ca). The students will start with their own classroom and, hopefully, the initiative will spread to include the collection of juice boxes from other classes in the school.

Introductory Project

- Begin by reading *One Well: The Story of Water on Earth*, by Rochelle Strauss. Reflect with the students on the issue of potable water. Many people in the world don't have access like we do to clean water; but there are ways we can help.
- Explain to the students that, while it is part of our responsibility to the Earth, recycling can also help us to earn money. "Each juice box is worth five cents. How many of you bring juice boxes in your lunches?" Introduce the idea that we could use the money from recycled juice boxes to help a community gain access to clean, drinkable water.
- Following are some sample questions to help students meet curricular objectives:
  - For estimation strategies
    - Number strand, objective 8:
      If Room 11 uses 9 drinking boxes a day and Room 15 uses 23, how many drinking boxes would that total in a day?
    - Number strand, objective 11:
      If our classroom uses 5 drinking boxes a day, how many would that be in a school week (5 days)? Each box is worth 5 cents. How much money would we have collected at the end of the week?
      If we collect 50 juice boxes on each of Monday, Tuesday, Wednesday, Thursday and Friday from the whole school, how many juice boxes would we have at the end of the week?
- Hand out the worksheet "How Many Boxes?" and work with the students to answer the questions. (The math is fairly advanced and requires decoding. Discussion is to be ongoing). Discuss time frame and class average of juice boxes: Who has a juice box today? Let's say that you bring the same number every day. How many days/weeks would it take for us to save $10.00? How will $10.00 make a difference?

  If the class averages about 15 boxes a day, or 75 cents a day:
  $15 \times 5 = 75$ juice boxes/school week
  $75 \times .05 = \$3.75$/ week
  $\$10.00 = 1{,}000$ cents
  1000 cents = 13.3 days (just over 13 days)
  75 cents

Students are encouraged to use their own strategies to solve this problem; eg, repeated addition, tens charts, students could use 5 cents instead of $0.05, and so on.

Making It Happen

The next step is to make it happen—we can do it! Explain the process, including descriptions of jobs. You may wish to put up a job description sheet on the overhead while you describe each job (ask why these jobs are important as you go through the descriptions; ask for any input the students might have regarding organization and expectations). Distribute tally sheets and let the students know that the sheets are to be kept up to date with all information filled in each week.

Job Descriptions

- **Box Collector**: Picks up boxes after lunch and takes them to classroom sink. This job may need more students depending on how many classrooms become involved. It will be better organized if one person is in charge.
- **Box Cutter**: Carefully cuts off the top of each box so that it can be easily rinsed.
- **Box Rinser**: Rinses each box to make sure there is no juice left inside. This is important to avoid the boxes getting mouldy.
- **Squisher/Packer**: Flattens all of the juice boxes and packs them into bags for the recycle depot. (If we flatten them, more will fit in each bag.)
- **Tallier**: Counts the boxes and tallies them on the classroom chart. Note: all students will have their own tally charts, in project folders, which they are expected to update daily and keep current.

Extensions

Hopefully, this activity will extend to other classrooms as well, which raises the students' level of responsibility as well as the level of mathematics (larger numbers). If that is the case, it might be helpful to create a bar graph that tracks the juice boxes contributed by each class.

Problems can be extended to include the school year (197 days) and can be worded in a way that requires more decoding. For example:

> If there are 25 students in the class and an average of 15 students bring juice boxes every day for lunch, how many juice boxes will there be at the end of the week? How much money will that be?

Modifications

- Do the project on a smaller scale—perhaps proceeds could be donated to a local charity.
- Start the project as a hypothetical application to gauge students' ability and to generate interest.

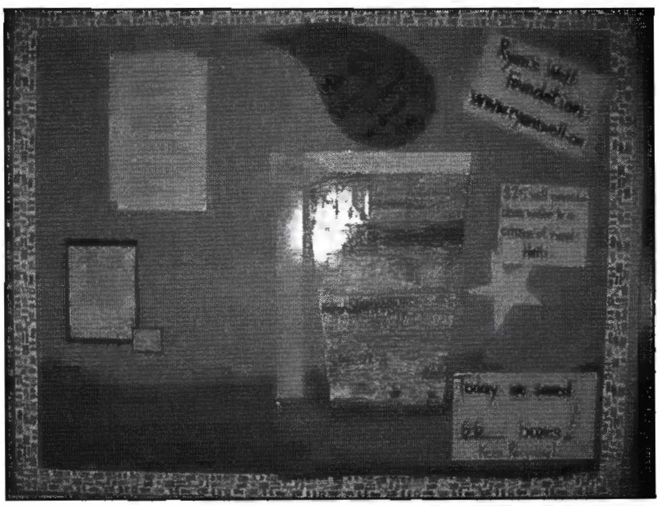
Vocabulary

Bar graph: A graph that uses bars to show data
Tally table: A table that uses tally marks to record data
Data: Information collected about people or things

Assessment

The aim of this project is to demystify mathematics by using it daily to reach a certain goal. The mathematics is not an end in itself but a means to achieving an end that is meaningful. Nevertheless, it is important that students understand the mathematics involved. Following are some ways in which you may assess students' understanding and whether or not the curricular objectives set out at the beginning of the lesson are being met.

- Check on the students' tally charts once a week:
  - Are the sheets being kept up to date?
  - Is there a total dollar amount for each week?
  - Can the student explain to you the significance of the tally marks?
    - What do the marks represent?
    - What is the purpose of keeping the chart?
- Mark the "How Many Boxes?" worksheets or have the students discuss them, in groups. During the discussion, move through the classroom and take note of students' responses.
  - How did they figure out the answers?
  - Are they making the connection between repeated addition and multiplication?
  - What tactics are they using? (For example, estimation strategies? Invented strategies? If so, can they explain them?)



Bulletin board: each day the water level rose in the glass based on the number of juice boxes collected. The scale on the slide is counted in dollar amounts, one notch per dollar.

Presentation at whole-school assembly. Glenna Haig and Nancy Espetveidt.

- Fill in the Job Duties Checklist for each student as he or she performs the assigned role for the week. This could aid in assessing attitude as well as gauging interest in the project.
- As a conclusion to the project, have the students reflect on their experience, their feelings, the math they used daily and so on. Students could write letters addressed to themselves or write journal entries, among many other possibilities.
- A rubric is included for consideration.

Reflection

Boxes for Water was successfully implemented as a whole-school project, with Grade 6 students taking turns collecting, counting, and tallying the juice boxes. To date, more than $150 has been raised and sent to Haiti through Ryan's Well Foundation. Ryan's Well sponsors several projects that need assistance; however, keep in mind that there are other organizations as well. Research the organization and project you would like to contribute to, or do the research and decide as a class. I chose Ryan's Well because it is Canadian, it was started by a student (students can relate to this) and it has a comprehensive website.

The goal of the project was to show students that by directing their energy and being aware, they could make a difference in the world. The students did not have to change their habits—they had always recycled their juice boxes, but now the proceeds were redirected and the results were astounding. The entire school was involved and was kept up to date with weekly announcements, daily updates to a bulletin board in the main hallway (managed by the Grade 6 students) and announcements at whole-school assemblies.

From the mathematics to the students' genuine interest, this project was a great success. Although it is time consuming to dedicate a part of each lunch hour (about 20 minutes) to counting, tallying and updating the bulletin board, the students were enthusiastic and the whole process became smoother as they became familiar with each job. I suggest that the bulletin board be updated daily, because it garners interest from passersby. Often, students of all grades and classes would gather around the board and discuss our progress. It was an excellent schoolwide team-building experience.

References

Alberta Learning. 2007. *Mathematics Kindergarten to Grade 9 Program of Studies*. Edmonton, Alta: Author.

Strauss, R. 2007. *One Well: The Story of Water on Earth*. Toronto, Ont: Kids Can.

Nancy Espetveidt graduated from the University of Alberta, where she is still involved in research projects. She is passionate about instructional design and enjoys sharing strategies to deliver curricular material in unique and engaging ways. Nancy is currently teaching Grade 4 at Bearspaw Elementary School, in Edmonton, Alberta.

Appendix A: Crosscurricular Connections

English Language Arts

- **General Outcome**: Discover and Explore
  - **Specific Outcomes:** Express ideas and develop understanding
    - Explain understanding of new concepts in own words
    - Explore ideas and feelings by asking questions, talking to others, and referring to oral, print, and other media texts
- **General Outcome**: Students will listen, speak, read, write, view and represent to manage ideas and information.
  - **Specific Outcome**: Plan and focus
    - Use self-questioning to identify information needed to supplement personal knowledge on a topic.
    - Identify facts and opinions, main ideas and details in oral, print and other media texts.
  - **Specific Outcome**: Determine information needs
    - Ask topic-appropriate questions to identify information needs.

Science

- **General Learner Expectation:** Demonstrate positive attitudes for the study of science and for the application of science in responsible ways.
  - **Specific Learner Expectation:** Students will show growth in acquiring and applying the following traits:
    - A willingness to work with others in shared activities and in sharing of experiences
    - Appreciation of the benefits gained from shared effort and cooperation
    - A sense of responsibility for personal and group actions
    - Respect for living things and environments, and commitment for their care

Social Studies

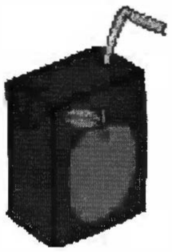
- **General Outcome**: Students will enrich their awareness and appreciation of how people live in other places.
  - **Specific Outcome:** Students will examine the geographic characteristics that shape communities in other parts of the world by exploring and reflecting upon the following questions for inquiry:
    - In what ways do the people in the communities depend on, adapt to and change the environment in which they live and work? (Economics and Resources; The Land, Places and People)
    - In what ways do the communities show concern for their natural environment? (Global Connections; The Land, Places and People)
    - How does the physical geography influence the human activities in the communities (eg, the availability of water, climate)? (Culture and Community; The Land, Places and People)
- **Global Citizenship General Outcome:** Students will demonstrate an understanding and appreciation of Canada's roles and responsibilities in global citizenship.
  - **Specific Outcomes—Values and Attitudes:** Students will
    - appreciate elements of global citizenship:
      - recognize how their actions might affect people elsewhere in the world and how the actions of others might affect them (Citizenship; Global Connections)
      - respect the equality of all human beings (Citizenship; Global Connections; Identity)
  - **Specific Outcomes—Knowledge and Understanding:** Students will
    - explore the concept of global citizenship by reflecting upon the following questions for inquiry:
      - What are some environmental concerns that Canada and communities around the world share? (Economics and Resources; Global Connections)
      - In what ways can individuals and groups contribute to positive change in the world? (Citizenship; Global Connections; Power, Authority and Decision Making)

Editor's note: Material in this appendix is from the Alberta Education programs of studies:

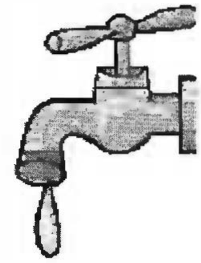
Alberta Education. 1996. *Science (Elementary) Program of Studies*. Edmonton, Alta: Author.

———. 2005. *Social Studies K–12 Program of Studies*. Edmonton, Alta: Author.

Alberta Learning. 2000. *English Language Arts (K–9) Program of Studies*. Edmonton, Alta: Author.



Appendix B: Project Worksheets and Rubric



How Many Boxes?

Each juice box is worth 5 cents. If our class saves 10 boxes in one day, how much money will we collect?

If we saved the same number of boxes each day, how many boxes would that be in a school week?

How much money would that be in a school week?

How many juice boxes will we need to save in order to collect $1.00?

How many juice boxes will we need to save in order to provide:

$10 = 1 person's weekly access to water

$50 = 1 household's daily needs

$100 = 1 household + 1 farm's needs

Week: ______________	
Job	**Name**
Box Collector Picks up boxes after lunch and takes them to classroom sink.	
Box Cutter Carefully cuts off the top of each box so it can be easily rinsed.	
Box Rinser Rinses each box to make sure there is no juice left over.	
Squisher/Packer Flattens all of the juice boxes and packs them into bags for the recycle depot.	
Tallier Counts the boxes and tallies them on our classroom chart.	

Week: ______________	**Number of Juice Boxes**
Monday	
Tuesday	
Wednesday	
Thursday	
Friday	

Total: ____________________

× $ 0.05

= $____________________

	Wow! That was amazing!	Great job!	Good work!	
How Many Boxes?	Your work is detailed and your answers are correct.	Your work is somewhat detailed and your answers are mostly correct.	Your answers are correct but you did not show your work.	Your answers need some correcting and you need to show your work.
Keeping a Tally Chart	Your project folder is very organized and complete. You kept your tally charts up to date and they are in order.	Your project folder is organized and mostly complete. Most of your tally charts up to date and they are in order.	Your project folder is somewhat organized and mostly complete. Some of your tally charts are up to date and they are in order.	Your project folder is incomplete and not organized. Your tally charts are not up to date and they are not in order.
Solving the Problem/ Repeated Addition	You are able to explain the math in our project very clearly, using your own words and ideas. You have made the connection between repeated addition and multiplication.	You are able to explain the math in our project fairly clearly and you give reasons for your answers. You show evidence of the connection between repeated addition and multiplication.	You are able to answer questions about the math in our project and you begin to share your ideas with the teacher's help. You show repeated addition, but do not connect it to multiplication.	You explain the math in our math project in a disorganized fashion that is difficult to follow. There is no evidence of repeated addition or its connection to multiplication.
Math Reflection	Excellent reflection. You communicated your feelings about the project and the math we practised very clearly.	Great reflection. You communicated your feelings about the project and the math we practised.	Good reflection. You wrote about the project and the math we practised.	Your reflection does not communicate your feelings about the project or the math we practised.

We can do it!

Alberta High School Mathematics Competition

Report on the First Round of the 52nd Contest

Andy Liu

Sponsors

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Individual Results

The first part of the 52nd Alberta High School Mathematics Competition was written on November 20, 2007, by 712 students—282 girls and 430 boys. The numbers of students in Grades 8, 9, 10, 11 and 12 are respectively 2, 35, 164, 195 and 316.

The top-scoring students are listed below. Unless otherwise indicated, the student was in Grade 12 at the time of the exam.

Rank	Score	Name	School
1	85	Frank Yang	Sir Winston Churchill High School, Calgary
		Linda Zhang	Western Canada High School, Calgary
3	84	Hunter Spink	Calgary Science School, Calgary (Grade 9)
4	80	Danny Shi	Sir Winston Churchill High School, Calgary (Grade 11)
		Jarno Sun	Western Canada High School, Calgary (Grade 11)
		Chong Shen	Sir Winston Churchill High School, Calgary
7	78	Karl Qin	Queen Elizabeth Jr/Sr High School, Calgary (Grade 11)
		Lucille Lu	Western Canada High School, Calgary (Grade 11)
		Zoe Cheung	J G Diefenbaker High School, Calgary
		Alex Chen	Sir Winston Churchill High School, Calgary
		Annie Xu	Old Scona Academic High School, Edmonton
		Darren Xu	Sir Winston Churchill High School, Calgary
13	77	Yuxiang Liu	Western Canada High School, Calgary
		Wen Wang	Western Canada High School, Calgary
15	76	Jaclyn Chang	Western Canada High School, Calgary (Grade 10)
		Chen Liu	Western Canada High School, Calgary (Grade 11)
		Taylor Hudson	J G Diefenbaker High School, Calgary
		Wen Song	Sir Winston Churchill High School, Calgary
		Kevin Tan	Sir Winston Churchill High School, Calgary
		Michael Wong	Tempo School, Edmonton

Rank	Score	Name	School
21	75	Mariya Sardarli	McKernan Junior High School, Edmonton (Grade 8)
		Andrew Qi	Vernon Barford Junior High School, Edmonton (Grade 9)
		Annie Wang	Sir Winston Churchill High School, Calgary (Grade 11)
		David Yu	Sir Winston Churchill High School, Calgary (Grade 10)
		Michael Zhou	Western Canada High School, Calgary
26	74	Stephanie Bohaichuk	Harry Ainlay High School, Edmonton (Grade 10)
		Jacky Tian	Western Canada High School, Calgary (Grade 11)
		Jonathan Wong	Western Canada High School, Calgary (Grade 11)
		Jared Gordon	Western Canada High School, Calgary
		Stephanie Laflamme	Bishop Carroll High School, Calgary
		Navid Nourian	Henry Wise Wood High School, Calgary
		Ben Wang	Sir Winston Churchill High School, Calgary
		Liz Yue	Sir Winston Churchill High School, Calgary
34	73	Maninder Longowal	Tempo School, Edmonton (Grade 11)
		David Szepesvari	Harry Ainlay High School, Edmonton (Grade 11)
		James Kim	Western Canada High School, Calgary
37	72	Spencer Boone	Western Canada High School, Calgary (Grade 11)
		Jessica Jiang	Old Scona Academic High School, Edmonton (Grade 11)
		Brett Baek	Western Canada High School, Calgary
		Victor Feng	Sir Winston Churchill High School, Calgary
		Lian Tang	Jasper Place High School, Edmonton
		Glen Wang	Western Canada High School, Calgary (Grade 11)
43	71	Di Mo	Sir Winston Churchill High School, Calgary (Grade 10)
		Anna Yu	Sir Winston Churchill High School, Calgary (Grade 11)
		Douglas Cheung	Old Scona Academic High School, Edmonton
		Brenna Pickell	Archbishop Jordan High School, Sherwood Park
47	**70**	Nafisah Tyebkhan	Tempo School, Edmonton (Grade 9)
		David Gordon	Western Canada High School, Calgary (Grade 10)
		Yuri Delanghe	Harry Ainlay High School, Edmonton (Grade 10)
		Alexander Neame	Harry Ainlay High School, Edmonton (Grade 10)
		Edward Xu	Henry Wise Wood High School, Calgary (Grade 10)
		Elsie Young	Western Canada High School, Calgary (Grade 10)
		Natasha Birchall	Tempo School, Edmonton (Grade 11)
		Steven Dien	Western Canada High School, Calgary (Grade 11)
		Mandi Xu	Western Canada High School, Calgary (Grade 11)
		Yingyu Yao	Sir Winston Churchill High School, Calgary (Grade 11)
		Min Bai	Western Canada High School, Calgary
		Topher Flanagan	Tempo School, Edmonton
		Naheed Jivra	Strathcona-Tweedsmuir School, Okotoks
		Philip Morin	St Francis High School, Calgary

Team Results

The contest was written by 34 schools. There were ten schools from Zone I (Calgary) with 314 students, six schools from Zone II (southern rural Alberta) with 73 students, ten schools from Zone III (Edmonton) with 174 students and eight schools from Zone IV (northern rural Alberta) with 151 students.

The top teams are listed below.

Rank	Score	Team Members and Manager
1	245	Sir Winston Churchill High School, Calgary—Frank Yan, Danni Shi and Chong Shen, managed by Neil Hamel
2	243	Western Canada High School, Calgary—Linda Zhang, Jarno Sun and Lucille Lu, managed by Renata Delisle
3	222	J G Diefenbaker High School, Calgary—Zoe Cheung, Taylor Hudson and Zinzhu Wei, managed by Terry Loschuk
4	221	Old Scona Academic High School, Edmonton—Annie Xu, Jessica Jiang and Douglas Cheung, managed by Ihor Lytviak
5	219	Tempo School, Edmonton—Michael Wong, Maninder Longowal and Nafisah Tyebkahn, managed by Lorne Rusnell
6	217	Harry Ainlay High School, Edmonton—Stephanie Bohaichuk, David Szepesvari and Yuri Delanghe/ Alexander Neame, managed by Jacqueline Coulas
7	210	Henry Wise Wood High School, Calgary—Navid Nourian, Edward Xu and Xin Zhang, managed by Michael Retallack
8	208	Queen Elizabeth Junior/Senior High School, Calgary—Karl Qin, Raphaell Masquillier and Fay Qian, managed by Sharon Reid
9	204	Jasper Place High School, Edmonton—Liang Tang, Duhao Meng and Jingchen Ge, managed by John MacNab
10	203	St Francis High School, Calgary—Philip Morin, Nicole Veltri and Kirsten Marshall, managed by Peter Walker

Other participating schools were

- Zone I (Calgary)
  - Bishop Carroll High School—Toni Fazio, manager
  - Calgary Science School—Scot Doehlar, manager
  - Central Memorial High School—Gerald Krabbe, manager
  - William Aberhart High School—James Kotow, manager
- Zone II (Southern Rural Alberta)
  - Crowsnest Consolidated High School (Crowsnest Pass)—Jodi Peebles, manager
  - Hughenden Public School—Crystal Chudley, manager
  - Oilfields High School (Turner Valley)—Chris Hughes, manager
  - Prairie Christian Academy (Three Hills)—Robert Hill, manager
  - St Gabriel the Archangel School (Chestermere)—Adrienne Busch, manager
  - Senator Gershaw School (Bow Island)—Linda Atwood, manager
  - Strathcona-Tweedsmuir School (Okotoks)—Nola Adam, manager
- Zone III (Edmonton)
  - Archbishop MacDonald High School—John Campbell, manager
  - Holy Trinity High School—Len Bonifacio, manager
  - McKernan Junior High School—Ward Patterson, manager
  - McNally High School—Brian Pike, manager
  - Vernon Barford Junior High School—Robert Wong, manager
  - Vimy Ridge Academy—Delcy Rolheiser, manager

  Ross Sheppard High School (JeremyKlassen, manager) registered for the contest, but was unable to hold it on the day.
- Zone IV (Northern Rural Alberta)
  - Archbishop Jordan High School (Sherwood Park)—Marge Hallonquist, manager
  - Ardrossan Junior Senior High School—Rebecca Gustafson, manager
  - École Secondaire Ste Marguerite d'Youville (St Albert)—Lisa La Rose, manager
  - Father Patrick Mercredi High School (Fort McMurray)—Ted Venne, manager
  - J A Williams High School (Lac La Biche)—Matt Dyck, manager
  - Leduc High School—Corlene Balding, manager
  - Paul Kane High School (St Albert)—Percy Zalasky, manager

Alberta High School Mathematics Competition

1. A positive integer has 1001 digits, all of which are 1s. When this number is divided by 1001, the remainder is
(a) 1 (b) 10 (c) 11 (d) 100 (e) none of these

2. Some cats have got into the pigeon loft because the total head count is 34 but the total leg count is 80. The number of cats among the pigeons is
(a) 6 (b) 12 (c) 17 (d) 22 (e) 28

3. In triangle ABC, $AB \leq 1 \leq BC \leq 2 \leq CA \leq 3$. The maximum area of triangle ABC is
(a) 1 (b) 3/2 (c) 2 (d) 5/2 (e) none of these

4. The number of ways in which five As and six Bs can be arranged in a row that reads the same backwards and forwards is
(a) 1 (b) 5 (c) 10 (d) 15 (e) none of these

5. Among twenty consecutive integers each at least 9, the maximum number of them that can be prime is
(a) 4 (b) 5 (c) 6 (d) 7 (e) 8

6. The non-negative numbers x and y are such that $2x + y = 5$. The sum of the maximum value of $x + y$ and the minimum value of $x + y$ is
(a) 0 (b) 5/2 (c) 5 (d) 15/2 (e) none of these

7. We wish to choose some of the positive integers from 1 to 1000 inclusive, such that no two differ by 3 or 5. The maximum number of positive integers we can choose is
(a) 200 (b) 300 (c) 333 (d) 500 (e) none of these

8. The number of polynomials p with integral coefficients such that $p(9) = 13$ and $p(13) = 20$ is
(a) 0 (b) 1 (c) 2 (d) 3 (e) infinitely many

9. The number of pairs (a, b) of positive integers such that all three roots of the cubic equation $x^3 - 10x^2 + ax - b = 0$ are positive integers is
(a) 3 (b) 8 (c) 10 (d) 66 (e) none of these

10. In the quadrilateral ABCD, AB = CD, AD = 2 and BC = 6. AD and BC are parallel lines at a distance 8 apart. The radius of the smallest circle that can cover ABCD is
(a) $\sqrt{18}$ (b) $\sqrt{20}$ (c) $\frac{\sqrt{85}}{2}$ (d) 5 (e) none of these

11. The real numbers x and y are such that $x + 2/y = 8/3$ and $y + 2/x = 3$. The value of xy is
(a) 3/2 (b) 4/3 (c) 2 (d) 4 (e) not uniquely determined

12. Let θ be an acute angle such that $\sec^2\theta + \tan^2\theta = 2$. The value of $\csc^2\theta + \cot^2\theta$ is
(a) 2 (b) 3 (c) 4 (d) 5 (e) none of these

13. The diameter AC divides a circle into two semicircular arcs. B is the midpoint of one these arcs, and D is any point on the other arc. If the area of ABCD is 16 square centimetres, the distance, in centimetres, from B to AD is
(a) 2 (b) $2\sqrt{2}$ (c) 4 (d) $4\sqrt{2}$ (e) dependent on the radius of the circle

14. Five students took part in a contest consisting of six true-or-false questions. Student #i gave the answer T to question #j if and only if $i \leq j$. The total number of incorrect answers is 8 or 9, and there are more incorrect answers of T than incorrect answers of F. The student who has both an incorrect answer of T and an incorrect answer of F is
(a) #1 (b) #2 (c) #3 (d) #4 (e) #5

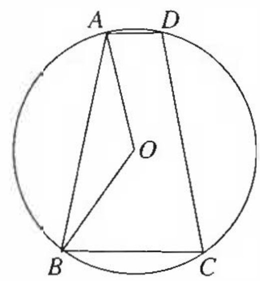
15. An integer n is randomly chosen from 10^{99} to $10^{100} - 1$ inclusive. The real number m is defined by $m = 9n/5$. Of the following five numbers, the one closest to the probability that $10^{99} \leq m \leq 10^{100} - 1$ is
(a) 1/3 (b) 4/9 (c) 1/2 (d) 5/9 (e) 2/3

16. The smallest value of the real number k such that $(x^2 + y^2 + z^2)^2 \leq k(x^4 + y^4 + z^4)$ holds for all real numbers x, y and z is
(a) 1 (b) 2 (c) 3 (d) 6 (e) 9

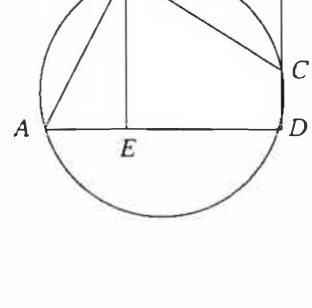
Alberta High School Mathematics Competition

Solution to Part I—2007

1. The number 111111 is divisible by 1001. Now $1001 = 6 \times 166 + 5$. Hence the desired remainder is the same when we divide 11111 by 1001—that is, 100. The answer is (**d**).
2. Tell the cats to put their front legs up. Now there are still 34 heads, but only $34 \times 2 = 68$ legs on the ground. Hence $80 - 68 = 12$ legs are up in the air, and each cat puts up 2 of them. It follows that the number of cats is $12 \div 2 = 6$. The answer is (**a**).
3. Since $AB \leq 1$ and $BC \leq 2$, the area of triangle ABC is at most 2. This maximum value can be attained when $AB = 1$ and $BC = 2$ and they are perpendicular to each other. Now $CA = \sqrt{5}$, and we indeed have $2 \leq CA \leq 3$. The answer is (**a**).
4. The middle symbol must be an A. The first five symbols consist of two As and three Bs, and they can be arranged in all possible ways. The last five symbols, consisting also of two As and three Bs, must be in reverse order with respect to the first five symbols. In the first five symbols, we only have to find the number of ways of choosing two of the five positions for the two As. This is given by $\binom{5}{2} = 10$. The answer is (**c**). An exhaustive analysis also works.
5. The integers from 11 to 30 include six primes, namely 11, 13, 17, 19, 23 and 29. In the ten odd numbers among any twenty consecutive integers each at least 7, at least three are multiples of 3 and exactly two are multiples of 5, but at most one can be a multiple of 15. Hence the maximum is indeed six. The answer is (**c**).
6. We have $2x + 2y = 5 + y \geq 5$, so that the minimum value of $x + y$ is 5/2, attained at $(x, y) = (5/2, 0)$. Also, $x + y = 5 - x \leq 5$, so that the maximum value of $x + y$ is 5, attained at $(x, y) = (0, 5)$. The answer is (**d**).
7. If we take all the even numbers, clearly no two will differ by 3 or 5. Hence, we can take at least 500 numbers. Now partition the integers from 1 to 1000 into blocks of 10. From each of the following five pairs, we can take at most one number: $(10n + 1, 10n + 4)$, $(10n + 2, 10n + 5)$, $(10n + 3, 10n + 8)$, $(10n + 6, 10n + 9)$ and $(10n + 7, 10n + 10)$. Hence we can take no more than 500 numbers. The answer is (**d**).
8. Suppose there exists such a polynomial p. Since $a^n - b^n$ is divisible by $a - b$ for all positive integers a, b and n with $a \neq b$, $13 - 9$ must divide $p(13) - p(9)$. However, 4 does not divide 7, and we have a contradiction. The answer is (**a**). A parity argument also works.
9. Let the positive integral roots be $r \leq s \leq t$. Then $x^3 - 10x^2 + ax + b = (x - r)(x - s)(x - t)$. Expansion yields $x^3 - (r + s + t)x^2 + (st + tr + rs)x - rst$. Hence $r + s + t = 10$. The possible partitions are (1,1,8), (1,2,7), (1,3,6), (1,4,5), (2,2,6), (2,3,5), (2,4,4) and (3,3,4). The answer is (**b**).
10. The centre O of the circle lies on the axis of symmetry of ABCD. Let y be its height above BC. Then $OB^2 = y^2 + 3^2$ while $OA^2 = (8 - y)^2 + 1^2$. Equating these two values yields $y = 7/2$. Hence the radius is $\sqrt{(7/2)^2 + 3^2} = \sqrt{85}/2$. The answer is (**c**).



11. Multiplying one equation by the other, we have $xy + 4 + \frac{4}{xy} = 8$. This may be rewritten as $0 = (xy)^2 - 4xy + 4 = (xy - 2)^2$. Hence $xy = 2$. The answer is (**c**).
12. Let $s = \sin^2\theta$ and $c = \cos^2\theta$. We have $\sec^2\theta + \tan^2\theta = \frac{1+s}{c} = 2$. Since $s + c = 1$, $2 - c = 2c$ so that $c = 2/3$. It follows that $s = 1/3$. Now $\csc^2\theta + \cot^2\theta = \frac{1+c}{s} = 5$. The answer is (**d**).
13. Let E be the point on AD such that BE is perpendicular to AD. Complete the rectangle BEDF. Now $AB = BC$, $\angle AEB = 90° = \angle CFB$ and $\angle ABE = 90° - \angle CBE = \angle CBF$. Hence ABE and CBF are congruent triangles and they have equal area. It follows that BEDF is a square, and its area is also 16. Hence $BE = 4$. The answer is (**c**).



14. The number of incorrect answers for each of questions 1 and 6 is 0 or 5. The number of incorrect answers for each of questions 2 and 5 is 1 or 4. The number of incorrect answers for each of questions 3 and 4 is 2 or 3. A total of 8 incorrect answers can only be made up from 0+1+3+3+1+0. However, we would have an equal number of incorrect answers of T and incorrect answers of F. Hence the total must be 9, and it can be made up from either 0+1+2+2+4+0 or 0+4+2+2+1+0. However, the latter yields more incorrect answers of F than incorrect answers of T. It follows that the correct answers for the six questions are T, F, T, F, F and F respectively. Only student #4 has both an incorrect answer of T (for question 2) and an incorrect answer of F (for question 3). The answer is (**d**).

15. We have $10^{99} \times 5/9 \leq n \leq (10^{100} - 1)5/9$. Since n is an integer, $5 \times 10^{98} < n < 5 \times 10^{99}$. However, we must eliminate those values of n where $5 \times 10^{98} < n < 10^{99}$. Thus the number of acceptable values of n is about 4.5×10^{99}. Since $10^{99} \leq n \leq 10^{100} - 1$, the desired probability is very close to ½. The answer is (**c**).

16. We can rewrite the inequality as $(k - 3)(x^4 + y^4 + z^4) + (y^2 - z^2)^2 + (z^2 - x^2)^2 + (x^2 - y^2)^2 \geq 0$, from which it is clear that $k \geq 3$. The answer is (**c**).

Editor's note: Andy Liu is a professor in the Department of Mathematical and Statistical Sciences at the University of Alberta. He enjoys working on research problems that are easy to understand but not so easy to solve.

Edmonton Junior High Mathematics Contest Winners

Susan Ludwig

Top Teams

First Place: Vernon Barford School—Kaiven Zhou, Joe Ou, Andrew Qi

Second Place: McKernan School—Jennifer Yu, Mariya Sardarli, Robert Luo

Third Place: Grandview Heights School—Lisa Wang, Qaasim Mian, Eric Xu, Stephanie Li

Top Three Individual Winners

First Place
Kaiven Zhou, Vernon Barford School
Jennifer Yu, McKernan School
Mariya Sardarli, McKernan School

Second Place
Joe Ou, Vernon Barford School

Third Place
Stephen Just, St Rose Junior High School

Edmonton Junior High Mathematics Contest 2008

Multiple-Choice Problems

1. The equation, shown below, which has **NO** solution is

A. $5x = 3x$

B. $x + 1 = x$

C. $\frac{x^2 - 1}{x - 1} = 0,\ x \neq 1$

D. $\frac{x + 1}{x} = 0,\ x \neq 0$

Solution B

Solution:
Subtracting x from both sides of the equation given in B gives $1 = 0$, which has NO solution in any set of numbers.

2. A quadrilateral drawn on the coordinate plane has the vertices R $(-4, 4)$, S $(3, 2)$, T $(3, -2)$ and U $(2, -3)$. The area of quadrilateral RSTU is

A. $49\frac{1}{2}$ units^2

B. $38\frac{1}{2}$ units^2

C. $38\frac{1}{2}$ units^2

D. $20\frac{1}{2}$ units^2

Solution D

Solution:
Area of surrounding square is 49 units^2
Subtract the areas of the triangles

$49 - 21 - 7 - \frac{1}{2} = 20\frac{1}{2}$ units^2

3. The Jones family averaged 90 km/h when they drove from Edmonton to their lake cottage. On the return trip, their average speed was only 75 km/h. Their average speed for the round trip is

A. 81.8 km/h

B. 82.5 km/h

C. impossible to determine because the distance from Edmonton to the cottage is not given

D. impossible to determine, because the driving time is not given

Solution A

Solution:

$d = st \qquad Average\ speed = \frac{\text{total distance}}{\text{total time}}$

$$Average\ time = \frac{2d}{t_{out} + t_{in}} = \frac{2d}{\frac{d}{90} + \frac{d}{75}} = \frac{2}{\frac{1}{90} + \frac{1}{75}} = 81.8 \text{ km/h}$$

4. The **four** answers shown below each contain 100 digits, with only the first 3 digits and the last 3 digits shown. The 100-digit number that could be a perfect square is

A. 512 ... 972
B. 493 ... 243
C. 793 ... 278
D. 815 ... 021

Solution D

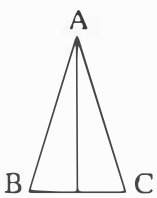
Solution:
By squaring each of the digits 0, 1, 2 ... 9 we see that squares cannot end in 8, 2, or 3.

5. Two sides of ΔABC each have a length of 20 cm and the third side has a length of 24 cm. The area of this triangle is

A. 192 cm^2
B. 173 cm^2
C. 141 cm^2
D. 72 cm^2

Solution A

Solution:



The triangle is isosceles; therefore the altitude drawn from A will bisect side BC at point D. The lengths of BD and CD are 12 cm. Apply Pythagorean Theorem to find the altitude.

$c^2 = a^2 + b^2$
$20^2 = 12^2 + b^2$
$256 = b^2$
$16 = b$

Calculate the Area

$A = \frac{1}{2} ab$
$= \times 16 \times 24$
$= 192 cm^2$

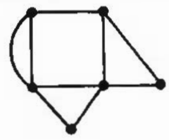
6. Only the **even** integers between 1 and 101 are written on identical cards, one integer per card. The cards are then placed in a box and mixed thoroughly. If a single card is drawn at random, then the probability that the number on the card is divisible by either 3 or 5, expressed as a decimal to the nearest hundredth, is

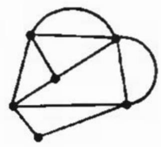
A. 0.50
B. 0.46
C 0.32
D. 0.20

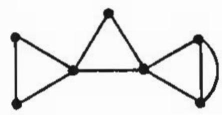
Solution B

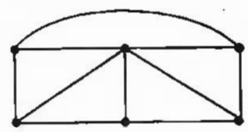
Solution:
The sequence 2, 4, 6, ..., 100 contains 50 terms, 23 of which are divisible by either 3 or 5.

7. Each of the following connected networks consists of segments and curves. A connected network is said to be *traversable* if we can trace the network without lifting our pencil from the paper. We must trace each of the segments or curves exactly once. Of the following connected networks, the one which is NOT traversable is

A.


B.


C.


D.


Solution: D

Solution:
A theorem of network theory is that if a connected network can be traversed, then it has at most two vertices of odd order. All of the networks above contain exactly two vertices of odd order, except for diagram D. The orders of the vertices in the diagrams (A though D) starting at top left and moving anticlockwise are

A. 3 – 4 – 2 – 4 – 2 – 3
B. 4 – 4 – 2 – 4 – 5 – 3
C. 2 – 2 – 4 – 2 – 4 – 3 – 3
D. 3 – 3 – 3 – 3 – 4

To traverse A, B or C, start at an odd vertex, and end at an odd vertex. Contest candidates may not know these theorems, but they should be able to traverse the networks by tracing.

8. The number of digits in the product $1^{2008} \times 25^{81} \times 2^{160}$ is

A. 2008
B. 2000
C. 162
D. 160

Solution C

Solution:

$1^{2008} \times 25^{81} \times 2^{160}$

$= 1 \times 25 \times 25^{80} \times 2^{80} \times 2^{80}$

$= 25 \times (25 \times 2 \times 2)^{80}$

$= 25 \times 100^{80}$

$= 25 \times 10^{160}$ (10^{160} is 161 digits)

$= 2.5 \times 10^{161}$

9. Each side of equilateral triangle ABC measures 5 units. On the base BC, we draw point P. From point P, we draw perpendiculars to the other two sides. The sum of the lengths of the perpendiculars is

A. $\frac{5\sqrt{3}}{2}$

B. $\frac{5\sqrt{2}}{2}$

C. $5\sqrt{3}$

D. $5\sqrt{2}$

Solution A

Solution:

Draw in AP = h

The area of the equilateral triangle is $\frac{1}{2}(5)(h)$ where h is the height of the triangle, or the area can be expressed as $\frac{1}{2}(5)m + \frac{1}{2}n$, where m and n are the lengths of the constructed perpendiculars. These 2 expressions must be equal, so

$\frac{1}{2}(5)(h) = \frac{1}{2}(5)m + \frac{1}{2}(5)n$

$h = m + n$

and $h = \frac{5\sqrt{3}}{2}$ by Pythagoras.

10. The sum of the ages of three brothers is 73. Tom is the oldest of the brothers, but he is less than 40 years old. The product of Tom's age and Michael's age is 750. The difference between Tom's age and Don's age is 7 more than the difference between Tom's age and Michael's age. Don is

A. 30 years old.
B. 21 years old.
C. 18 years old.
D. 8 years old.

Solution C

Solution:

Let T, D and M represent the age of Tom, Don and Michael respectively.

$$\begin{cases} T + D + M = 73 \\ T < 40 \\ TM = 750 \\ T - D - 7 = T - M \end{cases}$$

is a system of equations that interprets the given information.

Solve the system to obtain $T = 30$, $D = 18$ and $H = 25$.

Don is 18 years old.

11. The 9-digit number 6❑8,351,962 is divisible by 3, where ❑ represents a missing digit. The remainder when this number is divided by 6 is

A. 3
B. 2
C. 1
D. 0

Solution D

Solution:

We are told that the number is divisible by 3; since it ends in 2, it is also divisible by 2. If divisible by 3 and 2, it is divisible by 6.

12. A set of six numbers has an average of 47. If a seventh number is included with the original six numbers, then the average is 52. The value of the seventh number is

A. 99
B. 82
C. 49.5
D 32.9

Solution B

Solution:

$\frac{282+n}{7} = 52$

$282 + n = 364$

$n = 82$

13. A set of N real numbers has an average of N. A set of M real numbers, where $M < N$, taken from the original set of N numbers has an average of M. The average of the remaining $N - M$ numbers is

A. M
B. N
C. N + M
D. N − M

Solution C

Solution:

Pick an arbitrary number for M and N.
Set N has 10 numbers with an average of 10; the sum of set N will be 100. Set M has 8 numbers with an average of 8; the sum of set M will be 64. There will be 2 numbers remaining with the sum of the set being 36. The average will be 18. 18 is 10 + 8, or $M + N$.

OR

Set N has N numbers with an average of N; the sum of set N will be N^2. Set M has M numbers with an average of M; the sum of set M will be M^2. The numbers remaining will be $N^2 - M^2$.
The average will be $\frac{N^2 - M^2}{N - M} = N + M$.

14. A right-angled triangle has sides a, b and c, where c is the length of the hypotenuse. If we draw a line d from the right angle that is perpendicular to the hypotenuse, then an expression for d in terms of a, b and c is

A. $\frac{ab}{c}$
B. $\frac{bc}{a}$
C. $\frac{ac}{2b}$
D. $\frac{bc}{2a}$

Solution A

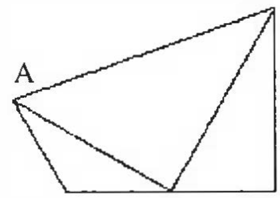
Solution:

Express the area of the right triangle in two ways:

$$\frac{1}{2}(a)(b) = \frac{1}{2}(d)(c)$$

$$d = \frac{ab}{c}$$

15. In the quadrilateral ABCD, AB = 1, BC = 2, CD = $\sqrt{3}$, ∠ABC = 120° and ∠BCD = 90°. E is the midpoint of BC. The perimeter of quadrilateral ABCD is



A. 5.16 units
B. 6.16 units
C. 6.38 units
D. 7.38 units

Solution D

Solution:

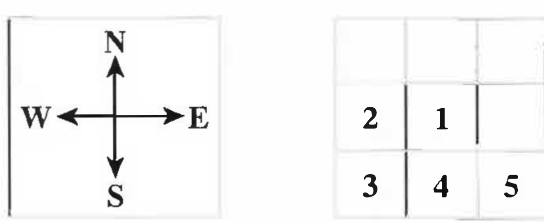
BE = 1 = EC since E is the midpoint of BC. A perpendicular from B to AE intersects AE at point F to create the 30° − 60° − 90° ΔBFE which gives EF = $\frac{\sqrt{3}}{2}$ and AE = $\sqrt{3}$. ΔECD is also 30° − 60° − 90° and since EC = 1 and CD = $\sqrt{3}$, we have ED = 2 by Pythagoras. In right triangle ΔAED, AD = $\sqrt{7}$ by Pythagoras. The perimeter of the quadrilateral is

$$1 + 2 + \sqrt{3} + \sqrt{7}$$
$$= 3 + \sqrt{3} + \sqrt{7}$$
$$\approx 7.38$$

Answers-Only Problems

Problem 1

On the partial grid shown below, positive integers are written in the following pattern: start with 1 and put 2 to its west. Put 3 south of 2, 4 to the east of 3 and 5 to the east of 4. Now 6 goes directly north of 5 and 7 to the north of 6. Then 8, 9 and 10 follow in that order to the west of 7 and so on, always moving in a counter-clockwise direction.



We have made a south turn at 2, an east turn at 3, a north turn at 5, and a west turn at 7. At which integer will we be making the fifth west turn?

Solution: 111

49	48	47	46	45	44	43
26	25	24	23	22	21	42
27	10	9	8	7	20	41
28	11	2	1	6	19	40
29	12	3	4	5	18	39
30	13	14	15	16	17	38
31	32	33	34	35	36	37

Consider concentric squares centred on 1. The next square is $3 \times 3 = 9 = 1 \times 8 + 1$, and this square has the integers from 2 through $(2+7=)$ 99 on its perimeter, with 7 at the northeast corner as the first west turn point.

The next square is $5 \times 5 = 25 = 3 \times 8 + 1$, and this square has the integers from 10 to $(10+15=)$ 25 on its perimeter, with 21 at the northeast corner as the second west turn number.

The next square is $7 \times 7 = 6 \times 8 + 1$, and this square has the integers from 26 to $(26+23=)$ 49 on its perimeter, with 43 at the northeast corner as the third west turn point. That is, all west turn points are on the northeast diagonal, and form the sequence 7, 21, 43, ..., and we could include 1 in this sequence to obtain 1, 7, 21, 43, ...

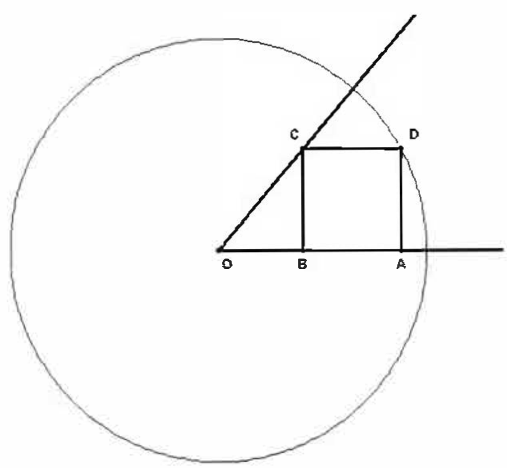
Considering the number sequence 1, 7, 21, 43, we can express this sequence as 1, 1+6, 7+14, 21+22, noting that the difference between successive terms is increasing by 8 units.

We can arrive at the same conclusion by noting that if A_n is the area of a given square, consider that $A_1 = 1$, $A_2 = 9$, $A_3 = 25$, $A_4 = 49, \ldots$, then $A_2 - A_1 = 9 - 1 = 8$, $A_3 - A_2 = 25 - 9 = 16$, $A_4 - A_3 = 49 - 25 = 24, \ldots$, which again shows that the difference between areas of successive squares is increasing by 8.

The difference, then, between 43 and the next integer in the sequence must be $22+8=30$, so that the fourth west turn point is $43+30=73$. The difference between 73 and next integer in sequence is $30+8=38$, so that the fifth west turn point is $73+38=111$.

Problem 2

Quadrilateral ABCD is a square. It is drawn so that points A and B are on $\overline{OA}$, and point D is on the circumference of a circle with its centre at point O. Point C is on $\overline{OC}$. If the radius of the circle is 10 units and $\angle COB = 45°$, then the area of square ABCD, to the nearest whole number, is



Solution: 20 units^2

Put point O on the origin of a coordinate plane so that OA is along the positive x-axis. Let the distance $OB = x = BC = BA = AD$, since ABCD is a square and $\angle COB = 45°$. Draw a radius from O to D.

In ΔODA, we have $OD = 10$, $OA = 2x$ and so that $(2x)^2 + x^2 = 10^2 \rightarrow 5x^2 = 100 \rightarrow x^2 = 20$, which is the area of square ABCD.

Problem 3

Let a, b and c represent three different positive integers whose product is 16. The maximum value of $a^b - b^c + c^a$ is

Solution: 263

Using factors of 16, a, b, and c are 1, 2 and 8. The maximum value of c^a is 2^8. Hence the maximum value of $a^b - b^c + c^a$ is $8^1 - 1^2 + 2^8 = 263$.

Problem 4

Let a, b and c represent any positive integers. The value of $\frac{1}{a}+\frac{1}{b}\left(1+\frac{1}{a}\right)+\frac{1}{c}\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)-\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)\left(1+\frac{1}{c}\right)$ is

Solution: −1

$$\frac{1}{a}+\frac{1}{b}\left(1+\frac{1}{a}\right)+\frac{1}{c}\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)-\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)\left(1+\frac{1}{c}\right)$$

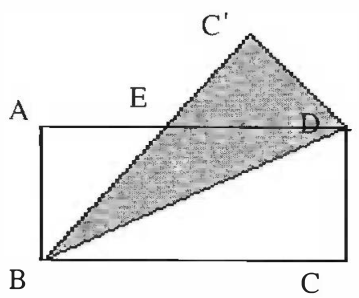
$$=\frac{1}{a}+\frac{1}{b}+\frac{1}{ab}+\frac{1}{c}+\frac{1}{ac}+\frac{1}{bc}+\frac{1}{abc}-\left(1+\frac{1}{b}+\frac{1}{a}+\frac{1}{ab}+\frac{1}{c}+\frac{1}{bc}+\frac{1}{ac}+\frac{1}{abc}\right)$$

$$=-1$$

Problem 5

A rectangular piece of paper ABCD is such that AB = 4 and BC = 8. It is folded along the diagonal BD so that triangle BCD lies on top of triangle BAD. C′ denotes the new position of C, and E is the point of intersection of AD and BC′.

The area of triangle BED is



Solution: 10

Triangles BAD and DC′B are congruent to each other. So triangles BAE and DC′E are also congruent to each other. Let AE = x. By Pythagoras's Theorem, BE = $\sqrt{4^2 + x^2}$. We also have BE = C′B − C′E = $8 - x$. Squaring both expressions and equating them, we have $16 + x^2 = 64 - 16x + x^2$, which simplifies to $16x = 48$ and $x = 3$.

Hence the area of triangle BED is $\frac{1}{2}$DC × DE = 2(8 − 3) = 10.

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