Mathematical Misconceptions: Diagnosis and Treatment

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A misconception, or incorrect conception, is generally a false idea. Students' misconceptions, a fundamental lack of understanding different from careless inaccuracy, are the natural result of efforts to construct new knowledge and can be viewed as an intellectual structure built on correct or incorrect previous knowledge. Although misconceptions are illogical, they make a lot of sense from students' perspectives. Students' mistakes are rational and meaningful efforts to learn mathematics (Ginsburg 1977).

Mathematics educators have observed and recognized students' misconceptions in practically every mathematics classroom from generation to generation. Students invent incorrect procedures to deal with new knowledge. To satisfy the need for calculating, simplifying or solving, students overgeneralize and distort a rule or procedure; for example, adding numerators and denominators to find the sum of two fractions, or assuming that division is distributive on addition because multiplication is. Such faulty mental models of mathematical concepts may be deeply ingrained in students' minds and are difficult to re-evaluate and remediate. They are quite persistent and may seriously interfere with the students' ability to learn mathematics if not detected and treated timely and appropriately. However, because mathematical misconceptions are unavoidable, they should not be eradicated but regarded as part of the normal learning process and as opportunities to enhance learning.

Commonly, correct new learning depends on previous correct learning, and incorrect new learning results from previous incorrect learning. However, incorrect new learning can also result from previous correct learning. This article provides examples and analyses of such cases.

Diagnosing Misconceptions

Analysis of misconceptions is critical to teaching and learning. Misconceptions are a natural part of students' conceptual structures that interact with new concepts and influence new learning in a negative way because they usually generate errors. Teachers must recognize and account for the causes of misconceptions. Diagnosing the origin of misconceptions is probably the most complex tasks of teachers. It requires in-depth knowledge about the structure of concepts, understanding of multiple representations and connections among subconcepts (Panasuk 2004). To identify the roots of misconceptions, teachers must look closely at students' prior experiences and current structures, and work at the level of specific details.

Different sources of misconceptions, including teaching methods, cause students to over-ride and overinterpret the rule. Students' misconceptions are not random but rather rule-governed and derive either from incorrect representation of the concepts or from procedures that have been taught (Ginsburg 1977). The following examples of overgeneralization of number properties are the most common underlying causes of students' misconceptions.

1. "Multiplication increases a number" ("division decreases a number") is a popular fallacious idea from elementary school. The origin of this misconception comes from a misrepresentation of the concept of multiplication or an incorrect generalization based on a limited set of multiplication examples that produce the larger number. Because a concept is a generalization that brings different elements or subconcepts into a basic relationship, it must be objectively true for all elements or subconcepts integrated in this idea. To convey the idea that multiplication increases a number, elementary teachers present only a subset of information because they are obviously dealing

with whole numbers. Stressing that multiplication of (positive) whole numbers produces a larger number is a critical detail that is unfortunately often overlooked. Although such overgeneralization of the multiplication concept at the elementary level might conveniently satisfy immediate needs, it is not only mathematically invalid but will negatively affect students' further learning. They will soon discover that the idea fails when a whole number is multiplied by a fraction, hence facing the problem of re-evaluating prior knowledge and rejecting it as incorrect. Inattention to important details that affect the development of correct understanding of mathematical concepts will cause confusion and generate new misconceptions.

2. When subtracting 18 from 43, elementary students usually produce the following:

$$\begin{array}{r} 43 \\ -\ \underline{18} \\ 35 \end{array}$$

In this example, students apply an incorrect and difficult cognitive structure, "subtract smaller from larger," to satisfy the need for calculating subtraction. The idea was likely emphasized in the classroom and became an element of the subtraction concept. Applying (incorrect) prior knowledge (subtract smaller from larger) to a new task (subtract 18 from 43), students distort the rule and develop (incorrect) new knowledge. When identifying the roots of the misconception, pointing out that students don't understand how to subtract whole numbers doesn't help. One must examine how this new knowledge (subtracting two-digit numbers when regrouping is required) connects to the previous cognitive structure (subtracting two-digit numbers when regrouping is not required) and embedded in a larger cognitive structure (the meaning of subtraction) students have already developed. The macro concept—subtraction—must be broken down into micro or subconcepts, such as place value, grouping, regrouping, anchoring to 10 and taking away, to better understand the structure of the macro concept and to consider all guiding principles from students' prior learning that they try to incorporate into a new idea.

Students often select information that they recognize from the problem to activate a seemingly appropriate existing cognitive structure, such as "subtract the smaller number from the larger number." If elementary students learn that a smaller number must be subtracted from a larger number without reference to a whole quantity, they attend only to the digits in the columns and ignore the place values of the digits. Students treat columns as a string of unrelated single digits. Treating subtraction with a set of rule-governed procedures violates the meaning of subtraction as deducting one whole quantity from another whole quantity. This approach is a typical example of overgeneralizing arithmetic operations and building incorrect new knowledge on previous incorrect knowledge.

There might be one more layer of this misconception. As previously stated, previous correct learning can influence incorrect new learning. For example, knowing that addition is commutative (correct conception), elementary students conclude that subtraction is also commutative (incorrect conception). Elementary teachers likely never explain that subtraction is not commutative, because they would need to explain that (8–3) is not the same as (3–8). They would need to address the meaning of (3–8), which is difficult without referring to the negative number concept. Only when negative numbers are introduced at the middle school level are students able to subtract 3 from 8 knowingly and explain the answer correctly.

To deal with this misconception teachers can provide students with remediation about the meaning of subtraction of one whole quantity from another whole quantity, the concept of regrouping and examples of real-life situations to demonstrate that taking three away from eight is not the same as taking eight away from three; that is, subtraction is not commutative. It might help to extend students' cognitive structure of subtraction with correct knowledge, conceptual understanding and, ideally, elimination of the conflicting rule.

3. Algebra students often reveal gaps in understanding and relationships between concepts studied in elementary and middle school. The following popular misconceptions are often seen in algebra classrooms:

$$\frac{a+b}{a}=b\ ;\ \frac{1}{a^{-1}+b^{-1}}=a+b\ ;\ 3\cdot\frac{x+1}{x-1}=\frac{3(x+1)}{3(x-1)}.$$

The analysis of such misconceptions is crucial and must not be limited to simply stating that students don't know how to deal with negative exponents, or they cannot cancel an addend in a sum, or they can cancel only a factor in a product, or by cancelling a they get the wrong answer. Such statements only scratch the surface and are not sufficient for correct identification of the roots of misconceptions. Rather, in-depth analysis of the structure of the problem, namely, "task analysis," is imperative and must be done to identify the sources of misconceptions (Panasuk 2004) to treat them properly and effectively.

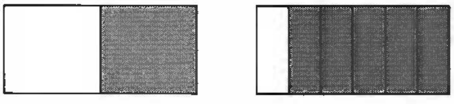
Interestingly, in each of these incorrect problems, the two prime suspects are (a) not knowing or paying attention to the order of operations and (b) the concept of fractions and operations with fractions. It is sometimes mistakenly assumed that the order of operations is only used for calculations at elementary school. The order of operations is a powerful concept that has its place and significance in algebra to convince students that the procedure is incorrect and to provide them with a solid tool to verify operations (Panasuk 2006).

Let us consider the common misconception of cancelling[1] a in the famous algebraic expression $\frac{a+b}{a}$.

When students cancel a, they likely lack knowledge and comprehension (Bloom 1956) of one or all of the following concepts:

i. Students might not have a solid foundation in the concept of equivalency of fractions, which is a prerequisite knowledge for reducing fractions and finding a common denominator; for example, if k is a common factor for both ak and bk, then $\frac{a \cdot k}{b \cdot k} = \frac{a}{b} \cdot \frac{k}{k} = \frac{a}{b} \cdot 1 = \frac{a}{b}$. It is typically emphasized that to obtain an equivalent fraction, both the numerator and the denominator are multiplied or divided by the same number. However, the reason for this is not emphasized. Teachers must ask students what would happen if the same number were added to or subtracted from both the numerator and the denominator. It is important to show that adding (or subtracting) the same number to the numerator and denominator does not produce an equivalent fraction. Such a demonstration relies on the logical principle that falsification of any hypothesis requires only one counter example. Accordingly, given the fraction $\frac{1}{2}$, adding 4 to both the numerator and the denominator will result in $\frac{1+4}{2+4} = \frac{5}{6}$. Clearly, the original fraction $\frac{1}{2}$ is not equivalent to $\frac{5}{6}$ because there is no unique number k that being multiplied by 1 produces 5 and being multiplied by 2 produces 6. It might be necessary to support the conclusion with a diagram that may well be more convincing to visual learners.

$\frac{1}{2}$ $\frac{5}{6}$



Although it is necessary to demonstrate examples when the rule works, demonstrating examples (numerical or algebraic) when the rule doesn't work is critical to prevent misconceptions and accentuate the rule.

ii. Students might only have a superficial knowledge (remembering the rule without connections to other concepts and details) of the addition of two fractions with the same denominator, which is not usually considered as a perplexing concept. It is likely that the students have practised only this identity $\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$ but have little or no experience to see that $\frac{a+b}{c}$ can be viewed as $\frac{a}{c} + \frac{b}{c}$. In other words, students must see that any fraction can be presented as a sum of several fractions with the same denominator. For example, $\frac{7}{12}$ can be seen as $\frac{1+6}{12} = \frac{1}{12} + \frac{6}{12}$ or $\frac{7}{12} = \frac{3+4}{12} = \frac{3}{12} + \frac{4}{12}$. Breaking down a fraction into two fractions with the same denominator is beneficial not only to prevent the misconception of cancelling but also to enhance students' knowledge of the concept of addition of fractions and related rules and procedures.

iii. Students might not have a firm definition of a fraction as a form of representation of a ratio of two quantities. In our case, the ratio of the quantity $(a+b)$ and the quantity a can be presented either as $(a+b) \div a$ or using a fraction bar, $\frac{a+b}{a}$. The students likely have not had enough practice converting fractions into division form and vice versa.

iv. Finally, the students might not have been encouraged to pay attention to the concept of order of operations. When $\frac{a+b}{a}$ is written as $(a+b) \div a$ it is easy to recognize the operations in their correct succession: calculate inside parentheses first and then divide. It also must be emphasized that to divide a sum by a number each addend must be divided by the number; that is, $(a+b) \div a = a \div a + b \div a$, which leads to $1 + b \div a = 1 + \frac{b}{a}$.

Often classroom teachers suggest replacing letters with numbers and performing calculations to show that cancelling a is an incorrect action. Replacing a letter with a number can be effective only if it is accompanied with the analysis of the structure of the problem, use of multiple representations (for example, representing the fraction $\frac{2+3}{2}$ as the division $(2+3) \div 2$, emphasizing correct order of operations and calculating inside the parentheses first:

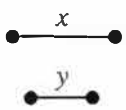
$(2+3)\div 2 = 5\div 2$, representing the division $(5 \div 2)$ as a fraction $\frac{5}{2}$, and meanings of operations and their order. Such a technique is likely to convince students of the correctness of the solution method and facilitate their understanding of the underlying concepts that support accurate procedures.

Assuming that $\frac{1}{a^{-1}+b^{-1}}$ is equal to $a+b$ is another example of fallacious reasoning and overgeneralization. There might be several causes of this misconception. First, it is a typical case of the *incorrect* application (distributing division over addition: $a\div(b+c)\neq a\div b+a\div c$) of the *correct* prior knowledge (distribution of multiplication over addition: $a(b+c)=ab+ac$). There also can be insufficient knowledge of the very concept of a fraction and the operations with fractions. Let us consider the following numerical fraction $\frac{15}{5+3}$, which can be rewritten as the division $15\div(5+3)$. Following the correct order of operations, we first calculate inside the parentheses, (5+3) = 8 and then divide 15 by 8. The result of the division can be shown in a fractional form $15\div 8=\frac{15}{8}$. If we try to distribute division over addition: $15\div(5+3)$, then we get $15\div 5+15\div 3=3+5=8$, which is not $\frac{15}{8}$. One numerical example is sufficient to show that $\frac{c}{a+b}$ is not equal to $\frac{c}{a}+\frac{c}{b}$. Based on the above, the expression $\frac{1}{a^{-1}+b^{-1}}$ needs to be written as $1\div(a^{-1}+b^{-1})$ to better recognize the order of operations. The first operation to be performed inside the parentheses is the application of the negative exponent and then addition of fractions with different denominators: $a^{-1}+b^{-1}=\frac{1}{a}+\frac{1}{b}=\frac{b+a}{ab}$; then $1\div\frac{a+b}{ab}=\frac{ab}{a+b}$

Another possible source of misconception is the false idea that $(a^{-1}+b^{-1})$ is equal to $(a+b)^{-1}$, based on the misconception that exponents can be distributed over addition, when in fact they can only be distributed over multiplication. A numerical example can be used to demonstrate the fallacy; that is, $2^{-1}+4^{-1}=\frac{1}{2}+\frac{1}{4}=\frac{3}{4}$, while $(2+4)^{-1}=6^{-1}=\frac{1}{6}$, thus $2^{-1}+4^{-1}\neq(2+4)^{-1}$.

The incorrect treatment of the multiplication of 3 by the algebraic fraction $\frac{x+1}{x-1}$ in the example $3\cdot\frac{x+1}{x-1}=\frac{3(x+1)}{3(x-1)}$ is likely to be a result of flawed prior knowledge related to some combination of (a) the concept of equivalent fractions, (b) the concept of multiplication of a whole number by a fraction and (c) the assumption that multiplication is distributive over division. Students might not recognize that the fraction $\frac{3(x+1)}{3(x-1)}$ can be presented as $\frac{3}{3}\cdot\frac{x+1}{x-1}$, which can be simplified to $\frac{x+1}{x-1}$. Multiplication of a whole number by a fraction has the same meaning as multiplication of two whole numbers, namely, repeated addition. This implies that $3\cdot\frac{x+1}{x-1}$ is equal to $\frac{x+1}{x-1}+\frac{x+1}{x-1}+\frac{x+1}{x-1}$. Since the denominators of the fractions are the same, we add the numerators only, to get $\frac{3(x+1)}{x-1}$. And finally, $3\cdot\frac{x+1}{x-1}$ can be viewed as $3\cdot[(x+1)\div(x-1)]$, which apparently is not the same as $3\cdot(x+1)\div 3\cdot(x-1)$ because multiplication is not distributive over division. If the symbolic representation looks complicated, a numerical example with a good dose of details might clarify confusion and reduce tension and anxiety; for example, $12\cdot\frac{(5+1)}{(5-1)}=12\cdot\frac{6}{4}=\frac{12\times 6}{4}=\frac{12}{4}\cdot 6=12\div 4\times 6=3\cdot 6=18$. However, $\frac{12(5+1)}{12(5-1)}=\frac{12\cdot 6}{12\cdot 4}=\frac{12\cdot 2\cdot 3}{12\cdot 2\cdot 2}=\frac{24}{24}\cdot\frac{3}{2}=\frac{3}{2}$. It is important to remember that this numerical example (as well as others) helps only to show that both expressions $12\cdot\frac{(5+1)}{(5-1)}$ and $\frac{12\cdot(5+1)}{12\cdot(5-1)}$ are not equal; it doesn't explain why they are not equal, and therefore other representations (such as repeated addition) of the concept are useful.

Students often combine $2x$ and $3y$ in the expression $(2x + 3y)$ into $5xy$. These students might have undeveloped understandings of the nature of like terms or coefficients, or they might misunderstand the idea of variables and the nature of arithmetic operations, particularly the order of operations. It is unlikely that algebra students would confuse $(x + y)$ and (xy) and think that these expressions are equal, and it is improbable that in the numerical example $2\cdot 4+3\cdot 7$ the students would attempt to add 2 and 3 and then multiply the sum by 4 and by 7. However, when dealing with the algebraic expression $2x + 3y$, they do sometimes consider it possible first to add the coefficients of unlike terms and then combine the variables in multiplication. Clearly, these students select some portions of information from problems they recognize and generate new strategies that seem appropriate to cope with the problem. The misconception can be treated using the variable-segment idea where two positive quantities x and y are presented as line segments (Panasuk 2004).

1. 

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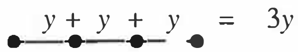
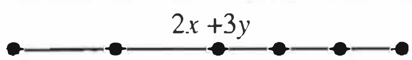
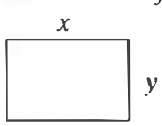
3. 

4. (2) and (3) together form a new segment



Meanwhile, the product of two positive quantities x and y can be illustrated as the area of the rectangle whose sides are x and y.



Area of the rectangle = xy

While $2x + 3y$ represents a length of a line segment (one dimensional figure), $5xy$ would represent the area of a rectangle (two-dimensional figure) whose side lengths are multiples of x and y; for example, a rectangle with side lengths $1x$ and $5y$.

Conclusion

It has been well documented that students' misconceptions are systematic and often result from overgeneralized or misapplied rules and algorithms (Ginsburg 1977; Resnick and Omanson 1987). In many cases, a misconception is just a symptom of a mathematical disorder that requires serious consideration and treatment. Knowing about the likelihood or possible existence of a misconception, teachers try to find ways to help students repair their faulty mental models. In doing so, the teachers answer the following questions:

- How do we know the misconception is present?
- How does it reveal itself?
- What is the underlying conceptual difficulty?
- In what way is the students' mental model faulty?
- How can we help students repair this particular faulty mental model?

Detailed diagnosing of misconceptions helps not only to identify the key subconcepts that students lack but also allows students to engage in cognitive analysis of the misconceptions. Students first need to be convinced (rather than just told) that the procedures they invented lead to incorrect answers and then encouraged to re-evaluate their prior knowledge and create correct conceptual understandings. Some misconceptions are the result of sterile presentation of rules or procedures; that is, how they are supposed to be done and when they work. Demonstration of the cases when the rules or procedures don't work or are not true is a powerful teaching method. To prevent misconceptions, it is pedagogically solid and warranted to show not only what works but also what doesn't. Providing counter examples and showing when the rule or procedure does not hold play essential roles in facilitating conceptual understanding and must accompany presentation of mathematical ideas, notions, rules and laws to amplify the idea of true versus false. Knowledge that a particular misconception is quite prevalent and likely to occur allows teachers to provide students with learning resources and activities that will assist them in remediation of the misconception and building a correct cognitive structure. Teachers may even consider changing the way they teach the topic to make it less likely that students will develop the misconception in the first place.

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