The Strange World of Continued Fractions

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What are continued fractions? Consider attempting to solve the quadratic equation

$$x^2 - 5x - 1 = 0$$

by dividing by x and writing the equation in the form

$$x = 5 + \frac{1}{x}.$$

The variable, x, is still found on the right side of the equation, but it can be replaced by its equal, $5 + 1/x$.

$$x = 5 + \frac{1}{x} = 5 + \cfrac{1}{5 + \cfrac{1}{x}}$$

If one repeats this several times, one gets

$$x = 5 + \cfrac{1}{5 + \cfrac{1}{5 + \cfrac{1}{5 + \cfrac{1}{5 + \cfrac{1}{x}}}}} \quad \ldots..$$

This type of multiple-decked fraction is known as a *simple continued fraction.* Continued fractions have been studied by many mathematicians in the past and are a subject of active investigation today. Almost every book on the theory of numbers includes a chapter on continued fractions.

An expression of the form

$$a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{a_3 + \cfrac{b_3}{a_4 + \ldots}}}$$

is called a *general continued fraction.* The numbers $a_1, a_2, a_3 \ldots b_1, b_2, b_3$ may be real or complex, and the number of terms may be finite or infinite.

In a simple continued fraction, all the numerators involved are 1s, and the first term, a_1, is usually a negative or positive number, while $a_2, a_3 \ldots$ are usually all positive.

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \ldots}}}$$

A *terminating* or *finite* simple continued fraction is a simple continued fraction that has a finite number of terms. It is in the form of

$$a + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ldots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}$$

Examples of these fractions have been found throughout mathematics for the past 2000 years, so its exact origin is hard to pinpoint, but the study of continued factions did not really begin until the later 1600s and early 1700s.

Traditionally, the origin of continued fractions is said to be at the time of the creation of Euclid's (ca 325 BC–ca 270 BC) algorithm, used to find the greatest common denominator of two numbers. Euclid's algorithm seems to have no connection to continued fractions, but by algebraically manipulating the algorithm, a simple continued fraction of p/q can be derived. Euclid and his predecessors probably never actually used this algorithm to discover continued fractions, but due to the close relationship it has to continued fractions, the creation of the algorithm is a very important step in its development.

For more than a thousand years, any work that used continued fractions was limited to specific examples. Indian mathematician Aryabhata (d 550 AD) used a continued fraction to solve a linear indeterminate equation. Rather than generalizing his method, he only used continued fractions in specific examples.

Traces of continued fractions were found in Greek and Arab mathematical writing as well, but were also only used in specific examples. Two Italians, Rafael Bombelli (1526–1572) and Pietro Cataldi (1548–1626), also contributed to this field, although, they still did not try to investigate the properties of continued fractions. Bombelli expressed $\sqrt{13}$ as a repeating continued fraction in 1572, and Cataldi did the same for $\sqrt{18}$ in 1613.

Continued fractions finally surfaced through the work of John Wallis (1616–1703). In his book, *Arithemetica Infinitorium* (1655), he developed and presented the identity

$$\frac{4}{\pi} = \frac{3 \bullet 3 \bullet 5 \bullet 5 \bullet 7 \bullet 7 \bullet 9}{2 \bullet 4 \bullet 4 \bullet 6 \bullet 6 \bullet 8 \bullet 9} \cdots$$

The first president of the Royal Society, Lord Brouncker, in about 1658, transformed this into:

$$\frac{4}{\pi} = 1 + \cfrac{1}{2 + \cfrac{9}{2 + \cfrac{25}{2 + \cfrac{49}{2 + \cfrac{81}{2 + \ldots}}}}} \quad \text{or} \quad \frac{4}{\pi} = 1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{2 + \cfrac{9^2}{2 + \ldots}}}}}$$

Wallis took an interest in this work and began the first steps to generalizing continued fraction theory. In his book *Opera Mathematica* (1695), Wallis laid a foundation for continued fractions. He was the first to use the term *continued fraction*.

The field of continued fractions began to flourish when mathematicians Leonhard Euler (1707–1783), Johann Lambert (1728–1777) and Joseph-Louis Lagrange (1736–1813) embraced the topic. Euler, in his work *De Fractionibus Continuis* (1737), was the first to prove that every rational number can be expressed as a terminating simple continued fraction.

Today, continued fractions serve as an important tool for new discoveries in the theory of numbers and in the field of Diophantine approximations. Analytic theory of continued fractions, an important generalization, is an extensive area for present and future research. Continued fractions have also been used within computer algorithms for computing rational approximations to real numbers, as well as solving indeterminate equations.

Rational Continued Fractions

There are two simple ways to write a finite simple continued fraction. It can be written as:

$$a_1 + \frac{1}{a_2 +} \frac{1}{a_3 +} \frac{1}{a_4 +} \cdots \frac{1}{+ a_n}$$

where the + signs after the first one are lowered to remind mathematicians of the real format of the continued fraction. A simple continued fraction can also be written by using the symbol $[a_1, a_2 \ldots a_n]$, so that

$$[a_1, a_2, \ldots a_n] = a_1 + \frac{1}{a_2 +} \frac{1}{a_3 +} \cdots \frac{1}{+ a_n}$$

The terms $a_1, a_2 \ldots a_n$ are called the *partial quotients* of the continued fraction.

Theorem: *Any rational number can be expressed as a finite simple continued fraction. Conversely, any finite simple continued fraction represents a rational number.*

Let p/q be any rational fraction, with $q>0$. Then

$$\frac{p}{q} = a_1 + \frac{r_1}{q}$$

where r_1 is the remainder and $0 < r_1 < q$, and a_1 is any positive or negative integer, or zero.

If $r_1 = 0$, then the process terminates. However, if $r_1 \neq 0$ then

$$\frac{p}{q} = a_1 + \cfrac{1}{\cfrac{q}{r_1}} \quad \text{where } 0 < r^1 < q$$

Repeating this process and dividing q again by r_1, we get

$$\frac{q}{r_1} = a_2 + \frac{r_2}{r_1} \quad \text{where } 0 < r^2 < r^1$$

These calculations continue until r_n equals 0. If p/q is indeed rational, then there has to be a point when $r_n = 0$ because the remainders form a sequence of decreasing non-negative integers where $q > r_1 > r_2 > r_3 \ldots$ and there cannot be an infinite number of positive integers between q and 0. By continually dividing, r_n eventually equals 0, and all the remainders cancel, leaving only the partial quotients of the finite simple continued fraction.

Therefore, any rational number, positive or negative, will give us $[a_1, a_2, a_3 \ldots a_n]$, where a_n is the terminating denominator of the continued fraction.

Let's examine the continued fraction expansion for the fraction 10/3. To write $\frac{10}{3}$ in the simple continued fraction form, where all the numerators in the chain of the expansions are 1s, we begin by writing $\frac{10}{3} = 3 + \frac{1}{3}$. We can, however, take it a step further and write 1/3 as 1/2+1/1, giving the number 10/3 the continued fraction expansion

$$\frac{10}{3} = 3 + \cfrac{1}{2 + \cfrac{1}{1}}$$

These representations can be recorded as < 3; 3 > or < 3; 2, 1 >.

Using the same procedure, we can change 75/31 into a simple continued fraction:

$$\frac{75}{31}=2+\frac{13}{31}=2+\cfrac{1}{\frac{31}{13}}=2+\cfrac{1}{2+\frac{5}{13}}=2+\cfrac{1}{2+\cfrac{1}{\frac{13}{5}}}=2+\cfrac{1}{2+\cfrac{1}{2+\frac{3}{5}}}$$

$$=2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{\frac{5}{3}}}}=2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{1+\frac{2}{3}}}}=$$

$$=2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{\frac{3}{2}}}}}=2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\frac{1}{2}}}}}=<2;2,2,1,1,2>$$

We could have also extended the process one more step and written the 1/ 2 as 1 + 1 / 1 + 1 / 1. This would give the expansion < 2; 2, 2, 1, 1, 1, 1 >.

Using $\frac{64}{27}$ as an example:

$$\frac{64}{27}=2+\frac{10}{27}$$

$$\frac{10}{27}=\cfrac{1}{\frac{27}{10}}=\cfrac{1}{2+\frac{7}{10}}$$

$$\frac{7}{10}=\cfrac{1}{\frac{10}{7}}=\cfrac{1}{1+\frac{3}{7}}$$

$$\frac{3}{7}=\cfrac{1}{\frac{7}{3}}=\cfrac{1}{2+\frac{1}{3}}$$

$$\frac{1}{3}=\cfrac{1}{\frac{3}{1}}=\frac{1}{3}$$

Since this fraction $\frac{1}{3}$ is the same as the original fraction on the last line, we know that we are finished. Therefore, we conclude that

$$\frac{64}{27}=2+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+\frac{1}{3}}}}\quad or\quad \frac{64}{27}=2+\frac{1}{2+}\frac{1}{1+}\frac{1}{2+}\frac{1}{3}$$

$$or\ \frac{64}{27}=[2,2,1,2,3].$$

These partial quotients [2, 2, 1, 2, 3], when assembled, will allow us to reconstruct the number 64/27.

$$2+\frac{1}{2+}\frac{1}{1+}\frac{1}{2+}\frac{1}{3}=2+\frac{1}{2+}\frac{1}{1}+\frac{3}{7}=2+\frac{1}{2+}\frac{7}{10}=2+\frac{10}{27}=\frac{64}{27}$$

Expansion of Irrational Continued Fractions

The method of finding the successive partial quotients of irrational numbers is repeated in order to expand the continued fraction. But *this* algorithm will never terminate or result in a rational number because when an integer is subtracted from an irrational number, the difference will still be irrational.

$$x=a_1+\frac{1}{x_2}$$

where, $x_2=\frac{1}{(x-a_1)}$

$$x_2=a_2+\frac{1}{x_3}$$

where, $x_3=\frac{1}{(x_2-a_2)}$

$$x_3=a_3+\frac{1}{x_4}$$

where, $x_4=\frac{1}{(x_3-a_3)}$

and so on.

Therefore,

$$x=a_1+\frac{1}{x_2}=a_1+\cfrac{1}{a_2+\frac{1}{x_3}}=a_1+\cfrac{1}{a_2+\cfrac{1}{a_3+\frac{1}{x_4}}}$$

or

$$x=[a_1,a_2,a_3,a_4\ldots]$$

As an example, the expansion of $\sqrt{3}=1.73205...$ will be used:

$$\sqrt{3}=1+\frac{1}{x_2}$$

where,

$$x_2=\frac{1}{\sqrt{3}-1}\bullet\frac{\sqrt{3}+1}{\sqrt{3}+1}=\frac{\sqrt{3}+1}{2}$$

$$\sqrt{3}=1+\cfrac{1}{\frac{\sqrt{3}+1}{2}}$$

Now the largest integer for the number $\frac{\sqrt{3}+1}{2}$ = 1.3660 ... is 1, so

$$\frac{\sqrt{3}+1}{2}=1+\frac{1}{x_3}$$

where,

$$x_3=\cfrac{1}{\frac{\sqrt{3}+1}{2}-1}=\frac{2}{\sqrt{3}-1}\bullet\frac{\sqrt{3}+1}{\sqrt{3}+1}=\sqrt{3}+1\,.$$

Then the largest integer for $\sqrt{3}+1 = 2.7320 \ldots$ is 2, so

$$\sqrt{3}+1 = 2 + \frac{1}{x_4}$$

where,

$$x_4 = \frac{1}{(\sqrt{3}+1)-2} = \frac{1}{\sqrt{3}-1} \bullet \frac{\sqrt{3}+1}{\sqrt{3}+1} = \frac{\sqrt{3}+1}{2}.$$

We see that this expansion will repeat and we can therefore conclude that the expression of $\sqrt{3}$ =[1, 1, 2, 1, 2, 1, 2 ...] *or* [1,1,2].

Theorem: *Any irrational number x can be expanded into an infinite simple continued fraction.*

Solving Quadratic Equations Using Continued Fractions

It is also possible to solve the quadratic equations of the form $ax^2 + bx + c = 0$ (where a, b and c are integers) using continued fractions. Using the equation $x^2 - 5x - 1 = 0$, we can rewrite this as $x^2 = 5x + 1$. Then $x = 5 + 1/x$. This means that whenever there is an x, we can replace it with a $5 + 1/x$. When we do this we get:

$$x = 5 + \frac{1}{x} = 5 + \cfrac{1}{5 + \cfrac{1}{x}} = [5, 5, 5, 5 \ldots]$$

We can continue to replace the x again and get a periodic continued fraction

$$x = 5 + \cfrac{1}{5 + \cfrac{1}{5 + \cfrac{1}{5 + \ldots}}}$$

Patterns in the Golden and Silver Means

The golden mean, Phi, is a root of the equation $x^2 + x - 1 = 0$, and as a continued fraction is expressed as Phi = [1,1,1,1...]. Silver means are the numbers that have the same repeating continued fraction pattern as Phi, such as [2, 2, 2, 2...],... [11, 11, 11, 11...] and so on.

The exact values of the silvers means, as well as the golden mean, can be calculated using simple algebra. For example, the quadratic equation $x^2 - 5x - 1 = 0$ contains a root that is a silver mean. When we solve this quadratic equation using the quadratic formula, we get $x = (5 \pm \sqrt{29})/2$. Since $(5 - \sqrt{29})/2$ is negative, the positive value, $(5 + \sqrt{29})/2$, is the value of the continued fraction [5, 5, 5, 5...], since all continued fractions are positive.

There is a pattern to the values of the continued fractions of the golden means as well. As we know, the quadratic equation that governs Phi is $x^2 + x - 1 = 0$. If we solve for x, we get $(1+\sqrt{5})/2$ as the value of the continued fraction [1, 1, 1...]. By looking at $\sqrt{2}$ = [1, 2, 2, 2...], we can determine that [2, 2, 2...] = $1 + \sqrt{2}$. Using the same reasoning, we can find that there is a pattern to the values of the continued fraction values of the golden and silver means:

[1, 1, 1, 1...] = $(1+\sqrt{5})/2$

[2, 2, 2, 2...] = $(2+\sqrt{8}/2) = 1+\sqrt{2}$

[3, 3, 3, 3...] = $(3+\sqrt{13})/2$

[4, 4, 4, 4...] = $(4+\sqrt{20})/2 = 2+\sqrt{5}$

[5, 5, 5, 5...] = $(5+\sqrt{29})/2$

[6, 6, 6, 6...] = $(6+\sqrt{40})/2 = 3+\sqrt{10}$

and so on.

Other Numbers with Patterns in Their Continued Fraction Expansions

"e," the base of natural logarithms, is the only number besides the square root expressions that yields this kind of continued fraction. Euler found many of these continued fraction expressions involving "e."

The continued fraction expansion of "e" is:

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{2}{3 + \cfrac{3}{4 + \cfrac{4}{5 + \ldots}}}}}$$

This is *not* a simple continued fraction (where the numerator always equals 1), so it cannot be expressed in bracket form. However, "e" can also be expanded so that it can be expressed as a simple continued fraction:

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{6 + \ldots}}}}}}}} = [2,1,2,1,1,4,1,1,6,1,1,8\ldots]$$

Euler also developed more continued fraction expansions involving "e" that were simple continued fractions:

$$\sqrt{e} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{5 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{9 + \ldots}}}}}} = [1,1,1,5,1,1,9,1,1,13,1,1,17\ldots]$$

$$e-1=1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{4+\ldots}}}}}=[1,1,2,1,1,4,1,1,6,1,1,8\ldots]$$

There are two other continued fraction expansions involving "e" developed by Euler:

$$\frac{e-1}{e+1}=\cfrac{1}{2+\cfrac{1}{6+\cfrac{1}{10+\cfrac{1}{14+\ldots}}}}=[0,2,6,10,14,18,22\ldots]$$

$$\frac{e-1}{2}=\cfrac{1}{1+\cfrac{1}{6+\cfrac{1}{10+\cfrac{1}{14+\ldots}}}}=[0,1,6,10,14,18,22\ldots]$$

This last expansion of Euler's allows us to approximate 'e' quickly. For example, the 7^{th} convergent to $\frac{(e-1)}{2}$ is $\frac{342762}{398959}$, so that

$$e=\frac{1084483}{398959}=2.718281828458\ldots$$

This number differs from the actual value of "e" by only one unit in the 12th decimal place.

Applications of continued fractions, as we have seen, have been closely tied to establishing rational approximations to irrational numbers, such as approximations to "e," or to the square root of n (where n is not a perfect square), as above. Another application of continued fractions arises in the area of mechanical engineering. Here, problems can be solved including the design of a gearbox that will take a given input of x revolutions per minute and deliver an output of y revolutions per minute. A third area of applications of continued fractions comes from the area of botany. Botanists have tried to understand the recurring appearance of the sequence 1, 2, 3, 5, 6, 13, 21, 34, 65,... in many natural settings. This sequence, the Fibonacci sequence, is found, for instance, in counting the patterns of seeds on a sunflower, or even leaves on a tree. Other applications deal with the calendar, the prediction of eclipses, chaos theory and the role that approximations play in designing musical instruments.

This topic is an excellent one to add to the mathematics curriculum of gifted students as an enrichment topic. Students studying this subject matter will gain additional facility in analyzing and generating number patterns, the representation of numbers in unique ways, increased proficiency with calculations, and the use of arithmetic and algebraic operations on real numbers. Students and teachers of mathematics alike can further explore the applications in mathematics and also such diverse fields as botany, astronomy and mechanics. Though its initial development seems to have taken a long time, once started, the field and its analysis grew rapidly. The fact that continued fractions are still being used signifies the long-term importance of the field.

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