The Banff Game: Probabilities Applied

A Craig Loewen

I am not entirely sure of the origin of this game. I remember playing it as a child, but it was only recently that I introduced it to my children on a muggy summer day in Banff.

We had already swum at the pool, watched TV, finished shopping and consumed the bear claw chocolates. Worse yet, it was raining and my children were bored. We desperately needed to find something for the kids to do. That's when I remembered this dice game, and we began to play. As we played we talked, and the kids asked me several interesting questions that led to some fun problem-solving challenges.

Here is how the game works.

Scoring

The chart below shows various combinations of dice, which when rolled *on a single roll* have the corresponding values.

⚀⚀⚀ =1000 pts	⚃⚃⚃ =400 pts
⚁⚁⚁ =200 pts	⚄⚄⚄ =500 pts
⚂⚂⚂ =300 pts	⚅⚅⚅ =600 pts
⚀⚁⚂⚃⚄⚅ =1500 pts	
⚀ =100 pts	⚄ =50 pts

For example, assume a player rolls all six dice and the following values result:

⚄⚅⚄⚀⚁⚄

This roll has a value of 600 points: 500 points are scored for the three ⚄s, and the single ⚀ scores a further 100 points.

The Rules

Players: This game is played with six regular six-sided dice and any number of players. (Note: It can also be played as a solitaire game testing to see how many turns are necessary to reach the final score).

1. On a turn a player starts by rolling all six dice.
2. The player now chooses to keep the points rolled (if any) or remove at least one die or combination of dice that have value according to the chart given, and then reroll the remaining dice.
3. The player continues to roll the dice, always removing one or more dice and rolling the remaining dice until one of the following happens:
   - The player chooses to stop and take the points rolled on this turn. Play passes to the left once the points are scored.
   - A combination of values is rolled that scores no points. If this happens, the player loses all the points accumulated on that turn, and play passes to the left.
   - All of the dice have been successfully rolled; that is, value combinations have been made using all six of the dice. If this happens, the player picks up all six dice and begins rolling again, adding to the total accumulated thus far on that turn.
4. The game ends when a player has reached 5000 points; however, all the other players get one final turn.
5. The player with the highest score wins.

An Example Turn

Assume a player rolls the following values with her first roll:

⚀⚀⚁⚂⚃⚃

The player must now remove at least one die. She would likely choose to remove the two ⚀s because they have a value of 200 points. The player must choose to either take the points or to risk them by rolling the remaining four dice. Assume the player chooses to risk the points and thus rerolls only the four dice that do not contribute to the point total (the ⚁⚂⚃⚃). Now let's assume the player gets the following results from the new roll: ⚀⚂⚃⚄. This second roll has produced a further 150 points (100 points for the ⚀, and 50 points for the ⚄). The player has three options:

- Stop rolling and take the 350 points accumulated over the two rolls.
- Take the ⚀ from the second roll (adding it to the two ⚀s from the first roll—a score of 300 points) and reroll the remaining three dice (the ⚂, ⚃ and ⚄) hoping for a higher score.
- Take the ⚀ and the ⚄ from the second roll (adding it to the two ⚀s from the first roll—a score of 350 points) and reroll the remaining two dice (the ⚂ and ⚃).

A turn does not end until a player chooses to stop and score the points collected, or rolls a combination of dice with no value. Remember, if at any time the player rolls a combination of dice that does not produce any points, she loses what she has collected on the turn thus far and play passes to her opponent.

A Few Questions

We were having a good time simply playing the game until my children started asking some interesting and rather difficult questions.

- What are my chances of rolling a ⚀ or a ⚄? At what point am I better off to stop than to roll again?
- How hard is it to roll a straight (⚀ through ⚅) in a single roll? Why is grandma the only one who seems to be able to roll a straight?
- Should I keep a ⚄ if I don't have to?

The last question was actually mine:

- What are the chances of rolling all six dice and getting nothing? And, why does this always happen to me?

A Few Answers

A sample space is a list of all of the possible outcomes from a chance event. When you roll a single die, the sample space is quite easy to determine. It looks like this:

[⚀ ⚁ ⚂ ⚃ ⚄ ⚅]

From this sample space we can determine a few things: any value ⚀ through ⚅ may possibly appear, and (if the die is fair) there is an equal chance of each value appearing. That chance is determined by counting the number of possibilities for any specific outcome (any given number), divided by the number of possible outcomes in the sample space. So, the chance of rolling a ⚃ with a single is 1/6 as there are six possible outcomes. The equation looks like this:

$$P(⚃) = \frac{\text{Number of outcomes containing ⚃}}{\text{Size of the sample space}} = \frac{1}{6}$$

It is more difficult to show the sample space for the rolling of two six-sided dice, but there are 36 possible outcomes, six possible outcomes on each of the two dice.

⚀⚀	⚀⚁	⚀⚂	⚀⚃	⚀⚄	⚀⚅
⚁⚀	⚁⚁	⚁⚂	⚁⚃	⚁⚄	⚁⚅
⚂⚀	⚂⚁	⚂⚂	⚂⚃	⚂⚄	⚂⚅
⚃⚀	⚃⚁	⚃⚂	⚃⚃	⚃⚄	⚃⚅
⚄⚀	⚄⚁	⚄⚂	⚄⚃	⚄⚄	⚄⚅
⚅⚀	⚅⚁	⚅⚂	⚅⚃	⚅⚄	⚅⚅

When you are rolling two dice, the chances of rolling a ⚃ on at least one die go way up. As it turns out, 11 different combinations of two dice contain at least one ⚃, and there are 36 possible outcomes in our sample space. It follows then that the probability of rolling at least one four is 11/36 or slightly less than 1/3. Likewise, we can calculate that there would be 216 possible outcomes with three dice ($6 \times 6 \times 6 = 6^3 = 216$), and there would be 1,296 (or 6^4) outcomes with four dice, 7,776 (or 6^5) outcomes with five dice, and 46,656 (or 6^6) outcomes with six dice.

We are now ready to tackle some of those important questions that should help us play this game more cleverly (and perhaps win).

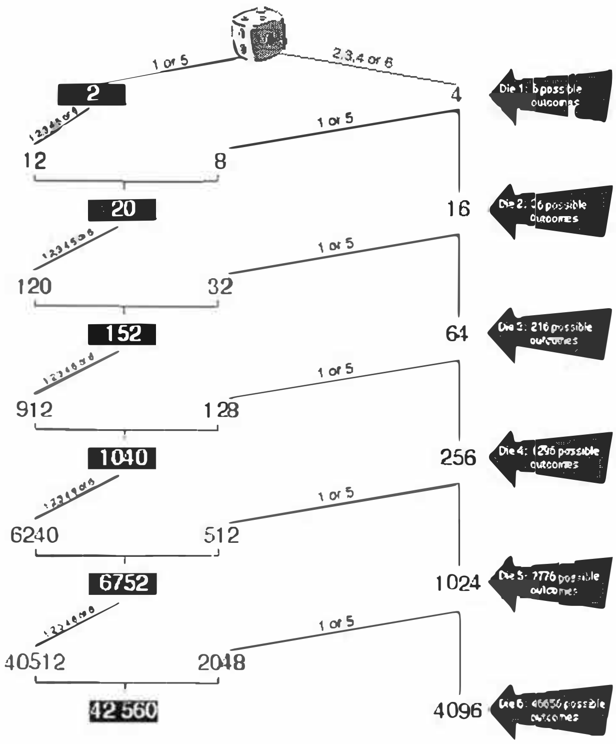
What are my chances of rolling a ⚀ or a ⚄? At what point am I better off to stop than to roll again?

The answers to these questions will obviously depend on how many dice you are rolling at the time. Let's start first with a simpler case; that is, where we are considering only a single outcome such as rolling a ⚃. As before, if the probability of rolling a ⚃ is 1/6 with one die and 11/36 with two dice, then presumably the more dice you have, the better are your chances of rolling a ⚃. The following table shows the chances of rolling at least one ⚃ with up to six dice.

No of Dice	Possible Outcomes	Positive Outcomes	Probability
1	6	1	0.167
2	36	11	0.306
3	216	91	0.421
4	1,296	671	0.517
5	7,776	4,651	0.598
6	46,656	31,031	0.665

We should be able to make similar computations with respect to rolling a ⚀ or a ⚄. This time we need to consider two outcomes, but the chance of rolling a ⚀ or ⚄ would be 2/6 (or 1/3) with a single die, and the chance of rolling at least one ⚀ or ⚄ with two dice would be 20/36. The chance has gone up from ⅓ to over ½ simply by adding the second die! This table shows the probability of rolling at least one ⚀ or one ⚄ with up to 6 dice.

No of Dice	Possible Outcomes	Positive Outcomes	Probability
1	6	2	0.333
2	36	20	0.556
3	216	152	0.704
4	1,296	1,040	0.802
5	7,776	6,752	0.868
6	46,656	42,560	0.912



The values in our chart can be verified using a tree diagram. Rolling six dice and hoping to get at least one ⚀ or one ⚄ is identical to the task of rolling a single die six times and hoping for at least one ⚀ or one ⚄ before your rolls are used up. The tasks are identical because each roll is independent from the others whether or not the dice are rolled simultaneously. The tree diagram shows the likelihood of success (rolling at least one ⚀ or one ⚄) if you are rolling a single die six times. In the diagram the number of possible successful rolls is shown with a black background. With a single die you have a 2 out of 6 chance of rolling at least one ⚀ or one ⚄, a 20 out of 36 chance with two dice, 152 out of 216 chance with three dice and so on.

We can apply what we know to build a strategy in playing this game: as long as we have two or more dice, the probabilities are on our side if we choose to roll again. But don't forget: having the probabilities on your side does not mean that you are guaranteed to roll what you want; there is still always the chance that you may "scratch."

Challenge: What is the probability of scratching if you are rolling only one die? Two dice?

Challenge: What is the probability of flipping a fair coin six times in a row and getting heads each time? Try to modify the tree diagram to calculate the probability.

How hard is it to roll a straight (a ⚀ through ⚅) in a single roll? Why is grandma the only one who seems to be able to roll a straight?

As it turns out, it is very difficult to roll a straight. We can use the same formula to compute our probability.

$$\text{Probability} = \frac{\text{Number of positive outcomes}}{\text{Size of the sample space}}$$

$$\text{P (straight)} = \frac{\text{Number of outcomes containing one of each}}{\text{Size of the sample space}}$$

Calculating the size of the sample space is relatively easy and has already been done above ($6^6 = 46{,}656$). Calculating the number of positive outcomes (that is, the outcomes that contain exactly one ⚀, one ⚁, one ⚂, one ⚃, one ⚄ and one ⚅) is more difficult. We can simplify this though by solving a similar and simpler problem: without repeating, how many numbers could you write using just two different digits; for example, just using a 1 and a 2? We can write only 12 and 21: there are two such numbers. How many could you write using three different digits (1, 2 and 3, no repeated digits)? We could write 123, 132, 231, 213, 312 and 321, a total of six in all. Keep going:

without repeating, how many ways are there of using four different digits (a 1, 2, 3 and 4) to build four-digit numbers? As it turns out, there are 24. Do you see a pattern?

$1 \times 2 = 2$
$1 \times 2 \times 3 = 6$
$1 \times 2 \times 3 \times 4 = 24$

If we continued our pattern, we would see that 120 numbers could be built using the digits 1 through 5 in various combinations, and 720 ways to use each of the numbers 1 through 6 without repeating a digit while writing unique 6-digit numbers. In other words, there are 720 different ways to arrange the digits 1 through 6. So, how is this like our dice experiment?

It is tempting to think there is only one way to roll a straight, but this would be a mistake. You see, we do not care which die gives us the ⚀, or which die gives us the ⚃ or ⚅. This means there are many different ways (720, in fact) to get one each of the values ⚀ through ⚅. We can put these numbers in our equation to calculate the probability of rolling a straight.

$$P\text{ (straight)} = \frac{\text{Number of outcomes containing one of each}}{\text{Size of the sample space}}$$

$$P\text{ (straight)} = \frac{720}{46\,656}$$

This value is obviously very small: it should be very difficult to roll a straight.

Here is a second way to think about this problem: we can imagine this situation by pretending that we are only rolling a single die, but rolling it six times. Each time we need to roll a value different from previous rolls. Any value will work for the first roll, but the second roll only has five possible values, and the third roll has four possible values and so on. The probability of rolling six consecutive usable values is calculated as follows:

$$P\text{ (straight)} = \frac{6}{6} \times \frac{5}{6} \times \frac{4}{6} \times \frac{3}{6} \times \frac{2}{6} \times \frac{1}{6} = \frac{720}{46656}$$

This computation produces a probability of approximately 0.015. It is fair to say that a player shouldn't be able to roll a straight very often. While I have no proof of cheating, perhaps grandma should explain why she is so uniquely fortunate at rolling straights when she plays this game!

Challenge: What is the chance of rolling an "almost straight;" that is, rolling six dice and getting exactly one pair?

Challenge: What is the chance of rolling a short straight (a ⚀ through ⚄, or a ⚁ through ⚅) with a single roll of six dice?

Should I keep a ⚄ if I don't have to?

This is an interesting question, and the answer may depend on a number of variables, such as a player's tolerance for risk, the number of dice this would leave for rerolling, the score in the game and so on. It is helpful, though, to consider some common scenarios in the game and to use our probabilities to help determine what the best choice may be in each case.

Scenario 1: Four dice have been successfully set aside, and rolling the remaining two dice produces a ⚀ and a ⚄. In this situation you do not need to take the ⚄, so you are faced with a decision: take the ⚀ and reroll the remaining die, or take both the ⚀ and ⚄ and roll all six dice. In each case you must consider the possible consequences of the choice you make.

If you choose to take the ⚀ and reroll the ⚄, what are the possible outcomes of this subsequent roll? If you roll a ⚀, your gamble worked. If you roll a ⚄, you are in exactly the same situation as if you had not chosen to roll (that is, you are no further ahead). If you roll a ⚁, ⚂, ⚃ or ⚅, the results are catastrophic. Obviously only one of the possible six outcomes improves your situation, and four of the possible six outcomes are disastrous. Clearly, the odds are not in your favour, and you should consider instead taking both the ⚀ and ⚄. Further, if you take the ⚀ and ⚄ (and thus have successfully scored all six dice), you may now pick up the dice and start over, and we have already seen that your chances of rolling further points in this situation exceed 90 per cent. This is the best option.

Scenario 2: A more interesting situation emerges if you have rolled your final three dice and received one ⚀, one ⚄ and another value you cannot use. In this case, are you better off to take the ⚀, or to take both the ⚀ and ⚄ even though you are not obligated to do so?

In this case the player's tolerance for risk is very important. Taking both the ⚀ and ⚄ leaves you in the unfavourable position of rerolling a single die. Taking only the ⚀ leaves you in a better position because your odds of succeeding with two dice are much better than with one; keep in mind, though, that the odds of succeeding with two dice are only slightly in your favour (55 per cent). The conservative player will probably opt to take both the ⚀ and ⚄, and pass the dice to his or her opponent. Of course, if you are not risking many points, you may just choose to roll anyway. In this game, as in life, risk can be tolerable or not, depending on both the odds and what is to be gained or lost.

Challenge: Are you better off to roll again or to take three ⚁s if you don't have to?

What are the chances of rolling all six dice and getting nothing? And, why does it always happen to me?

This question is probably the most difficult to answer. We are essentially looking for the number of possible combinations that have no ⚀, no ⚄ and no triplets (or 4, 5 or 6 of a kind). Of the 46,656 possible outcomes, there are 45,216 possible safe rolls with six dice, and thus only 1,440 ways to scratch while rolling six dice. Using our formula we can calculate that there is only about a 3 per cent chance of scratching while rolling all six dice! There is therefore no explanation as to why it happens to me so often. I'm just (un)lucky I guess!

Challenge: How do you calculate the number of safe rolls with six dice according to the rules of this game?

Variations

It is always fun to change the rules of a game just to see how it affects the game's outcome. The following variations simply increase the challenge or introduce other ways to score points.

A player must have a score of 450 to start the game. If the player does not get at least that many points he or she simply scratches the turn and play passes to the left.

Rolling four of a kind "doubles the triple;" that is, if a player rolls four ⚄s all at the same time, she or he has rolled a value of 1,000. Put another way, the first three ⚄s are worth 500 points, and the fourth ⚄ doubles that score for a final value of 1,000 points.

Play with five rather than six dice. A straight can be either a ⚁ through ⚅, or a ⚀ through ⚄ in a single roll.

A Word About Problem Solving

The Banff Game is genuinely enjoyable and worth playing for that reason alone; however, what it does have in common with other games is that it is a type of extended problem.

All problems have a list of conditions and restrictions as well as a purpose or goal. Solving the problem always requires inventing or selecting and applying a strategy. In comparison, games have conditions (the rules, how to roll the dice, how to score points), goals (reaching 5,000 points) and strategies (considering the probabilities): games qualify as problems in every way.

Anyone can play a game without being particularly strategic, but by asking a few questions a game can also become an opportunity to create an interesting problem-solving context. By applying problem-solving knowledge, we build new skills and develop mathematical understanding. The Banff game provides one context in which we may explore concepts related to probability and chance.

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