delta-k

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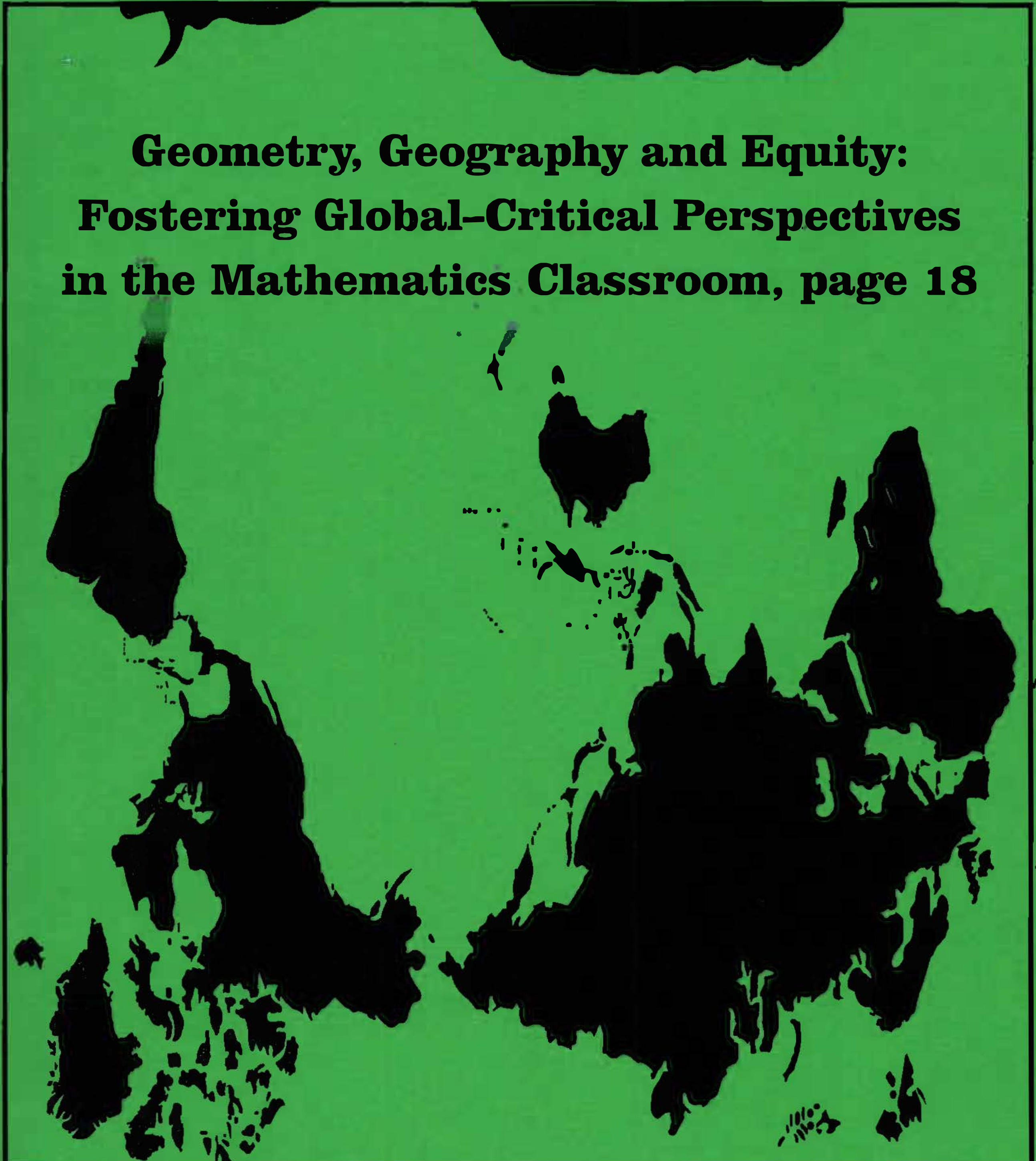

Geometry, Geography and Equity: Fostering Global-Critical Perspectives in the Mathematics Classroom, page 18

Guidelines for Manuscripts

delta-K is a professional journal for mathematics teachers in Alberta. It is published twice a year to

- promote the professional development of mathematics educators, and
- stimulate thinking, explore new ideas, and offer various viewpoints.

Submissions are requested that have a classroom as well as a scholarly focus. They may include

- personal explorations of significant classroom experiences;
- descriptions of innovative classroom and school practices;
- reviews or evaluations of instructional and curricular methods, programs or materials;
- discussions of trends, issues or policies;
- a specific focus on technology in the classroom; or
- a focus on the curriculum, professional and assessment standards of the NCTM.

Suggestions for Writers

1. *delta-K* is a refereed journal. Manuscripts submitted to *delta-K* should be original material. Articles currently under consideration by other journals will not be reviewed.
2. If a manuscript is accepted for publication, its author(s) will agree to transfer copyright to the Mathematics Council of the Alberta Teachers' Association for the republication, representation and distribution of the original and derivative material.
3. All manuscripts should be typewritten, double-spaced and properly referenced. All pages should be numbered.
4. The author's name and full address should be provided on a separate page. If an article has more than one author, the contact author must be clearly identified. Authors should avoid all other references that may reveal their identities to the reviewers.
5. All manuscripts should be submitted electronically, using Microsoft Word format.
6. Pictures or illustrations should be clearly labelled and placed where you want them to appear in the article. A caption and photo credit should accompany each photograph.
7. References should be formatted consistently using *The Chicago Manual of Style*'s author–date system.
8. If any student sample work is included, please provide a release letter from the student's parent/guardian allowing publication in the journal.
9. Articles are normally 8–10 pages in length.
10. Letters to the editor or reviews of curriculum materials are welcome.
11. Send manuscripts and inquiries to the editor: Gladys Sterenberg, 195 Sheep River Cove, Okotoks, AB T1S 2L4; e-mail gladyss@ualberta.ca.

MCATA Mission Statement

Providing leadership to encourage the continuing enhancement of teaching, learning and understanding mathematics.

delta-K

Volume 46, Number 2 June 2009

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EDITORIAL

Recently, I've been thinking extensively about change. According to the *Oxford English Dictionary* (1989), *change* can mean to exchange, to bend or to become different. In Alberta, we are in an era of change in mathematics education. What are we exchanging? How are we bending? Will we become different as mathematics teachers? Already we have seen the effect of change as we are asked to engage in a multitude of professional development opportunities.

When considering change, I am reminded of the importance of resiliency. Wang (1998, 12) describes resiliency as "the capacity to successfully overcome personal vulnerabilities and environmental stressors, to be able to 'bounce back' in the face of potential risks, and to maintain well-being." It is resiliency that gives each of us the ability to respond in generative ways to change. Bernard (1995) suggests that schools fostering resilience also allow children to develop the internal assets necessary for resilience, such as problem-solving skills, autonomy, a purposeful, constrictive, and optimistic outlook on the future, and effective communication and relationship skills. Resiliency is linked to problem-solving skills, autonomy and communication skills: a rationale for teaching mathematics in the way suggested by the curriculum changes we encounter.

In this issue of *delta-K*, change is an underlying theme. Regina Panasuk offers her opinion of how mathematical misconceptions can be rethought. Sandra Pulver presents an interesting way of thinking about fractions, and Daniel Jarvis and Immaculate Namukasa propose ideas of how we can change our focus of mathematics education to one of equity. Olive Chapman, Krista Letts and Lynda MacLellan describe their experiences of embracing change through lesson study, an emerging professional development approach. Craig Loewen provides practical applications of games and problems to be used in the changing mathematics classroom.

I wish you all the best this summer as you reflect on past experiences and look forward with optimism to the changes you will encounter in upcoming months. Enjoy!

References

Bernard, B. 1995. *Fostering Resilience in Children*. ERIC/EECE Digest EDO-PS99, 1–6.

Wang, M. 1998. "Building Educational Resilience." *Phi Delta Kappa Fastbacks* 430, 7–61.

Gladys Sterenberg

From the President's Pen

As another school year comes to an end, it is interesting to examine the changes that have occurred in our lives. Whether you changed positions, taught the new curriculum or a new course, or used a different approach, something new may have happened in your mathematical life this year. Students change every year, and the world around us changes at a phenomenal rate. How do we keep up? Do we hold on to the old, embrace the new or combine the two? With new curriculum being introduced, it is often hard to decide if our old tools are still applicable or if we need new tools.

Mathematical concepts remain the same. The laws of mathematics have not changed, but the methodology is quite different. I remember using the slide rule, and logarithmic and trigonometric tables in high school. I could multiply and divide using logarithms but never quite understood why I couldn't just use a pencil and paper. When I finally used a scientific calculator to do trigonometric calculations, I wondered why I had wasted my time with tables. Looking back, though, learning how to use any table was not a waste of time. Students often ask us why they need to know how to do things in mathematics. The dreaded "When am I ever going to use this?" question is asked as often today as it was years ago, but students today expect a better answer. With the information age, students expect instant answers to their questions. Learning for the sake of learning is not valued like it once was—or is it?

Teachers are expected to learn more now than ever. It is not enough to simply understand the material and know how to present it in many different ways. We need to understand how each student learns and differentiate for all. We must embrace all new ideas and stay on top of the latest educational research. New methodology and new technology mean that we must learn. Will SMART boards simplify mathematics? Do graphing calculators explain concepts? Of course not, but these tools help students make connections and visualize the mathematics. We have moved beyond having students memorize a single way of approaching a problem; they must use a variety of personal strategies to solve new problems. The world is changing, and we must change with it. Many things never change: the passion that mathematics teachers have for students and for the subject matter, the basic foundation that students must have to understand higher level mathematics and the sense of accomplishment that we all feel when we finally reach the student who is having difficulty.

Take time this summer to recharge and relax. Learn something new just for yourself. Thank you for all you do to make mathematics meaningful for students.

Sharon Gach

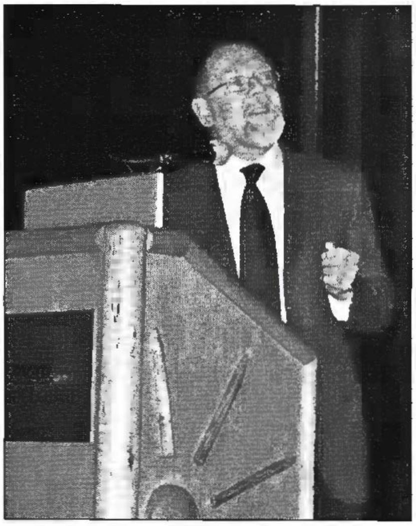
MCATA Annual Conference

The 2008 conference reflected our vision for mathematics learners with deep mathematical understanding and was organized around the theme "Exploring New Depths, Reaching New Heights." We had two keynote addresses. Dr Ron Eglash spoke about his experiences exploring the geometry of the indigenous villages, ruins and artifacts of African peoples and North American First Nations. Dr Tom Archibald, a historian of mathematics, shared his insights into how mathematics was used to create an educated elite and the implications this historical role plays in mathematics education today. The photographic memories demonstrate the healthy and vibrant engagement of mathematical professionals at the conference.

We are now planning the 2009 conference. Please join us October 23–25 at the River Cree Resort, Enoch, Alberta (Enoch borders west Edmonton). Our theme, "Mathematics Teaching for Real Learning," builds on the ATA's Real Learning First Initiative. The conference program will address teaching, learning, assessment and curricula, all focused on real learning first. Our speakers are among the best mathematics teachers and curriculum leaders we have in Alberta. As well, we will have the opportunity to learn with teacher educators, educational curriculum designers and resource producers from across Canada. Watch the MCATA website (www.mathteachers.ab.ca) for registration and program information.

Elaine Simmt

Keynote Speakers



Dr Tom Archibald

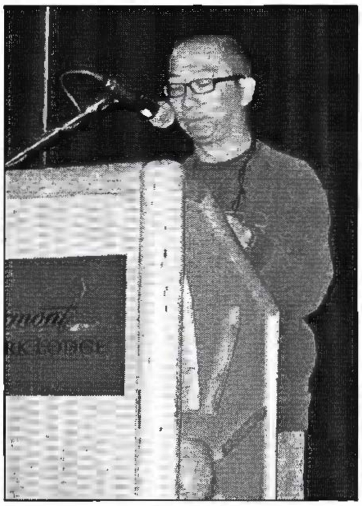
Dr Ron Eglash

Photographic Memories
MCATA Conference 2008

Math Educator Awards presented by MCATA president Sharon Gach

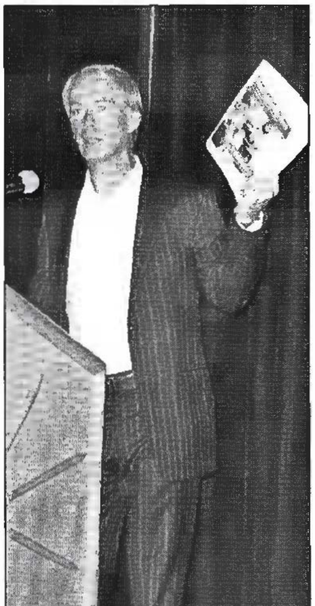
Janine Klevgaard

Lori Bell



Robert Wong,
MCATA web technician

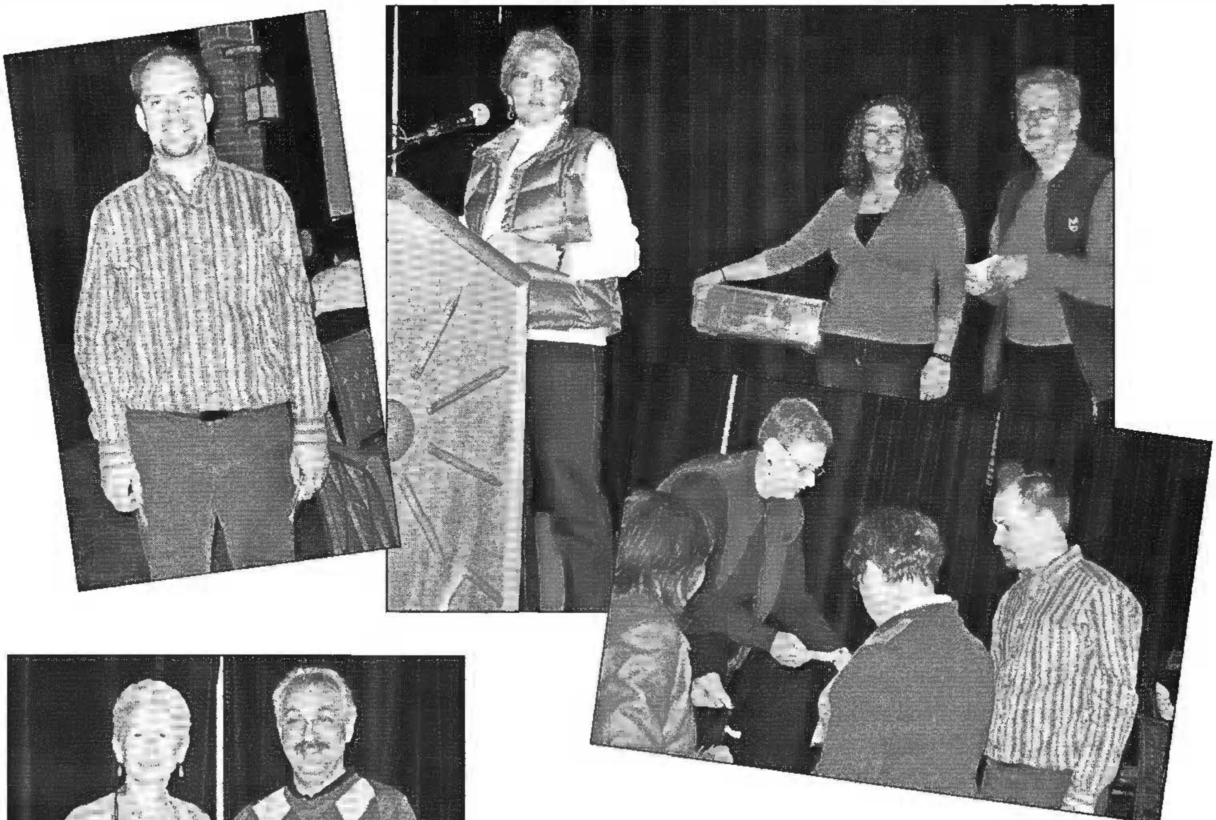
Conference registration: Sharon Gach, Mona Borle and Janis Kristjansson

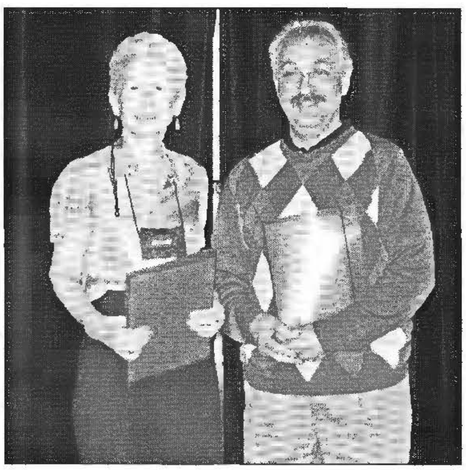


J-C Couture

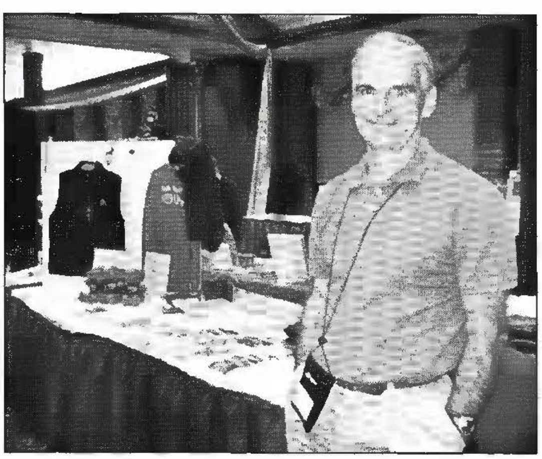
MCATA executive

Door prizes

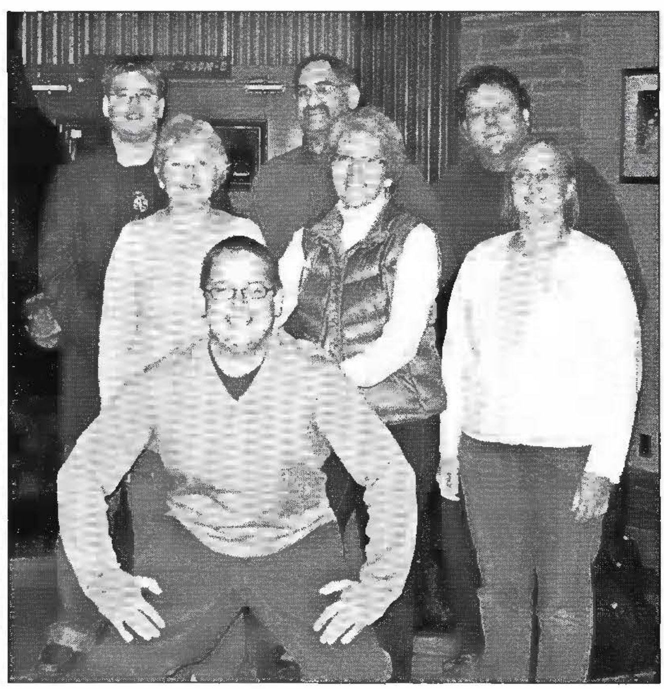




Friends of MCATA Award: Marie Hauk and Bryan Quinn



MCATA display: Darryl Smith



Conference committee (left to right, back to front): David Martin, Rod Lowry, Daryl Chichak, Janis Kristjansson, Elaine Simmt, Mona Borle and Tom Janzen.
Missing: Carmen Wasylynuik, Donna Chanasyk, Pat Chichak, Sharon Gach, Rebecca Steel, Susan Ludwig

The Right Angle: Report from Alberta Education

Provincial implementation of the revised K–9 mathematics program of studies started in September 2008 for kindergarten, Grades 1, 4 and 7. Grades 2, 5 and 8 were available for optional implementation in September 2008 and will be in provincial implementation next year. Grades 3, 6, 9 and 10 will be provincially implemented in September 2010. The following table summarizes the K–10 implementation schedule.

	September 2008	September 2009	September 2010
Provincial	K, 1, 4, 7	2, 5, 8	3, 6, 9, 10
Optional	2, 5, 8	3, 6, 9	

As you review the revised Alberta programs of studies for mathematics, remember that individual outcomes are only part of the program; the underlying philosophy provides the context for interpreting the outcomes. This underlying philosophy is described in the first few pages of the programs of studies. For instance, the following two sections are taken from the *Mathematics Kindergarten to Grade 9 Program of Studies* (2007).

Beliefs About Students And Mathematics Learning

Students are curious, active learners with individual interests, abilities and needs. They come to classrooms with varying knowledge, life experiences and backgrounds. A key component in successfully developing numeracy is making connections to these backgrounds and experiences. Students learn by attaching meaning to what they do, and they need to construct their own meaning of mathematics. This meaning is best developed when learners encounter mathematical experiences that proceed from the simple to the complex and from the concrete to the abstract. Through the use of manipulatives and a variety of pedagogical approaches, teachers can address the diverse learning styles, cultural backgrounds and developmental stages of students, and enhance within them the formation of sound, transferable mathematical understandings. At all levels, students benefit from working with a variety of materials, tools and contexts when constructing meaning about new mathematical ideas. Meaningful student discussions provide essential links among concrete, pictorial and symbolic representations of mathematical concepts. The learning environment should value and respect the diversity of students' experiences and ways of thinking, so that students are comfortable taking intellectual risks, asking questions and posing conjectures. Students need to explore problem-solving situations in order to develop personal strategies and become mathematically literate. They must realize that it is acceptable to solve problems in a variety of ways and that a variety of solutions may be acceptable.

Goals for Students

The main goals of mathematics education are to prepare students to

- use mathematics confidently to solve problems,
- communicate and reason mathematically,
- appreciate and value mathematics,
- make connections between mathematics and its applications,
- commit themselves to lifelong learning and
- become mathematically literate adults, using mathematics to contribute to society.

Students who have met these goals will

- gain understanding and appreciation of the contributions of mathematics as a science, philosophy and art,
- exhibit a positive attitude toward mathematics,
- engage and persevere in mathematical tasks and projects,
- contribute to mathematical discussions,
- take risks in performing mathematical tasks and
- exhibit curiosity.

By looking at the front end of the program of studies teachers can see how beliefs about students and mathematics learning provide a foundation for the general and specific outcomes that are detailed in the reminder of the document and support the goals for students.

Reference

Alberta Learning. 2007. *Mathematics Kindergarten to Grade 9 Program of Studies*. Edmonton, Alta: Alberta Learning.

Jennifer Dolecki

Mathematical Misconceptions: Diagnosis and Treatment

Regina Panasuk

A misconception, or incorrect conception, is generally a false idea. Students' misconceptions, a fundamental lack of understanding different from careless inaccuracy, are the natural result of efforts to construct new knowledge and can be viewed as an intellectual structure built on correct or incorrect previous knowledge. Although misconceptions are illogical, they make a lot of sense from students' perspectives. Students' mistakes are rational and meaningful efforts to learn mathematics (Ginsburg 1977).

Mathematics educators have observed and recognized students' misconceptions in practically every mathematics classroom from generation to generation. Students invent incorrect procedures to deal with new knowledge. To satisfy the need for calculating, simplifying or solving, students overgeneralize and distort a rule or procedure; for example, adding numerators and denominators to find the sum of two fractions, or assuming that division is distributive on addition because multiplication is. Such faulty mental models of mathematical concepts may be deeply ingrained in students' minds and are difficult to re-evaluate and remediate. They are quite persistent and may seriously interfere with the students' ability to learn mathematics if not detected and treated timely and appropriately. However, because mathematical misconceptions are unavoidable, they should not be eradicated but regarded as part of the normal learning process and as opportunities to enhance learning.

Commonly, correct new learning depends on previous correct learning, and incorrect new learning results from previous incorrect learning. However, incorrect new learning can also result from previous correct learning. This article provides examples and analyses of such cases.

Diagnosing Misconceptions

Analysis of misconceptions is critical to teaching and learning. Misconceptions are a natural part of students' conceptual structures that interact with new concepts and influence new learning in a negative way because they usually generate errors. Teachers must recognize and account for the causes of misconceptions. Diagnosing the origin of misconceptions is probably the most complex tasks of teachers. It requires in-depth knowledge about the structure of concepts, understanding of multiple representations and connections among subconcepts (Panasuk 2004). To identify the roots of misconceptions, teachers must look closely at students' prior experiences and current structures, and work at the level of specific details.

Different sources of misconceptions, including teaching methods, cause students to over-ride and overinterpret the rule. Students' misconceptions are not random but rather rule-governed and derive either from incorrect representation of the concepts or from procedures that have been taught (Ginsburg 1977). The following examples of overgeneralization of number properties are the most common underlying causes of students' misconceptions.

1. "Multiplication increases a number" ("division decreases a number") is a popular fallacious idea from elementary school. The origin of this misconception comes from a misrepresentation of the concept of multiplication or an incorrect generalization based on a limited set of multiplication examples that produce the larger number. Because a concept is a generalization that brings different elements or subconcepts into a basic relationship, it must be objectively true for all elements or subconcepts integrated in this idea. To convey the idea that multiplication increases a number, elementary teachers present only a subset of information because they are obviously dealing

with whole numbers. Stressing that multiplication of (positive) whole numbers produces a larger number is a critical detail that is unfortunately often overlooked. Although such overgeneralization of the multiplication concept at the elementary level might conveniently satisfy immediate needs, it is not only mathematically invalid but will negatively affect students' further learning. They will soon discover that the idea fails when a whole number is multiplied by a fraction, hence facing the problem of re-evaluating prior knowledge and rejecting it as incorrect. Inattention to important details that affect the development of correct understanding of mathematical concepts will cause confusion and generate new misconceptions.

2. When subtracting 18 from 43, elementary students usually produce the following:

$$\begin{array}{r} 43 \\ -\underline{18} \\ 35 \end{array}$$

In this example, students apply an incorrect and difficult cognitive structure, "subtract smaller from larger," to satisfy the need for calculating subtraction. The idea was likely emphasized in the classroom and became an element of the subtraction concept. Applying (incorrect) prior knowledge (subtract smaller from larger) to a new task (subtract 18 from 43), students distort the rule and develop (incorrect) new knowledge. When identifying the roots of the misconception, pointing out that students don't understand how to subtract whole numbers doesn't help. One must examine how this new knowledge (subtracting two-digit numbers when regrouping is required) connects to the previous cognitive structure (subtracting two-digit numbers when regrouping is not required) and embedded in a larger cognitive structure (the meaning of subtraction) students have already developed. The macro concept—subtraction—must be broken down into micro or subconcepts, such as place value, grouping, regrouping, anchoring to 10 and taking away, to better understand the structure of the macro concept and to consider all guiding principles from students' prior learning that they try to incorporate into a new idea.

Students often select information that they recognize from the problem to activate a seemingly appropriate existing cognitive structure, such as "subtract the smaller number from the larger number." If elementary students learn that a smaller number must be subtracted from a larger number without reference to a whole quantity, they attend only to the digits in the columns and ignore the place values of the digits. Students treat columns as a string of unrelated single digits. Treating subtraction with a set of rule-governed procedures violates the meaning of subtraction as deducting one whole quantity from another whole quantity. This approach is a typical example of overgeneralizing arithmetic operations and building incorrect new knowledge on previous incorrect knowledge.

There might be one more layer of this misconception. As previously stated, previous correct learning can influence incorrect new learning. For example, knowing that addition is commutative (correct conception), elementary students conclude that subtraction is also commutative (incorrect conception). Elementary teachers likely never explain that subtraction is not commutative, because they would need to explain that (8–3) is not the same as (3–8). They would need to address the meaning of (3–8), which is difficult without referring to the negative number concept. Only when negative numbers are introduced at the middle school level are students able to subtract 3 from 8 knowingly and explain the answer correctly.

To deal with this misconception teachers can provide students with remediation about the meaning of subtraction of one whole quantity from another whole quantity, the concept of regrouping and examples of real-life situations to demonstrate that taking three away from eight is not the same as taking eight away from three; that is, subtraction is not commutative. It might help to extend students' cognitive structure of subtraction with correct knowledge, conceptual understanding and, ideally, elimination of the conflicting rule.

3. Algebra students often reveal gaps in understanding and relationships between concepts studied in elementary and middle school. The following popular misconceptions are often seen in algebra classrooms:

$$\frac{a+b}{a}=b \; ; \; \frac{1}{a^{-1}+b^{-1}}=a+b \; ; \; 3\cdot\frac{x+1}{x-1}=\frac{3(x+1)}{3(x-1)}.$$

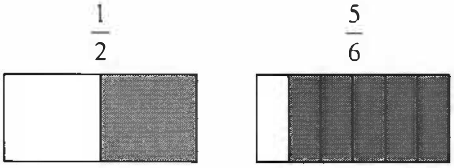
The analysis of such misconceptions is crucial and must not be limited to simply stating that students don't know how to deal with negative exponents, or they cannot cancel an addend in a sum, or they can cancel only a factor in a product, or by cancelling *a* they get the wrong answer. Such statements only scratch the surface and are not sufficient for correct identification of the roots of misconceptions. Rather, in-depth analysis of the structure of the problem, namely, "task analysis," is imperative and must be done to identify the sources of misconceptions (Panasuk 2004) to treat them properly and effectively.

Interestingly, in each of these incorrect problems, the two prime suspects are (a) not knowing or paying attention to the order of operations and (b) the concept of fractions and operations with fractions. It is sometimes mistakenly assumed that the order of operations is only used for calculations at elementary school. The order of operations is a powerful concept that has its place and significance in algebra to convince students that the procedure is incorrect and to provide them with a solid tool to verify operations (Panasuk 2006).

Let us consider the common misconception of cancelling[1] a in the famous algebraic expression $\frac{a+b}{a}$.

When students cancel a, they likely lack knowledge and comprehension (Bloom 1956) of one or all of the following concepts:

i. Students might not have a solid foundation in the concept of equivalency of fractions, which is a prerequisite knowledge for reducing fractions and finding a common denominator; for example, if k is a common factor for both ak and bk, then $\frac{a \cdot k}{b \cdot k} = \frac{a}{b} \cdot \frac{k}{k} = \frac{a}{b} \cdot 1 = \frac{a}{b}$. It is typically emphasized that to obtain an equivalent fraction, both the numerator and the denominator are multiplied or divided by the same number. However, the reason for this is not emphasized. Teachers must ask students what would happen if the same number were added to or subtracted from both the numerator and the denominator. It is important to show that adding (or subtracting) the same number to the numerator and denominator does not produce an equivalent fraction. Such a demonstration relies on the logical principle that falsification of any hypothesis requires only one counter example. Accordingly, given the fraction $\frac{1}{2}$, adding 4 to both the numerator and the denominator will result in $\frac{1+4}{2+4} = \frac{5}{6}$. Clearly, the original fraction $\frac{1}{2}$ is not equivalent to $\frac{5}{6}$ because there is no unique number k that being multiplied by 1 produces 5 and being multiplied by 2 produces 6. It might be necessary to support the conclusion with a diagram that may well be more convincing to visual learners.



Although it is necessary to demonstrate examples when the rule works, demonstrating examples (numerical or algebraic) when the rule doesn't work is critical to prevent misconceptions and accentuate the rule.

ii. Students might only have a superficial knowledge (remembering the rule without connections to other concepts and details) of the addition of two fractions with the same denominator, which is not usually considered as a perplexing concept. It is likely that the students have practised only this identity $\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$ but have little or no experience to see that $\frac{a+b}{c}$ can be viewed as $\frac{a}{c} + \frac{b}{c}$. In other words, students must see that any fraction can be presented as a sum of several fractions with the same denominator. For example, $\frac{7}{12}$ can be seen as $\frac{1+6}{12} = \frac{1}{12} + \frac{6}{12}$ or $\frac{7}{12} = \frac{3+4}{12} = \frac{3}{12} + \frac{4}{12}$. Breaking down a fraction into two fractions with the same denominator is beneficial not only to prevent the misconception of cancelling but also to enhance students' knowledge of the concept of addition of fractions and related rules and procedures.

iii. Students might not have a firm definition of a fraction as a form of representation of a ratio of two quantities. In our case, the ratio of the quantity $(a+b)$ and the quantity a can be presented either as $(a+b) \div a$ or using a fraction bar, $\frac{a+b}{a}$. The students likely have not had enough practice converting fractions into division form and vice versa.

iv. Finally, the students might not have been encouraged to pay attention to the concept of order of operations. When $\frac{a+b}{a}$ is written as $(a+b) \div a$ it is easy to recognize the operations in their correct succession: calculate inside parentheses first and then divide. It also must be emphasized that to divide a sum by a number each addend must be divided by the number; that is, $(a+b) \div a = a \div a + b \div a$, which leads to $1 + b \div a = 1 + \frac{b}{a}$.

Often classroom teachers suggest replacing letters with numbers and performing calculations to show that cancelling a is an incorrect action. Replacing a letter with a number can be effective only if it is accompanied with the analysis of the structure of the problem, use of multiple representations (for example, representing the fraction $\frac{2+3}{2}$ as the division $(2+3) \div 2$, emphasizing correct order of operations and calculating inside the parentheses first:

$(2+3)\div 2 = 5 \div 2$, representing the division $(5 \div 2)$ as a fraction $\frac{5}{2}$, and meanings of operations and their order. Such a technique is likely to convince students of the correctness of the solution method and facilitate their understanding of the underlying concepts that support accurate procedures.

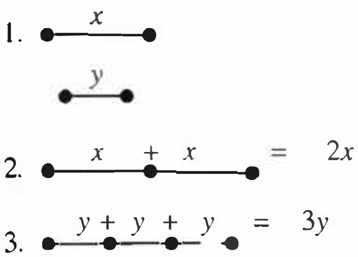
Assuming that $\frac{1}{a^{-1}+b^{-1}}$ is equal to $a+b$ is another example of fallacious reasoning and overgeneralization. There might be several causes of this misconception. First, it is a typical case of the *incorrect* application (distributing division over addition: $a \div (b+c) \neq a \div b + a \div c$) of the *correct* prior knowledge (distribution of multiplication over addition: $a(b+c) = ab + ac$). There also can be insufficient knowledge of the very concept of a fraction and the operations with fractions. Let us consider the following numerical fraction $\frac{15}{5+3}$, which can be rewritten as the division $15 \div (5+3)$. Following the correct order of operations, we first calculate inside the parentheses, (5+3) = 8 and then divide 15 by 8. The result of the division can be shown in a fractional form $15 \div 8 = \frac{15}{8}$. If we try to distribute division over addition: $15 \div (5+3)$, then we get $15 \div 5 + 15 \div 3 = 3 + 5 = 8$, which is not $\frac{15}{8}$. One numerical example is sufficient to show that $\frac{c}{a+b}$ is not equal to $\frac{c}{a}+\frac{c}{b}$. Based on the above, the expression $\frac{1}{a^{-1}+b^{-1}}$ needs to be written as $1 \div (a^{-1}+b^{-1})$ to better recognize the order of operations. The first operation to be performed inside the parentheses is the application of the negative exponent and then addition of fractions with different denominators: $a^{-1}+b^{-1} = \frac{1}{a}+\frac{1}{b} = \frac{b+a}{ab}$; then $1 \div \frac{a+b}{ab} = \frac{ab}{a+b}$

Another possible source of misconception is the false idea that $(a^{-1} + b^{-1})$ is equal to $(a + b)^{-1}$, based on the misconception that exponents can be distributed over addition, when in fact they can only be distributed over multiplication. A numerical example can be used to demonstrate the fallacy; that is, $2^{-1}+4^{-1} = \frac{1}{2}+\frac{1}{4} = \frac{3}{4}$, while $(2+4)^{-1} = 6^{-1} = \frac{1}{6}$, thus $2^{-1}+4^{-1} \neq (2+4)^{-1}$.

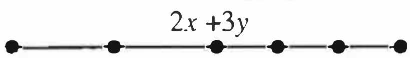
The incorrect treatment of the multiplication of 3 by the algebraic fraction $\frac{x+1}{x-1}$ in the example $3 \cdot \frac{x+1}{x-1} = \frac{3(x+1)}{3(x-1)}$ is likely to be a result of flawed prior knowledge related to some combination of (a) the concept of equivalent fractions, (b) the concept of multiplication of a whole number by a fraction and (c) the assumption that multiplication is distributive over division. Students might not recognize that the fraction $\frac{3(x+1)}{3(x-1)}$ can be presented as $\frac{3}{3} \cdot \frac{x+1}{x-1}$, which can be simplified to $\frac{x+1}{x-1}$. Multiplication of a whole number by a fraction has the same meaning as multiplication of two whole numbers, namely, repeated addition. This implies that $3 \cdot \frac{x+1}{x-1}$ is equal to $\frac{x+1}{x-1}+\frac{x+1}{x-1}+\frac{x+1}{x-1}$. Since the denominators of the fractions are the same, we add the numerators only, to get $\frac{3(x+1)}{x-1}$. And finally, $3 \cdot \frac{x+1}{x-1}$ can be viewed as $3 \cdot [(x+1) \div (x-1)]$, which apparently is not the same as $3 \cdot (x+1) \div 3 \cdot (x-1)$ because multiplication is not distributive over division. If the symbolic representation looks complicated, a numerical example with a good dose of details might clarify confusion and reduce tension and anxiety; for example, $12 \cdot \frac{(5+1)}{(5-1)} = 12 \cdot \frac{6}{4} = \frac{12 \times 6}{4} = \frac{12}{4} \cdot 6 = 12 \div 4 \times 6 = 3 \cdot 6 = 18$.

However, $\frac{12(5+1)}{12(5-1)} = \frac{12 \cdot 6}{12 \cdot 4} = \frac{12 \cdot 2 \cdot 3}{12 \cdot 2 \cdot 2} = \frac{24}{24} \cdot \frac{3}{2} = \frac{3}{2}$. It is important to remember that this numerical example (as well as others) helps only to show that both expressions $12 \cdot \frac{(5+1)}{(5-1)}$ and $\frac{12 \cdot (5+1)}{12 \cdot (5-1)}$ are not equal; it doesn't explain why they are not equal, and therefore other representations (such as repeated addition) of the concept are useful.

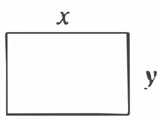
Students often combine $2x$ and $3y$ in the expression $(2x + 3y)$ into $5xy$. These students might have undeveloped understandings of the nature of like terms or coefficients, or they might misunderstand the idea of variables and the nature of arithmetic operations, particularly the order of operations. It is unlikely that algebra students would confuse $(x + y)$ and (xy) and think that these expressions are equal, and it is improbable that in the numerical example $2 \cdot 4 + 3 \cdot 7$ the students would attempt to add 2 and 3 and then multiply the sum by 4 and by 7. However, when dealing with the algebraic expression $2x + 3y$, they do sometimes consider it possible first to add the coefficients of unlike terms and then combine the variables in multiplication. Clearly, these students select some portions of information from problems they recognize and generate new strategies that seem appropriate to cope with the problem. The misconception can be treated using the variable-segment idea where two positive quantities x and y are presented as line segments (Panasuk 2004).



4. (2) and (3) together form a new segment



Meanwhile, the product of two positive quantities x and y can be illustrated as the area of the rectangle whose sides are x and y.



Area of the rectangle = xy

While $2x + 3y$ represents a length of a line segment (one dimensional figure), $5xy$ would represent the area of a rectangle (two-dimensional figure) whose side lengths are multiples of x and y; for example, a rectangle with side lengths $1x$ and $5y$.

Conclusion

It has been well documented that students' misconceptions are systematic and often result from overgeneralized or misapplied rules and algorithms (Ginsburg 1977; Resnick and Omanson 1987). In many cases, a misconception is just a symptom of a mathematical disorder that requires serious consideration and treatment. Knowing about the likelihood or possible existence of a misconception, teachers try to find ways to help students repair their faulty mental models. In doing so, the teachers answer the following questions:

- How do we know the misconception is present?
- How does it reveal itself?
- What is the underlying conceptual difficulty?
- In what way is the students' mental model faulty?
- How can we help students repair this particular faulty mental model?

Detailed diagnosing of misconceptions helps not only to identify the key subconcepts that students lack but also allows students to engage in cognitive analysis of the misconceptions. Students first need to be convinced (rather than just told) that the procedures they invented lead to incorrect answers and then encouraged to re-evaluate their prior knowledge and create correct conceptual understandings. Some misconceptions are the result of sterile presentation of rules or procedures; that is, how they are supposed to be done and when they work. Demonstration of the cases when the rules or procedures don't work or are not true is a powerful teaching method. To prevent misconceptions, it is pedagogically solid and warranted to show not only what works but also what doesn't. Providing counter examples and showing when the rule or procedure does not hold play essential roles in facilitating conceptual understanding and must accompany presentation of mathematical ideas, notions, rules and laws to amplify the idea of true versus false. Knowledge that a particular misconception is quite prevalent and likely to occur allows teachers to provide students with learning resources and activities that will assist them in remediation of the misconception and building a correct cognitive structure. Teachers may even consider changing the way they teach the topic to make it less likely that students will develop the misconception in the first place.

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The Strange World of Continued Fractions

Sandra M Pulver

What are continued fractions? Consider attempting to solve the quadratic equation

$$x^2 - 5x - 1 = 0$$

by dividing by x and writing the equation in the form

$$x = 5 + \frac{1}{x}.$$

The variable, x, is still found on the right side of the equation, but it can be replaced by its equal, $5 + 1/x$.

$$x = 5 + \frac{1}{x} = 5 + \cfrac{1}{5 + \cfrac{1}{x}}$$

If one repeats this several times, one gets

$$x = 5 + \cfrac{1}{5 + \cfrac{1}{5 + \cfrac{1}{5 + \cfrac{1}{5 + \cfrac{1}{x}}}}} \quad \ldots\ldots$$

This type of multiple-decked fraction is known as a *simple continued fraction.* Continued fractions have been studied by many mathematicians in the past and are a subject of active investigation today. Almost every book on the theory of numbers includes a chapter on continued fractions.

An expression of the form

$$a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{a_3 + \cfrac{b_3}{a_4 + \ldots}}}$$

is called a *general continued fraction.* The numbers $a_1, a_2, a_3 \ldots b_1, b_2, b_3$ may be real or complex, and the number of terms may be finite or infinite.

In a simple continued fraction, all the numerators involved are 1s, and the first term, a_1, is usually a negative or positive number, while $a_2, a_3 \ldots$ are usually all positive.

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \ldots}}}$$

A *terminating* or *finite* simple continued fraction is a simple continued fraction that has a finite number of terms. It is in the form of

$$a + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ldots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}$$

Examples of these fractions have been found throughout mathematics for the past 2000 years, so its exact origin is hard to pinpoint, but the study of continued factions did not really begin until the later 1600s and early 1700s.

Traditionally, the origin of continued fractions is said to be at the time of the creation of Euclid's (ca 325 BC–ca 270 BC) algorithm, used to find the greatest common denominator of two numbers. Euclid's algorithm seems to have no connection to continued fractions, but by algebraically manipulating the algorithm, a simple continued fraction of p/q can be derived. Euclid and his predecessors probably never actually used this algorithm to discover continued fractions, but due to the close relationship it has to continued fractions, the creation of the algorithm is a very important step in its development.

For more than a thousand years, any work that used continued fractions was limited to specific examples. Indian mathematician Aryabhata (d 550 AD) used a continued fraction to solve a linear indeterminate equation. Rather than generalizing his method, he only used continued fractions in specific examples.

Traces of continued fractions were found in Greek and Arab mathematical writing as well, but were also only used in specific examples. Two Italians, Rafael Bombelli (1526–1572) and Pietro Cataldi (1548–1626), also contributed to this field, although, they still did not try to investigate the properties of continued fractions. Bombelli expressed $\sqrt{13}$ as a repeating continued fraction in 1572, and Cataldi did the same for $\sqrt{18}$ in 1613.

Continued fractions finally surfaced through the work of John Wallis (1616–1703). In his book, *Arithemetica Infinitorium* (1655), he developed and presented the identity

$$\frac{4}{\pi}=\frac{3\bullet 3\bullet 5\bullet 5\bullet 7\bullet 7\bullet 9}{2\bullet 4\bullet 4\bullet 6\bullet 6\bullet 8\bullet 9}.$$

The first president of the Royal Society, Lord Brouncker, in about 1658, transformed this into:

$$\frac{4}{\pi}=1+\cfrac{1}{2+\cfrac{9}{2+\cfrac{25}{2+\cfrac{49}{2+\cfrac{81}{2+\ldots}}}}} \text{ or } \frac{4}{\pi}=1+\cfrac{1^2}{2+\cfrac{3^2}{2+\cfrac{5^2}{2+\cfrac{7^2}{2+\cfrac{9^2}{2+\ldots}}}}}$$

Wallis took an interest in this work and began the first steps to generalizing continued fraction theory. In his book *Opera Mathematica* (1695), Wallis laid a foundation for continued fractions. He was the first to use the term *continued fraction*.

The field of continued fractions began to flourish when mathematicians Leonhard Euler (1707–1783), Johann Lambert (1728–1777) and Joseph-Louis Lagrange (1736–1813) embraced the topic. Euler, in his work *De Fractionibus Continuis* (1737), was the first to prove that every rational number can be expressed as a terminating simple continued fraction.

Today, continued fractions serve as an important tool for new discoveries in the theory of numbers and in the field of Diophantine approximations. Analytic theory of continued fractions, an important generalization, is an extensive area for present and future research. Continued fractions have also been used within computer algorithms for computing rational approximations to real numbers, as well as solving indeterminate equations.

Rational Continued Fractions

There are two simple ways to write a finite simple continued fraction. It can be written as:

$$a_1+\frac{1}{a_2+}\frac{1}{a_3+}\frac{1}{a_4+}\cdots\frac{1}{+a_n}$$

where the + signs after the first one are lowered to remind mathematicians of the real format of the continued fraction. A simple continued fraction can also be written by using the symbol $[a_1, a_2 \ldots a_n]$, so that

$$[a_1,a_2,\ldots a_n]=a_1+\frac{1}{a_2+}\frac{1}{a_3+}\cdots\frac{1}{+a_n}$$

The terms $a_1, a_2 \ldots a_n$ are called the *partial quotients* of the continued fraction.

Theorem: *Any rational number can be expressed as a finite simple continued fraction. Conversely, any finite simple continued fraction represents a rational number.*

Let p/q be any rational fraction, with $q>0$. Then

$$\frac{p}{q}=a_1+\frac{r_1}{q}$$

where r_1 is the remainder and $0<r_1<q$, and a_1 is any positive or negative integer, or zero.

If $r_1=0$, then the process terminates. However, if $r_1 \neq 0$ then

$$\frac{p}{q}=a_1+\frac{1}{\frac{q}{r_1}} \text{ where } 0<r^1<q$$

Repeating this process and dividing q again by r_1, we get

$$\frac{q}{r_1}=a_2+\frac{r_2}{r_1} \text{ where } 0<r^2<r^1$$

These calculations continue until r_n equals 0. If p/q is indeed rational, then there has to be a point when $r_n=0$ because the remainders form a sequence of decreasing non-negative integers where $q>r_1>r_2>r_3 \ldots$ and there cannot be an infinite number of positive integers between q and 0. By continually dividing, r_n eventually equals 0, and all the remainders cancel, leaving only the partial quotients of the finite simple continued fraction.

Therefore, any rational number, positive or negative, will give us $[a_1,a_2,a_3\ldots a_n]$, where a_n is the terminating denominator of the continued fraction.

Let's examine the continued fraction expansion for the fraction 10/3. To write $\frac{10}{3}$ in the simple continued fraction form, where all the numerators in the chain of the expansions are 1s, we begin by writing $\frac{10}{3}=3+\frac{1}{3}$. We can, however, take it a step further and write 1/3 as 1/2+1/1, giving the number 10/3 the continued fraction expansion

$$\frac{10}{3}=3+\cfrac{1}{2+\cfrac{1}{1}}$$

These representations can be recorded as < 3; 3 > or < 3; 2, 1 >.

Using the same procedure, we can change 75/31 into a simple continued fraction:

$$\frac{75}{31}=2+\frac{13}{31}=2+\frac{1}{\frac{31}{13}}=2+\cfrac{1}{2+\frac{5}{13}}=2+\cfrac{1}{2+\cfrac{1}{\frac{13}{5}}}=2+\cfrac{1}{2+\cfrac{1}{2+\frac{3}{5}}}$$

$$=2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{\frac{5}{3}}}}=2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{1+\frac{2}{3}}}}}=$$

$$=2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{\frac{3}{2}}}}}=2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\frac{1}{2}}}}}=\ <2;2,2,1,1,2>$$

We could have also extended the process one more step and written the 1/ 2 as 1 + 1 / 1 + 1 / 1. This would give the expansion < 2; 2, 2, 1, 1, 1, 1 >.

Using $\frac{64}{27}$ as an example:

$$\frac{64}{27}=2+\frac{10}{27}$$

$$\frac{10}{27}=\frac{1}{\frac{27}{10}}=\cfrac{1}{2+\frac{7}{10}}$$

$$\frac{7}{10}=\frac{1}{\frac{10}{7}}=\cfrac{1}{1+\frac{3}{7}}$$

$$\frac{3}{7}=\frac{1}{\frac{7}{3}}=\cfrac{1}{2+\frac{1}{3}}$$

$$\frac{1}{3}=\frac{1}{\frac{3}{1}}=\frac{1}{3}$$

Since this fraction $\frac{1}{3}$ is the same as the original fraction on the last line, we know that we are finished. Therefore, we conclude that

$$\frac{64}{27}=2+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+\frac{1}{3}}}}\quad or\quad \frac{64}{27}=2+\frac{1}{2+}\frac{1}{1+}\frac{1}{2+}\frac{1}{3}$$

$$or\ \frac{64}{27}=[2,2,1,2,3].$$

These partial quotients [2, 2, 1, 2, 3], when assembled, will allow us to reconstruct the number 64/27.

$$2+\frac{1}{2+}\frac{1}{1+}\frac{1}{2+}\frac{1}{3}=2+\frac{1}{2+}\frac{1}{1}+\frac{3}{7}=2+\frac{1}{2+}\frac{7}{10}=2+\frac{10}{27}=\frac{64}{27}$$

Expansion of Irrational Continued Fractions

The method of finding the successive partial quotients of irrational numbers is repeated in order to expand the continued fraction. But *this* algorithm will never terminate or result in a rational number because when an integer is subtracted from an irrational number, the difference will still be irrational.

$$x=a_1+\frac{1}{x_2}$$

where, $x_2=\frac{1}{(x-a_1)}$

$$x_2=a_2+\frac{1}{x_3}$$

where, $x_3=\frac{1}{(x_2-a_2)}$

$$x_3=a_3+\frac{1}{x_4}$$

where, $x_4=\frac{1}{(x_3-a_3)}$

and so on.

Therefore,

$$x=a_1+\frac{1}{x_2}=a_1+\cfrac{1}{a_2+\frac{1}{x_3}}=a_1+\frac{1}{a_2+}\frac{1}{a_3+\frac{1}{x_4}}$$

or

$$x=[a_1,a_2,a_3,a_4\ldots]$$

As an example, the expansion of $\sqrt{3}=1.73205...$ will be used:

$$\sqrt{3}=1+\frac{1}{x_2}$$

where,

$$x_2=\frac{1}{\sqrt{3}-1}\bullet\frac{\sqrt{3}+1}{\sqrt{3}+1}=\frac{\sqrt{3}+1}{2}$$

$$\sqrt{3}=1+\cfrac{1}{\frac{\sqrt{3}+1}{2}}$$

Now the largest integer for the number $\frac{\sqrt{3}+1}{2}=1.3660\ldots$ is 1, so

$$\frac{\sqrt{3}+1}{2}=1+\frac{1}{x_3}$$

where,

$$x_3=\cfrac{1}{\frac{\sqrt{3}+1}{2}-1}=\frac{2}{\sqrt{3}-1}\bullet\frac{\sqrt{3}+1}{\sqrt{3}+1}=\sqrt{3}+1\ .$$

Then the largest integer for $\sqrt{3}+1 = 2.7320\ldots$ is 2, so

$$\sqrt{3}+1 = 2 + \frac{1}{x_4}$$

where,

$$x_4 = \frac{1}{(\sqrt{3}+1)-2} = \frac{1}{\sqrt{3}-1} \bullet \frac{\sqrt{3}+1}{\sqrt{3}+1} = \frac{\sqrt{3}+1}{2}.$$

We see that this expansion will repeat and we can therefore conclude that the expression of $\sqrt{3}$ =[1, 1, 2, 1, 2, 1, 2 ...] *or* [1,1,2].

Theorem: *Any irrational number x can be expanded into an infinite simple continued fraction.*

Solving Quadratic Equations Using Continued Fractions

It is also possible to solve the quadratic equations of the form $ax^2 + bx + c = 0$ (where a, b and c are integers) using continued fractions. Using the equation $x^2 - 5x - 1 = 0$, we can rewrite this as $x^2 = 5x + 1$. Then $x = 5 + 1/x$. This means that whenever there is an x, we can replace it with a $5 + 1/x$. When we do this we get:

$$x = 5 + \frac{1}{x} = 5 + \frac{1}{5 + \frac{1}{x}} = [5, 5, 5, 5\ \ldots]$$

We can continue to replace the x again and get a periodic continued fraction

$$x = 5 + \cfrac{1}{5 + \cfrac{1}{5 + \cfrac{1}{5 + \ldots}}}$$

Patterns in the Golden and Silver Means

The golden mean, Phi, is a root of the equation $x^2 + x - 1 = 0$, and as a continued fraction is expressed as Phi = [1,1,1,1...]. Silver means are the numbers that have the same repeating continued fraction pattern as Phi, such as [2, 2, 2, 2...],... [11, 11, 11, 11...] and so on.

The exact values of the silvers means, as well as the golden mean, can be calculated using simple algebra. For example, the quadratic equation $x^2 - 5x - 1 = 0$ contains a root that is a silver mean. When we solve this quadratic equation using the quadratic formula, we get $x = (5 \pm \sqrt{29})/2$. Since $(5 - \sqrt{29})/2$ is negative, the positive value, $(5 + \sqrt{29})/2$, is the value of the continued fraction [5, 5, 5, 5...], since all continued fractions are positive.

There is a pattern to the values of the continued fractions of the golden means as well. As we know, the quadratic equation that governs Phi is $x^2 + x - 1 = 0$. If we solve for x, we get $(1+\sqrt{5})/2$ as the value of the continued fraction [1, 1, 1...]. By looking at $\sqrt{2}$ = [1, 2, 2, 2...], we can determine that [2, 2, 2...] = $1 + \sqrt{2}$. Using the same reasoning, we can find that there is a pattern to the values of the continued fraction values of the golden and silver means:

[1, 1, 1, 1...] = $(1+\sqrt{5})/2$

[2, 2, 2, 2...] = $(2+\sqrt{8}/2) = 1+\sqrt{2}$

[3, 3, 3, 3...] = $(3+\sqrt{13})/2$

[4, 4, 4, 4...] = $(4+\sqrt{20})/2 = 2+\sqrt{5}$

[5, 5, 5, 5...] = $(5+\sqrt{29})/2$

[6, 6, 6, 6...] = $(6+\sqrt{40})/2 = 3+\sqrt{10}$

and so on.

Other Numbers with Patterns in Their Continued Fraction Expansions

"e," the base of natural logarithms, is the only number besides the square root expressions that yields this kind of continued fraction. Euler found many of these continued fraction expressions involving "e."

The continued fraction expansion of "e" is:

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{2}{3 + \cfrac{3}{4 + \cfrac{4}{5 + \ldots}}}}}$$

This is *not* a simple continued fraction (where the numerator always equals 1), so it cannot be expressed in bracket form. However, "e" can also be expanded so that it can be expressed as a simple continued fraction:

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{6 + \ldots}}}}}}}} = [2,1,2,1,1,4,1,1,6,1,1,8\ldots]$$

Euler also developed more continued fraction expansions involving "e" that were simple continued fractions:

$$\sqrt{e} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{5 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{9 + \ldots}}}}}} = [1,1,1,5,1,1,9,1,1,13,1,1,17\ldots]$$

$$e-1 = 1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{4+\ldots}}}}} = [1,1,2,1,1,4,1,1,6,1,1,8\ldots]$$

There are two other continued fraction expansions involving "e" developed by Euler:

$$\frac{e-1}{e+1} = \cfrac{1}{2+\cfrac{1}{6+\cfrac{1}{10+\cfrac{1}{14+\ldots}}}} = [0,2,6,10,14,18,22\ldots]$$

$$\frac{e-1}{2} = \cfrac{1}{1+\cfrac{1}{6+\cfrac{1}{10+\cfrac{1}{14+\ldots}}}} = [0,1,6,10,14,18,22\ldots]$$

This last expansion of Euler's allows us to approximate 'e' quickly. For example, the 7^{th} convergent to $\frac{(e-1)}{2}$ is $\frac{342762}{398959}$, so that

$$e = \frac{1084483}{398959} = 2.718281828458\ldots$$

This number differs from the actual value of "e" by only one unit in the 12th decimal place.

Applications of continued fractions, as we have seen, have been closely tied to establishing rational approximations to irrational numbers, such as approximations to "e," or to the square root of n (where n is not a perfect square), as above. Another application of continued fractions arises in the area of mechanical engineering. Here, problems can be solved including the design of a gearbox that will take a given input of x revolutions per minute and deliver an output of y revolutions per minute. A third area of applications of continued fractions comes from the area of botany. Botanists have tried to understand the recurring appearance of the sequence 1, 2, 3, 5, 6, 13, 21, 34, 65,... in many natural settings. This sequence, the Fibonacci sequence, is found, for instance, in counting the patterns of seeds on a sunflower, or even leaves on a tree. Other applications deal with the calendar, the prediction of eclipses, chaos theory and the role that approximations play in designing musical instruments.

This topic is an excellent one to add to the mathematics curriculum of gifted students as an enrichment topic. Students studying this subject matter will gain additional facility in analyzing and generating number patterns, the representation of numbers in unique ways, increased proficiency with calculations, and the use of arithmetic and algebraic operations on real numbers. Students and teachers of mathematics alike can further explore the applications in mathematics and also such diverse fields as botany, astronomy and mechanics. Though its initial development seems to have taken a long time, once started, the field and its analysis grew rapidly. The fact that continued fractions are still being used signifies the long-term importance of the field.

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Geometry, Geography and Equity: Fostering Global–Critical Perspectives in the Mathematics Classroom

Daniel Jarvis and Immaculate Namukasa

A significant part of the equity debate relates to the sensitivity that teachers show toward the different social, cultural and linguistic traditions that students bring to the mathematics classroom. This article combines geometry, geography and equity by discussing a variety of world map projections and five K–10 mathematics activities that promote increased global awareness and critical analysis skills.

Introduction

The fact that equity is at the top of the *Principles and Standards for School Mathematics* (NCTM 2000) underscores its importance: "Excellence in mathematics education requires equity—high expectations and strong support for all students" (p 11). Beyond curriculum content, pedagogical accommodation, differentiated assessment and technological access issues, another significant part of the equity debate relates to the sensitivity that teachers show toward social, cultural and linguistic traditions that students bring to the mathematics classroom (D'Ambrosio 1985, 1997).

Activities in the mathematics classroom that foster curricular connections, while raising awareness of global issues and traditions and promoting the development of critical analysis skills, represent worthwhile investigations. This article presents five student activities ranging from kindergarten to Grade 10. The first activity integrates geometry and geography as elementary students analyze the shapes of the continents, then position the seven polygons on an activity sheet and locate themselves on a world map template. The second activity integrates geometry and geography for Grades 4–6 students as they critically compare grid-based, student-generated representations of the common Mercator and less common Gall-Peters world map projections. In the final three activities, Grades 8–10 students further examine issues of cartographic distortion in terms of length, area, shape and angle as they explore various types of map projections. All five activities connect mathematics with the real world (and in this case with the "nonreal" projective world) and can form the basis for expanded discussions on cultural, historical and equity-based issues across the elementary and secondary school curriculum.

World Map Projections

Perhaps the most common world map projection, at least for North American, European and African students, is the Mercator projection (Figure 1).

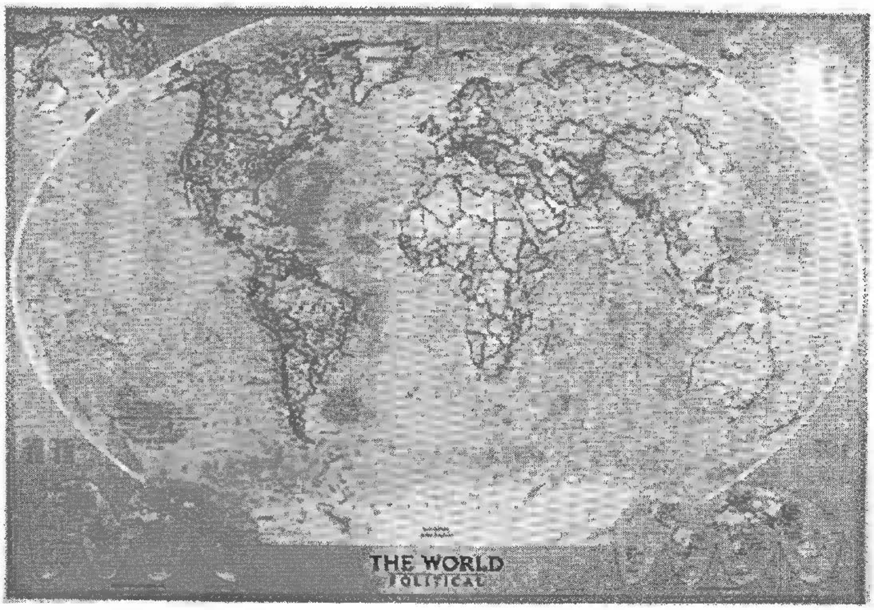

Figure 1: Common Mercator world map found in most North American schools

In fact, this particular representation of the planet Earth is so popular and pervasive that it is often difficult for students (and for most adults) to accept any other type of world map projection as legitimate. For example, one of the authors has the following three alternative world maps displayed in his office, and these maps never fail to elicit interest, surprise and questions from visiting students and colleagues.

The Upside-Down world map (Figure 2)---drawn in 1970 by a 12-year-old Australian student, Stuart McAuthur—is really a misnomer for two reasons:

1. From outer space there is no up or down in terms of the Earth's position—north and south poles have been arbitrarily selected to aid in navigation but are arguably rather meaningless in the larger universe context.
2. For many students living in the southern hemisphere, particularly in Oceania, this is the common world map displayed in schools and is therefore not considered upside down at all.

North Americans are surprised at the position of their country (that is, lower-left corner) and are uncomfortable with the map's strange and "incorrect" upside-down perspective. For Canadians, one small comfort is that the St Lawrence River flows down the map and into the Atlantic Ocean.

The Pacific-centred map (Figure 3) is the second alternative world map projection in which the international date line located within the large Pacific Ocean is depicted as central to the map, rather than being the line selected to form both the peripheral boundaries, as in many Mercator projection maps. One of the authors first encountered this world map while teaching at an international school in northern Thailand. Many classrooms in Thailand, and presumably in other Asian and perhaps South American countries, would feature such maps as normal and familiar representations of the planet.

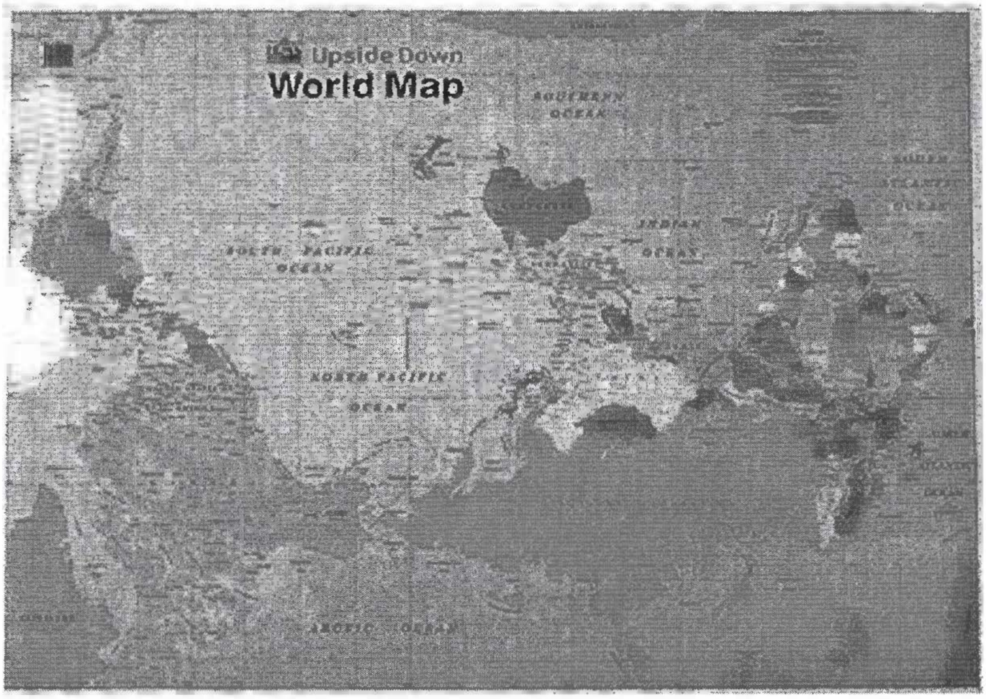

Figure 2: Upside-Down, Reversed or South-Up world map common in the southern hemisphere

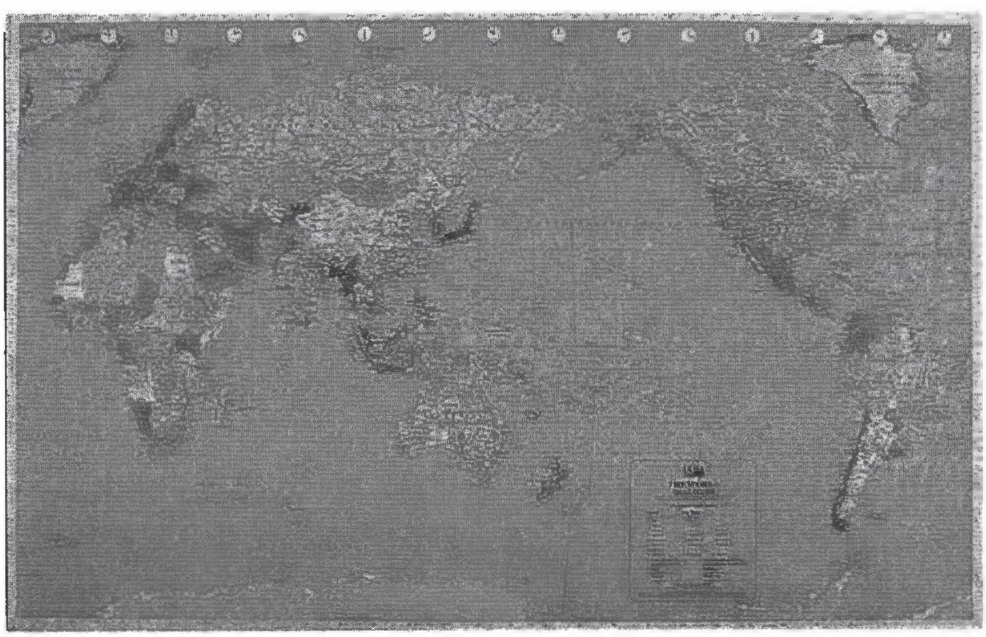

Figure 3: "Pacific-centred" map common in many Asian schools

The Gall-Peters equal area world map projection forms yet another significant and different representation of our three-dimensional earth on a two-dimensional surface.

In this equal-area cylindrical map—named after both the Scottish clergyman James Gall (1885) and the historian/activist Arno Peters (who presented his version in 1973)—the land masses are presented in their actual (or at least much closer to the actual) surface areas, without the major distortion that occurs toward the poles in other projections. For example, in the Mercator projection the size of regions is increasingly inflated according to their distance from the equator. As a result, Greenland sometimes appears larger than Africa, whereas in reality Africa is 14 times larger than Greenland. One of the authors, a native of Uganda, can understand Peters's emphasis on the political implications of such representations and how this often makes developing regions or countries near the equator appear much smaller and less significant than they really are. Although not entirely accurate in terms of its proportions, the Gall-Peters map presents a different and useful perspective of the world.

Taken together, the Mercator, reversed, Pacific-centred and Gall-Peters world map projections[1] can help students construct a more accurate and richer understanding of the earth. To further build on this increased understanding, and with a view to integrating geometry and geography, we present two mathematics activities that teachers can implement in the elementary curriculum.

Me and My Geometrical World

In this activity, K–3 students are escorted to the world map presumably hanging in a school hallway, library or classroom. Students have likely passed by this large map many times, but they are about to look at it in a new way that will make it difficult to ever pass by mindlessly again. Once the students are seated on the floor, the teacher asks them to name some basic geometric shapes; for example, triangles, circles, squares, rectangles and parallelograms. Then students are instructed to squint until the map details become fuzzy in order to identify any of these large geometric shapes. With a pointer, student volunteers take turns pointing out a few such perceived shapes on the map. This activity may also be done on an interactive whiteboard with an Internet connection.

Students then return to class and paste colourful, precut (or for older grades, student-generated and cut out) geometric shapes onto a template (Figure 6). While the template (see Student Activity: Me and My Geometrical World) obviously does not include direct visual reference to the world's 190 plus countries, it does provide an interesting perspective of the seven major continents and their approximate spatial relationship. Finally, students are asked to locate (in atlases, classroom maps or on the Internet) their school or city on the shapes they have pasted on the page and to mark the location with a coloured dot. If many international students or ESL immigrant children are in the class, the teacher may have students locate and mark their countries of origin on the world map,

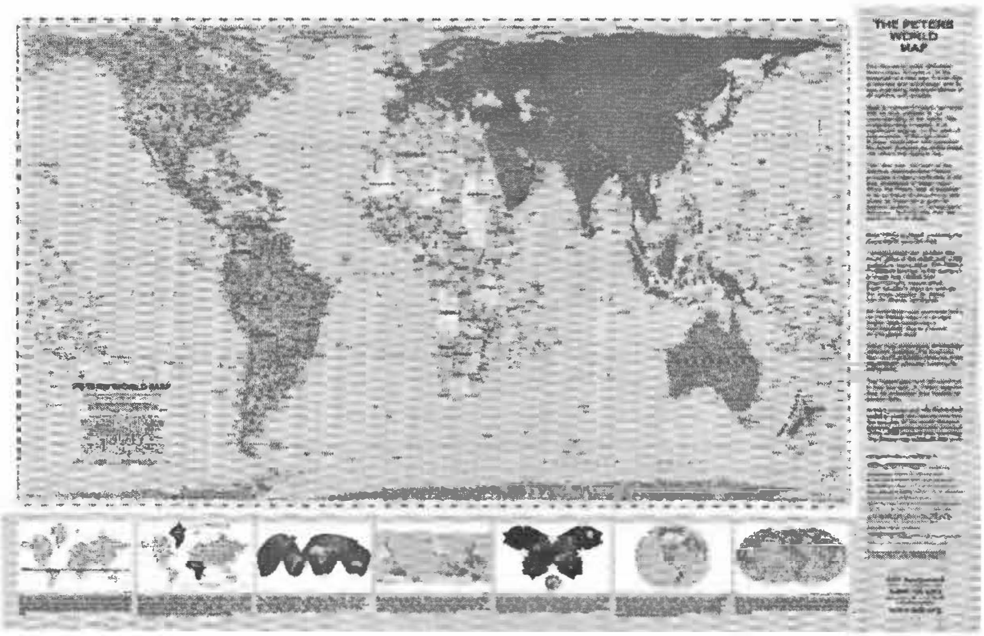

Figure 4: Gall-Peters equal-area map showing more accurate continental proportions

asking students if they would like to share their maps with partners or with the whole class. Having done this activity with a kindergarten class as an invited guest speaker, one of the authors was impressed with the students' keen curiosity and the teacher's positive comments during the lesson.

How Big Is My Country, Really?

In the second activity, Grades 4–6 students begin with Mercator and Gall-Peters world map projections. Using grid paper overlays (11-x-14-inch works better than 8.5-x-11-inch map/paper, but both are doable), students colour either full or partial squares (for example, half a square can be shaded for half a land mass area viewed within the square) to represent the approximate land mass areas found within the two different projections. For both projections, students count the total coloured-square area for the entire map and for each separate continent, then calculate the percentage of the world's land area for each continent using a simple division operation (see Figure 6). Class results can then be collected on the board or on a spreadsheet to examine average values. Average values may be rounded off to calculate areas of each continent in terms of fractions of the total world area. These average values may then be compared with the percentage of population for each continent (available on Wikipedia) and per capita figures. For enrichment, students can redraw the world map more critically and examine alternative maps based on population characteristics.

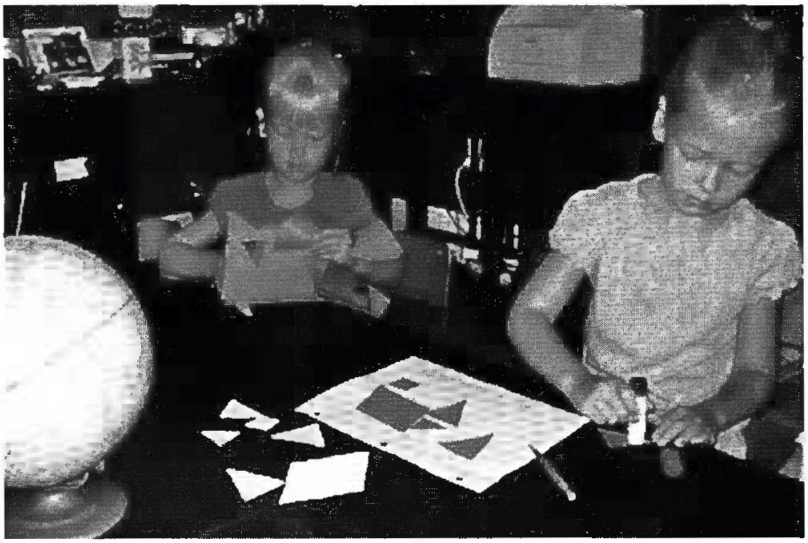

Figure 5: Students completing activity 1 template and locating self on world map

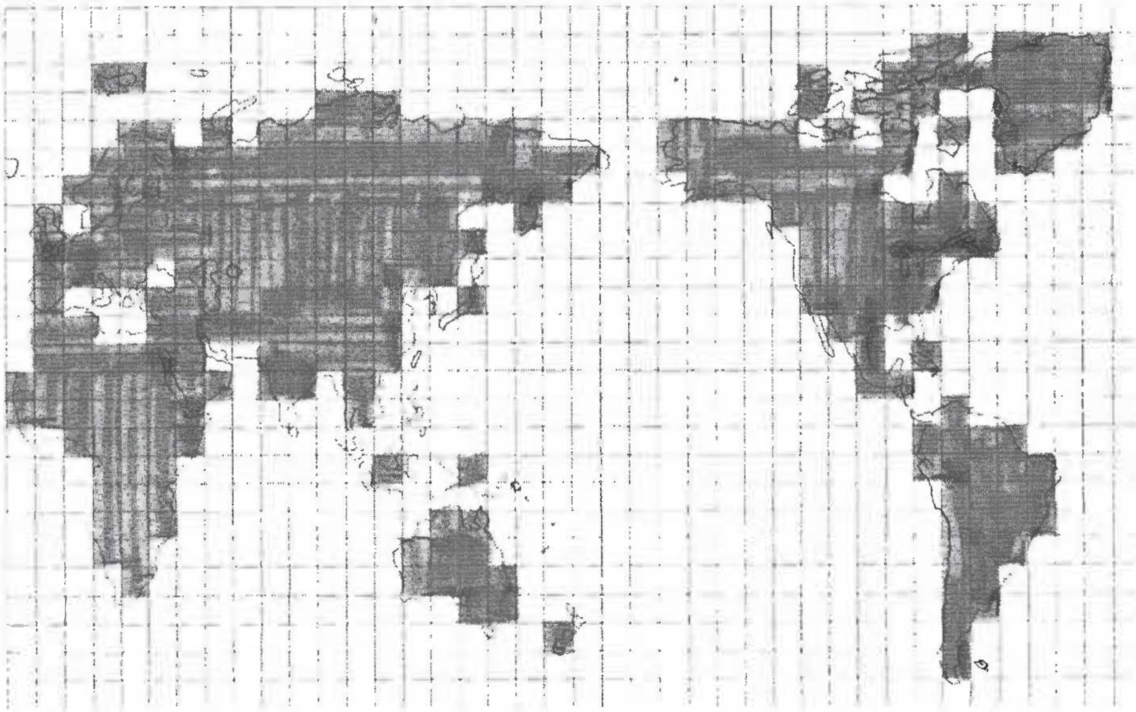

Figure 6: Students use grid paper laid over world maps to capture approximate areas

As part of an integrated geography, history, literacy and mathematics assignment, students can research the backgrounds of the Mercator and Gall-Peters world map projections focusing on when, where and why each map has developed and how each has been used internationally. This can lead to an interesting exploration, and the discussions that ensue can involve issues of politics, finance, trade, navigation, colonization, technology and social equity. This second can foster a deeper understanding of earth's land masses and an increased awareness that how people have chosen to visually represent these land masses has had a significant and often controversial effect on earth's various populations. For instance, students may begin to understand that even what appear to be fixed locations, such as west and east, north and south, are not really fixed, nor are they arbitrary, but that they represent given points of view.

Length, Area, Shape and Angle Distortion

The following three Grades 8–10 activities use different kinds of maps to measure angles, length and areas. The goal is to explore map projections as a mathematical activity, thereby understanding why cartographers and countries select various projections and the far-reaching consequences that these decisions can have.

In the first activity, the teacher asks how to flatten a globe; that is, how to project the globe onto a map. Students may be given spherical objects with which to work. A spherical- or ellipsoid-shaped ball (one that bulges more at the centre) or even an onion demonstrates this well. Cut one onion into halves vertically and remove layers to reconstruct several hollow onions. In attempting to flatten two halves of an onion, students soon realize that this produces an interrupted shape as the onions tear. Cutting the onion into quarters or eighths reduces the tear. A teacher shows an interrupted map of the world centred on or near the Greenwich meridian; for example, the Interrupted Mollweide (Greenwich-centred) map featuring two lobes in the northern hemisphere and three in the southern hemisphere (see Figure 7).

Using such a map, the teacher asks the following questions: What purpose does this map serve? Are there any distortions? What other centre longitudes could be used for interrupted maps? Students may look at other interrupted maps and state which ones they prefer, such as interruptions that do not cut across continents.

To expand on this activity, the teacher may introduce other (hollow) polyhedra and transparent plastic cylinders, such as a tumbler. The teacher explains that had the earth been a different sort of solid, such as a prism, pyramid or other polyhedron that has no curvature, then a map would have been drawn by opening up the polygons to form flat, connected patterns or nets. For example, a cube would be represented by six connected squares; a cuboid (rectangular prism) would form adjoining rectangles; a cylinder would feature rectangles and two circles; and a tetrahedron would form a series of connected triangles. Area and length would be preserved, and some angles and

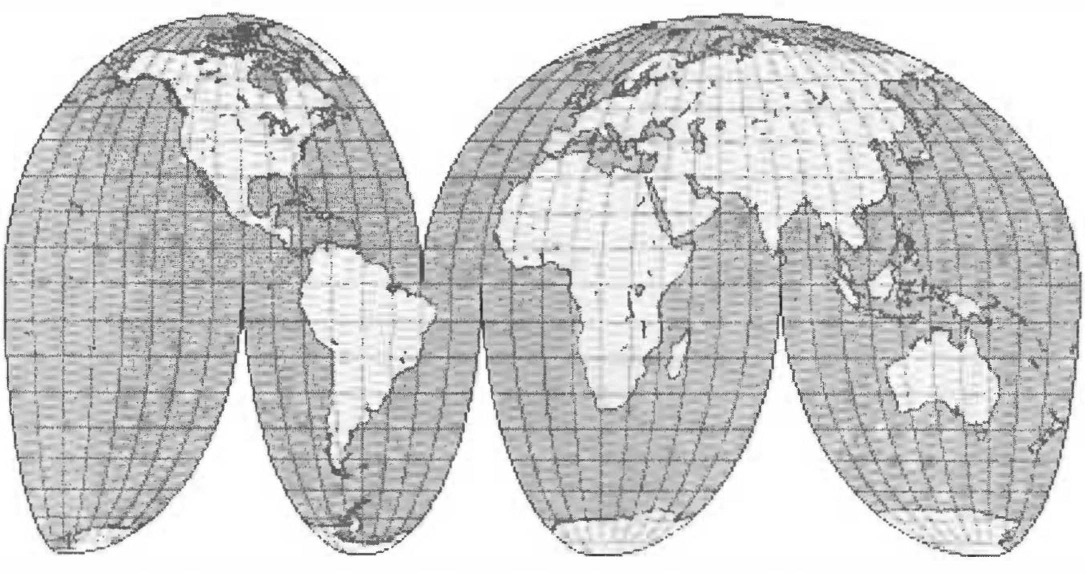

Figure 7: Greenwich-centred "Interrupted" Mollweide world map projection

directions would be preserved on these interrupted nets. Continents hypothetically located at the corners of these nets would be split apart as in other interrupted maps. A teacher may show Buckminster Fuller's Dymaxion map that is based on a 20-sided polyhedron—an icosahedron (see Figure 8)—and raise questions about purpose, distortions and other possible Dymaxion maps.

The second activity is based on two specific questions: What did cartographers do to avoid interrupted maps? and How can we produce a rectangular map from an interrupted map? Students may have noticed that if they stretch some parts and shrink others of a flattened sphere or ellipsoid, they may be able to fill the gaps or interruptions. A teacher may use a plastic ball to show that with some stretching, a rectangular shape can be formed from a flattened sphere. What are the different ways of stretching (and shrinking) that produce rectangular maps? A teacher may then show the class different maps that were produced with stretches farther away from the equator or those that stretch or shrink as one moves away from a meridian in the northern hemisphere; for example, 45° north (and its southern hemisphere counterpart—45° south). A teacher would introduce the concept of the standard latitude (no stretching or shrinking is done; for example, the Lambert [1772] map has standard latitude 0° and aspect ratio 3.141:1). Students would explore where the most distortion occurs on this map. For the Lambert map, students would likely notice that maximum distortion happens the farther one moves away from the equator, as widths are stretched to fill in the rectangle (that is, a horizontal line of equal length as one drawn on the equator will be shorter in reality the farther one is away from the equator). Horizontal lines at the north and south poles would be almost infinitely times longer. Students might then be asked, based on the Lambert map, to imagine a map with aspect ratio 1:1—a square map—and think about what the standard latitudes would be—nearer to or farther from the equator? What would happen to the shapes of the continents—would they be longer and thinner or shorter and thicker? What might be done with the lines of longitude that have been stretched to preserve area? To compensate for this distortion many cartographers use mathematics formulas to adjust the spacing of the meridians (that is, distort the length of vertical lines) so that in the end equality in area is maintained. This question introduces the idea of equal area maps such as the Gall /Peters map (standard latitude 45°; aspect ratio 1.571:1). Students then discuss the advantages and disadvantages of equal area maps.

The third activity introduces the Mercator map as an equal angular map constructed for navigation purposes. Students may estimate and then measure distances and bearings between their city and another using a Mercator map. The teacher looks up this distance and the relevant bearings beforehand in preparation for the activity. Students notice which measurement is in error (that is, Mercator will produce errors of distance but not angle). Students might also estimate, measure and compare the lengths of Greenland and Africa, and South America and Europe on this map. Although these two pairs of land masses appear to be in 1:1 ratios, the actual figures prove otherwise. Students can also be introduced to the concept of rhumb line (or loxodrome), which represents a path of constant bearing, and a conformal map which has a function that preserves angles.

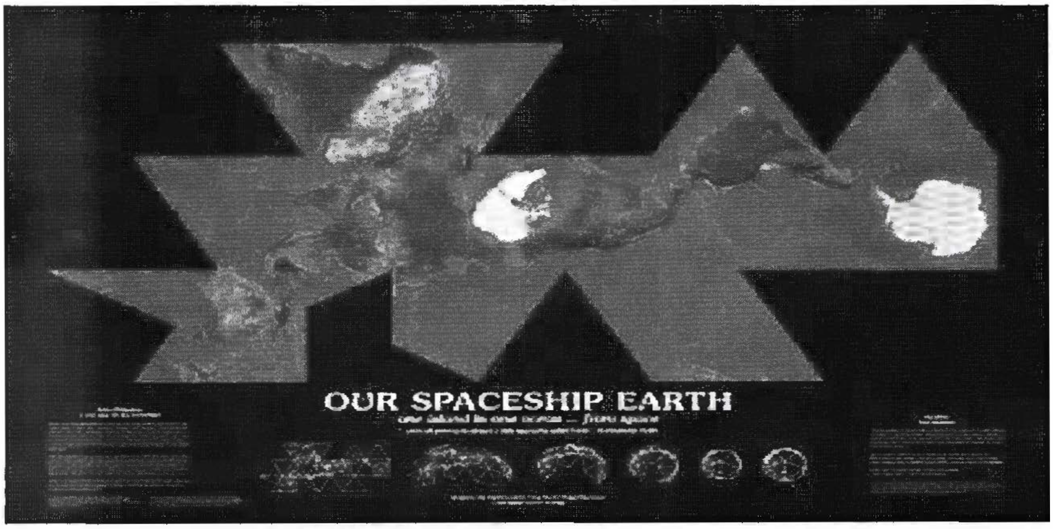

Figure 8: Buckminster Fuller's Dymaxion map world map projection

Highlights from the history of map projection may include early Egyptian recordings of the first estimates of the circumference of the earth; map drawings on Babylonian clay tables; the Greek mathematician Archimedes inscribing a sphere in a cylinder to figure out the surface area of the sphere; Eratosthenes' and Ptolemy's measurements of the earth's circumference; Mercator in the 16th century; Newton's 17th-century explanation that the earth is a spheroid; and the establishment of the Greenwich meridian in the 19th century. Also the teacher may relate, or have students research, actual stories that relate to map projection controversies. For example, the San Francisco federal court case in which the National Resource Defence Council (NRDC) fought against Navy and National Marine Fisheries (NMFS) to determine boundary lines, both parties presenting different map projections to substantiate their respective claims (NRDC used the Mercator projection; NMFS used the Behrman equal area map) (Propen 2007).

For other map-based mathematics activities we recommend Koirala and Goodwin (2002) on comparing areas of states or provinces using a map of a continent; Liben (2008) on helping students understand maps as representations; Haslam and Taylor (1996) and Wood, Kaiser, Abramms (2006) on hands-on activities for projecting globes and early maps, including Inuit coastal maps and a Toronto-centred circular map; Gutstein (2001) on maps and misrepresentations; Wilkins and Hicks (2001) on estimating the earth's water coverage using varied map projections; McLaughlin (2006) on creating maps for the areas in which students live; and Hawkins (2003) on lesson ideas that involve comparing proportions using varied maps. For more secondary school and university activities using maps, refer to Feeman (2000) who discusses how rectangular maps are constructed using calculus and trigonometry.

Ethnomathematics and the Global Student Citizen

Ethnomathematics educators encourage developing students' critical awareness in mathematics classrooms (see D'Ambrosio 2007). They examine interactions between culture, politics, economics and mathematics, and challenge those that perpetuate social injustice and inequities. D'Ambrosio observes that "social justice allows us not only to know what has been decided about ourselves and society (which is the objective of 're-productory' and imitative education), but calls us to participate in decisions about ourselves and society (which is the objective of creative critical education)" (Foreword).

This article illustrates how using a global artifact—a world map—can raise critical awareness among students. Historically, the map—originally conceived and drawn in the 16th century—was the first artifact to show the continents connected. The use of different maps, with one version becoming so dominant (that is, the Mercator), shows that the map, like mathematics, can significantly affect students' views of the world. The world map is an early facilitator of globalization. Using a map in a mathematics lesson is a way of drawing, writing and reading the world (Freire and Macedo 1987; Gutstein 2006). Mathematics in this case is used as a tool to interpret our society. Skovsmose and Valero (2002) refer to the types of activities described above as culturally and politically powerful mathematical ideas because they offer learners critical and emancipatory interpretations.

Concluding Thoughts

With impressive advancements in web-based technology, the availability of open-source (free) dynamic image software, such as Google Earth, and the growing availability of interactive whiteboards in many school districts, the above-described activities can be greatly enhanced within the classroom or computer lab. For example, lines or shapes can be drawn overtop of actual, updated satellite imagery, and the mouse can be used to rotate one's perspective at any level of magnification.

Such power placed in the hands of a young learner can make global explorations highly motivating and informative. The increasing international access to high-speed Internet connections and the use of such things as open-source software, no doubt will affect technology-based equity in mathematics education. Furthermore, the student-centred activities described in this article provide teachers with starting points from which integrated projects can grow, by which understanding of self and society can be deepened, and through which the equity principle of the National Council of Teachers of Mathematics can be realized in a variety of interesting and meaningful ways.

Note

1. For a full and more detailed discussion of the various world map projections, the reader is referred, for example, to the Wikipedia entry entitled, Map Projection (http://en.wikipedia.org/wiki/Map_projection).

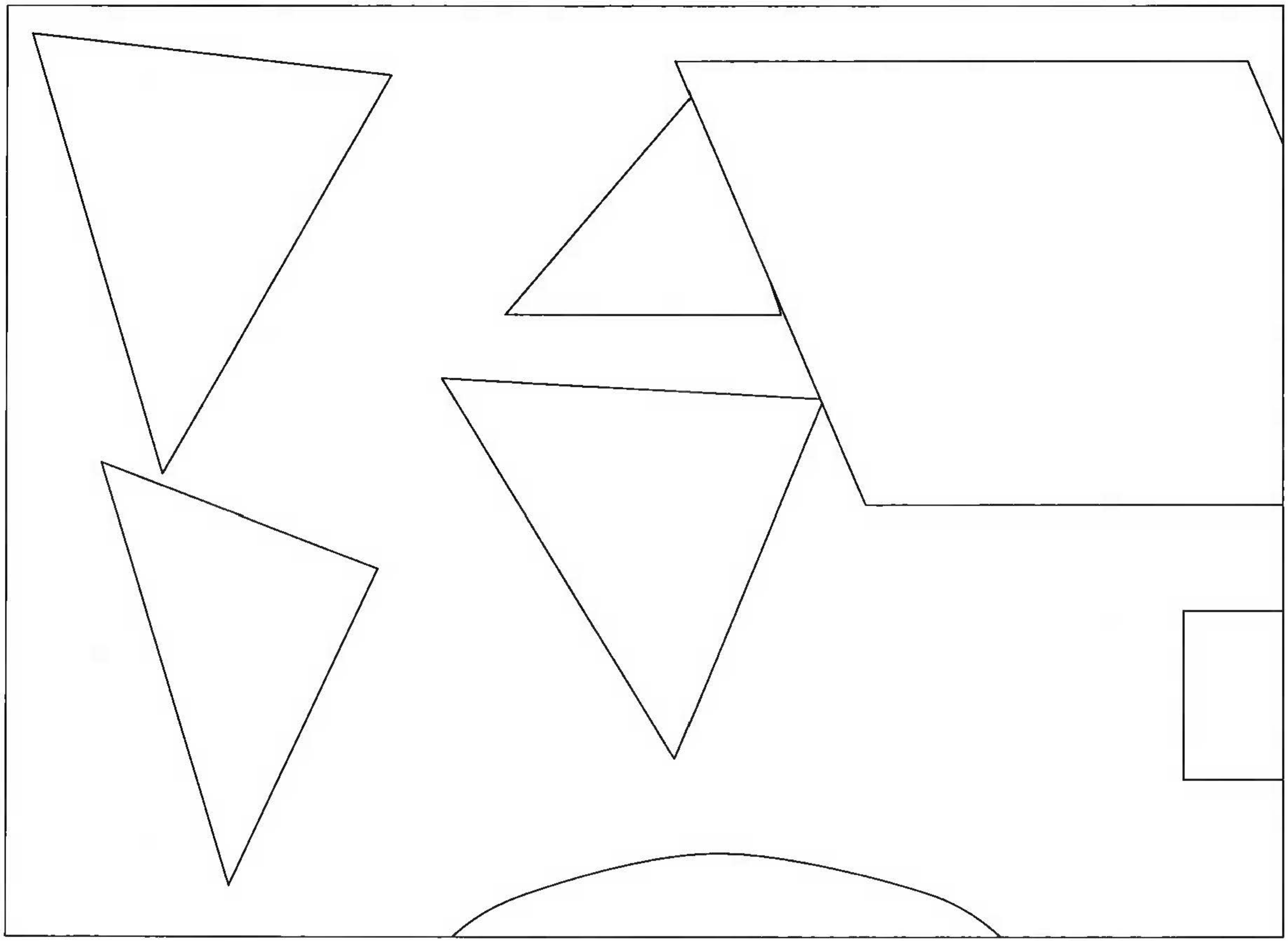

Figure 9: Student Activity: Me and My Geometrical World

Description: Analyzing the Mercator world map projection for large, common geometric shapes; labelling the seven major land masses (continents); and locating one's school on the world map.

Instructions: Once you have cut out, pasted and labelled the seven continents/polygons, find the approximate location of your school on the map and mark it with the small sticker provided. Turn your map around and view the world from different sides of the page.

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Developing Inquiry-Based Teaching Through Lesson Study

Olive Chapman, Krista Letts and Lynda MacLellan

Inquiry-based teaching is widely promoted in mathematics education to help students develop a deep understanding of mathematics and mathematical thinking. In inquiry classrooms, students construct mathematical meaning through reasoning, communicating, exploring and collaborating with peers and teachers while working on inquiry-oriented tasks, including nonalgorithmic problems and investigations (NCTM 1991). However, adopting an inquiry perspective is challenging for teachers because they were taught mathematics that way. This can create a difficult situation when preparing today's students and tomorrow's workers for the demands of the 21st century unless teachers are provided with meaningful learning opportunities to deepen their understanding of inquiry-oriented pedagogy. This article describes such a learning opportunity based on the experience of a group of elementary teachers. Specifically, it discusses a lesson-study approach and the type of knowledge of inquiry teaching and actual teaching resulting from the experience.

Lesson Study

Japanese teachers use lesson study as professional development to systematically examine their teaching to become more effective in teaching mathematics (Stigler and Hiebert 1999; Fernandez and Chokshi 2002). It offers teachers a structure to transmit, reformulate and share craft knowledge through practice and collaboration with peers (Shimahara 2002). It has recently been introduced to North America and is becoming popular for promoting teacher-driven instructional change. Teachers use it to enhance their mathematical knowledge and pedagogical skills, which leads to student-centred teaching in the classroom (Stigler and Hiebert 1999). It fosters improvement in increased development of instructional knowledge and understanding of subject matter (Lewis, Perry and Hurd 2004).

Lesson study involves recursive cycles of planning, teaching, analyzing and revising of lessons that contribute to a continuous improvement model of professional development for teachers. A small group of teachers work collaboratively to plan, teach, observe and analyze the lessons. They start by identifying a goal or problem to explore. This is followed by a recursive cycle composed of four phases: collaboratively developing a lesson plan, implementing the lesson while colleagues and other experts observe, analytically reflecting on the teaching and learning that occurred, and revising the lesson for reimplementation (Curcio 2002; Stigler and Hiebert 1999). During each cycle of implementation, a different teacher teaches the lesson to students in a normal classroom setting while the other group members observe and take notes. Finally, the teachers produce a report on what they have learned from the study lessons, particularly with respect to their goal.

Finding the time to engage in lesson study is a challenge. Unlike North American teachers, Japanese teachers build time into their weekly teaching schedule for professional development. However, the experience of the group of teachers described below shows that lesson study can be adopted effectively.

The Study Group

In spring 2002, 14 Calgary elementary teachers formed a mathematics study group as part of their school's requirement to develop and implement a professional growth plan. Their broad goal was to develop and foster a commitment by all staff to participate in professional growth opportunities to improve the quality of teaching and learning in classrooms. To achieve this, the initial intent was to reflect on their teaching during group meetings. Divisions I and II teachers were represented in the group. Three teachers assumed the role of group leaders to organize meetings and activities. A mathematics education professor joined the group as a mentor in fall 2002. The group met once every three weeks for about one and a half hours at the end of the day.

To highlight the role of the group leaders, the professor will be referred to as the *mentor*, the group leaders as the *team* and the participants as the *group*.

The mentor's role was to provide support, theoretical validation and help in abstracting general ideas from the teachers' thinking and experiences. In general, the mentor acted as a colleague and group participant in sharing ideas and learning from the experience as opposed to being in authority. The mentor had no previous experience with lesson study or teaching at the elementary school level but had taught mathematics methods courses for prospective elementary teachers.

The mentor introduced the idea of lesson study to the group as an approach that can be used to design collaborative learning experiences and encourage growth and continuous improvement. After reading the description of lesson study in Stigler and Hiebert (1999) and with the support of the principal, the group decided to try it. Because this was a new idea for all involved and time was an issue, they decided that the team would participate in the first round of the lesson study and provide feedback to the group. If the approach proved effective, the others would participate in future rounds. The team was granted two days of release time from teaching during the 2003 winter term to implement the first round. This time was allocated as follows:

1. Half day for initial planning to identify a focus, process and schedule
2. Full day for researching, developing relevant ideas and planning
3. Half day for classroom implementation, observation and debriefing

These sessions occurred at different times during the term to accommodate the mentor's schedule and arrangements for substitute teachers for the team.

The positive result of this first round of lesson study on the team's learning led to planning the other rounds to include the whole group. The after-school meetings were used to take the group through the process. In addition, all the teachers received a half-day release time from teaching, during which they planned the mathematics lessons to be studied. Each team member worked with a subgroup of teachers to plan the lessons. The teachers covered each other's classes to obtain the release time to observe the lessons. This way, over three years, 22 teachers had the experience and learned from it. This number was a result of teachers leaving because of other commitments and others entering.

The Lesson-Study Approach

This section describes the different stages of the lesson-study approach in the initial round and summarizes the follow-up rounds.

Identifying a Topic to Study

The team members were interested in inquiry-based teaching but wanted to focus on a specific feature for the lesson study. They identified this feature by examining the philosophy section of the mathematics curriculum document *Alberta Program of Studies for K–9. Western Canadian Protocol for Collaboration in Basic Education* (Alberta Education 1996), which incorporates a perspective of mathematics learning that supports inquiry-based teaching. This document states: "Students learn by attaching meaning to what they do; and they must be able to construct their own meaning of mathematics. The meaning is best developed when learners encounter mathematical experiences that proceed from the ... concrete to the abstract" (p 2). The goals for students include using "mathematics confidently to solve problems and communicate and reason mathematically" (p 2). The curriculum also emphasizes seven mathematical processes (p 4): communication, connections, estimation and mental mathematics, problem solving, reasoning, technology and visualization, all of which can play an important role in inquiry-based teaching if they are interpreted as intended. Thus, the curriculum provided validation and a basis for the starting point to identify a topic to study.

The team members focused on the seven mathematical processes in the curriculum, and after a lengthy discussion and reflection on their teaching, concluded that communication in an inquiry-teaching context was the key process that they would like to study in the first round. As stated in the curriculum, "Students need to communicate mathematical ideas clearly and effectively, orally and in writing" (Alberta Education 1996, 4). But this communication is different from the traditional mathematics classroom where the focus is on transmitting information, copying notes and recording solutions to exercises or routine problems. The team was interested in focusing on other features of communication that allowed students to think and actively engage in their learning. Once a topic for the lesson study was established, the next stage was determining specific features of communication and inquiry-based teaching to use in designing the lesson to be studied.

Video Case Study

The team and mentor discussed possible ways to obtain relevant information about specific features of communication and inquiry-based teaching. Instead of beginning with reading theory on these topics, the team members preferred to study a video as the basis of their learning. The mentor suggested the DVD series, *Mathematics: With Manipulatives* (Burns 1988). The team chose *Pattern Blocks* and *Cuisenaire Rods*, two of the six DVDs in the series. To have focus in studying the videos, the team and mentor attended to the following: lesson goal, students' role, teacher's role, specific questions posed by the teacher to stimulate and extend students' thinking, classroom environment, nature of inquiry tasks and key features of the inquiry lesson.

After studying the videos, the following key components for inquiry-based lessons were identified:

1. Three modes of communication: oral, written and nonverbal (for example, observing, listening and acting/gesturing)
2. Tasks that allow for open and guided exploration, prediction, discussion and evaluation
3. Student roles, teacher roles and environment (specific features identified are provided in Figure 1)
4. Questions and prompts (examples are provided in Figure 2)

Following the video study, the team later read and discussed the National Council of Teachers of Mathematics (2001) standards on discourse. However, the outcome of the video study formed a key basis for planning the experimental inquiry-based lesson.

Planning the Experimental Lesson

The Grade 1 teacher on the team volunteered her classroom for the first round of lesson testing, thus the team chose a topic from the Grade 1 curriculum to develop an inquiry-based lesson. The topic "explore and classify 3-D objects according to their properties" was selected to correspond with the teacher's class schedule. The team brainstormed different approaches to teach the topic. Each teacher on the team described what she might do in her class.

Teacher One

- Observe objects in classroom
- Discuss why these objects have certain shapes
- Post pictures of objects in the real world around the classroom and use to identify shapes
- Name geometric objects
- Link to objects in class
- Refer to chart with formal names
- Investigate attributes
- Relate to real world—why things have certain shapes

Figure 1: Roles and Environment

Student Role	**Teacher Role**	**Classroom Environment**
Communicate in all three modes	Pose questions and prompts	Manipulatives
Engage in inquiry	Allow time for exploration	Small groups
Collaborate	Select appropriate tasks	Whole-class sharing
Take risks	Allow time for communication	Supportive/risk free
Show curiosity	Observe and listen to	Lots of mathematical talk
Reflect	Support/encourage	

Figure 2: Questions and Prompts

1. What do you notice?
2. What else do you notice that is different?
3. Who can explain how (or why) this makes sense?
4. What do you think the answer (or pattern or outcome) could be? How do you know?
5. How do you know it will (will not) work?
6. Where (or when) would you use this ________________?
7. Suppose I want to______________, how can I start?
8. Who can describe it so that I can do it?
9. Present your idea.
10. Explain the problem to your partner (the class).
11. What do you know about ________________ (for example, this topic)?
12. Can you make a general statement about ________________?

Teacher Two

- Describe geometric objects in groups or pairs
- List names of objects and descriptive words on a chart
- Build a model of one object (a skeleton representation)
- Discuss, comparing skeleton and actual object
- Introduce formal names

Teacher Three

- Pose a problem, for example, build a house with this object
- Discuss attributes
- Explore attributes
- Classify attributes
- Describe common features

Reflecting on these approaches and the outcome of the video study, the team and mentor identified the following set of key components for inquiry-based teaching to use to structure the experimental lesson:

Goal

- Prerequisite
- Free exploration and discussion
- Prediction of properties of concept
- Application of concept
- Testing predictions
- Evaluation of knowledge of concept
- Extension of concept to new situations

Using this structure and key questions to promote inquiry-based communication, the team designed a plan for the experimental Grade 1 lesson on introduction to geometric solids. Figure 3 provides an abbreviated outline of the plan.

Figure 3: Outline of Experimental Lesson Plan

- Free exploration objects (10 minutes) (Talk, experiment and observe in small groups.)
- Discussion (5 minutes). (What did you notice? Record answers.)
- Prediction (5 minutes). (Will objects roll or slide? Record individually.)
- Pose problem/real world application (5 minutes). (Suppose I want to build a house on a mountain, what would I need to know about objects?)
- Test prediction, record results (5 minutes).
- Comparison (5 minutes). (Discuss solutions with partner and support answers.)
- Evaluation (10 minutes). (Venn diagram, sort shapes and make general statements about "What I know about 3D objects.")
- Extension (homework). (Look for things at home and around school that roll or slide.)

Conducting and Observing the Experimental Lesson

The Grade 1 teacher of the team taught the lesson to her students in her classroom. The other team members and the mentor observed and made notes on an observation sheet (Figure 4). The intent was to focus on how the lesson was conducted and how effective the questions and prompts were for communicating the components of the lesson to facilitate meaningful and worthwhile inquiry and learning of the mathematics concepts.

Figure 4: Observation Sheet

Notice?	
Make sense?	
Predict?	
How know?	
Make connections?	
Describe/explain	
Generalize/summarize	
Other	

Analyzing the Experimental Lesson

The lesson was analyzed immediately after it concluded. The observers and the teacher of the lesson shared notes, focusing on communication and the components of the lesson. For the most part, the lesson went as planned. It consisted of the following sequence of activities: brief introduction to set the tone; free exploration (in small groups) of 11 3-D geometric objects; whole-class discussion; individual prediction using worksheet with pictures of the 11 3-D objects and columns for rolls only, slides only and rolls and slides; comparison with a partner; whole-class discussion of an application (think of self as builder); prediction if all will agree; focused exploration to test predictions; comparison and discussion of findings with others; whole-class discussion of findings; 3-D vocabulary of objects and building of Venn diagram on whiteboard with pictures; evaluation/reflection/generalization in relation to goal of lesson; and an application task for homework. These components and activities were effective in creating a learner-centred classroom and promoting inquiry and rich mathematical communication. The children were actively involved throughout the lesson doing mathematics, and thinking and communicating mathematically. The teacher of the lesson kept posing questions that stimulated or revealed their thinking as she circulated during the lesson and during the whole-class discussion. This allowed her and observers to learn from and about the children based on their ways of thinking. The team was amazed and impressed with what the children were able to do and the richness of their thinking.

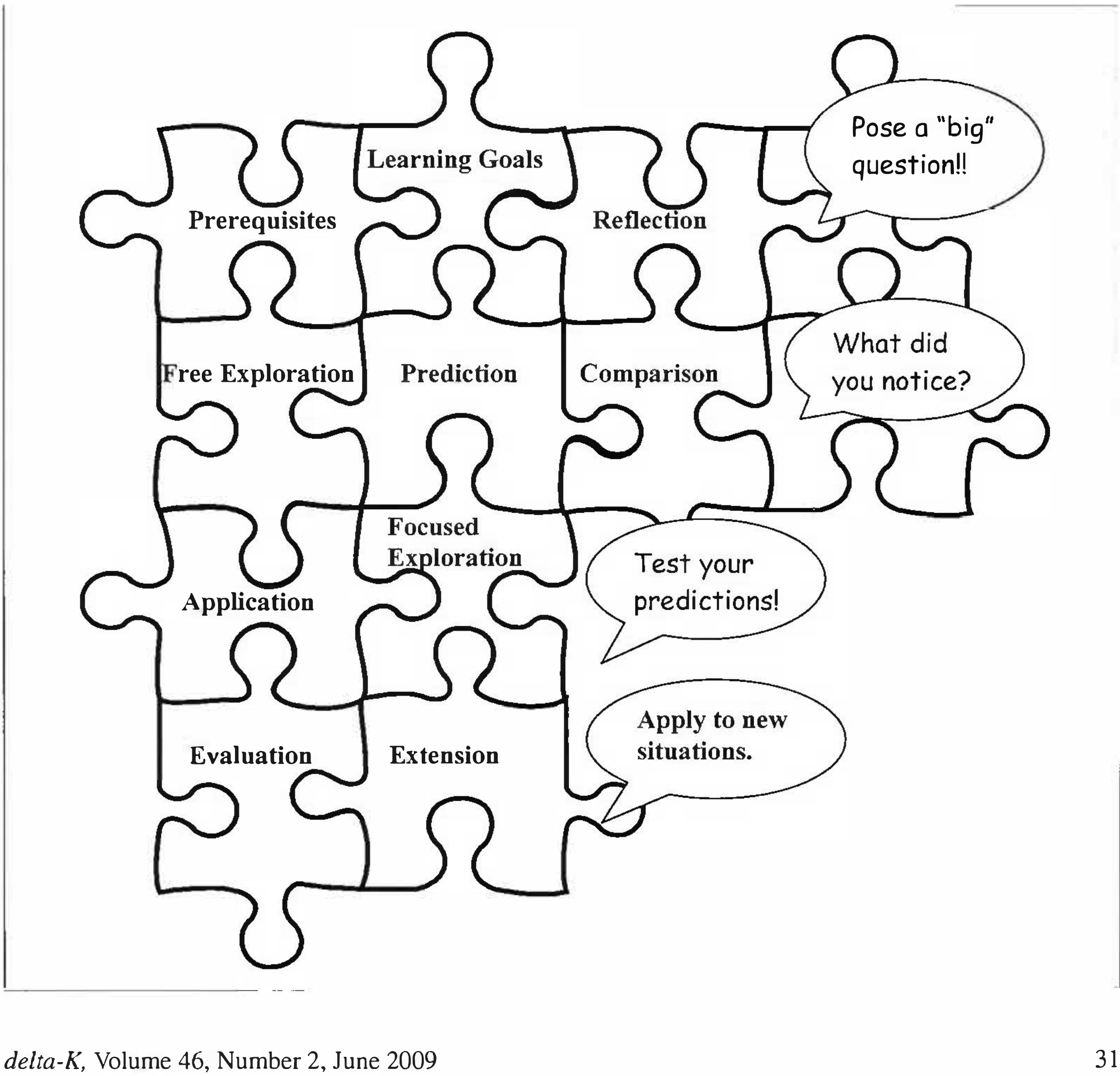
The only change that was identified was to clarify two more components to an inquiry lesson: comparison and reflection. The team members concluded that their inquiry-teaching model was workable, and they felt comfortable and confident implementing it in their teaching. They emphasized that the structure of the lesson is flexible as to which components are used and how they are sequenced. Thus they decided to represent the approach in the form of a jigsaw puzzle, which they called the Jigsaw Teaching Model. The experimental lesson was one way of applying this model.

The Jigsaw Teaching Model

Figure 5 is the final version of the Jigsaw Teaching Model developed by the team to represent the key components of an inquiry-based classroom and key questions and prompts. The model is in the form of a jigsaw because the pieces do not have to follow a linear pattern but can be organized differently, combined or omitted depending on the topic being taught. The pieces represent different components that are important in inquiry teaching.

The model requires the teacher to (1) identify learning goals that include conceptual understanding, (2) expose students' prerequisite knowledge/conceptions for the concept being taught in an inquiry way, (3) have students make predictions about possible

Figure 5: The Jigsaw Teaching Model



outcomes related to the concept, (4) allow students to engage in free exploration of the concept (through discourse and/or using manipulatives), (5) engage students in focused exploration, (6) have students work on application of concept, (7) engage students in comparison, evaluation and reflection of their learning and (8) suggest extension of concept to other situations or related concepts.

Follow-Up Rounds of Lesson Study

The team did not follow the recursive cycle of the lesson study in terms of revising and teaching the same lesson. Instead, they made the Jigsaw Teaching Model the basis for subsequent rounds of the lesson study with the group. These rounds of experimental lessons included (1) the meaning of five (kindergarten); (2) estimation with mass (Grade 3), representing a multidigit number in different ways (Grade 3) with goals that students will understand why the value of a digit changes depending on its position in a number and the meaning of regrouping among hundreds, tens and ones; (3) mental arithmetic (Grade 5) and (4) measurement (Grade 6), for example, perimeter and area.

Later, the teachers also worked on exploring lessons on problem solving in their classrooms. Figure 7 describes a lesson taught four years later by the Grade 1 teacher who taught the first experimental lesson, which shows that the approach was being sustained. The teacher explained:

> The students' thinking started us off on a new game of trying to stump the rest of the class with our ordering rules. Students began to use mass, capacity, width, height, cost, temperature, time and so on to place the objects in order. The discussions that followed were lively and informed. We all learned more than the original activity, and it was a fabulous culminating activity. The following day students were presented with a puddle question about how they could measure a puddle after a rainy day. They recorded their answers in their math journals in the form of a mind map or brainstorming diagram. They used words, pictures, diagrams and so on to show their thinking. Finally they shared their ideas with several partners and discussed similarities and differences. When we listed our combined ideas on a chart, they included length (using their feet), width (using their hands), depth (using pile of rocks), area (using boxes), capacity (using cup measure), weight (using scale to weigh cup and multiplying or repeated addition), temperature (using a thermometer) and time (counting how many minutes/hours for sun to dry up puddle).

This shows what these children are able to accomplish when *their* thinking, not only the teacher's, becomes the focus of the lesson.

Figure 7: Grade 1 Lesson (four years later)

Learning goals: Students will use knowledge of nonstandard measurement to order objects, recognize that objects can be sorted in a variety of ways and think flexibly when dealing with size.

Free exploration challenge: Find three objects that you can order by size.

Prediction: Is there more than one way to order the three objects?

Comparison: pair/share—Can you guess my measurement sorting rule?

Focused exploration/application: Which measurement sorting rule would change my order? Which rule would not change the order? Where would a fourth object fit? How do you know?

Recording: Record your favourite sorting rule in your math journal. What did you notice about measurement and sorting in this activity? Did you notice any patterns?

Extension: How could you measure a water puddle after a rainy day?

Conclusion

The lesson-study approach helped these teachers create shifts in their thinking and teaching, and develop useful knowledge of mathematics teaching. They reported deeper and more meaningful understanding of the following inquiry teaching:

- questioning techniques that guide and enrich student thinking;
- open-ended, thought-provoking questions to motivate students to discuss and understand mathematics at a deeper level;
- student-centred strategies for listening to students and observing their problem-solving behaviours; and
- strategies that allow students to assume ownership of their knowledge and knowledge construction.

With these shifts, students were more involved in mathematics as a way of problem solving, reasoning, communicating and connecting to their world.

The teachers also valued the collaboration during the lesson study; that is, being a part of the group process of planning, observing, debriefing and sharing specific examples from classroom work. As one teacher explained, "It was a powerful experience to hear the student communication that an observer can record, but which the teacher misses as she focuses on presenting the lesson and maintaining classroom management." In general, their collegial focus on mathematics instruction increased, and they articulated and demonstrated a renewed interest in mathematics. Most important, they learned an approach that they can use for ongoing learning and growth in their teaching. They offer their experience in the article as an example of an approach other teachers could adapt in their quest to inquiry-based teaching.

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The Banff Game: Probabilities Applied

A Craig Loewen

I am not entirely sure of the origin of this game. I remember playing it as a child, but it was only recently that I introduced it to my children on a muggy summer day in Banff.

We had already swum at the pool, watched TV, finished shopping and consumed the bear claw chocolates. Worse yet, it was raining and my children were bored. We desperately needed to find something for the kids to do. That's when I remembered this dice game, and we began to play. As we played we talked, and the kids asked me several interesting questions that led to some fun problem-solving challenges.

Here is how the game works.

Scoring

The chart below shows various combinations of dice, which when rolled *on a single roll* have the corresponding values.

⚀⚀⚀ =1000 pts	⚃⚃⚃ =400 pts
⚁⚁⚁ =200 pts	⚄⚄⚄ =500 pts
⚂⚂⚂ =300 pts	⚅⚅⚅ =600 pts
⚀⚁⚂⚃⚄⚅ =1500 pts	
⚀ =100 pts	⚄ =50 pts

For example, assume a player rolls all six dice and the following values result:

⚄⚅⚄⚀⚁⚄

This roll has a value of 600 points: 500 points are scored for the three ⚄s, and the single ⚀ scores a further 100 points.

The Rules

Players: This game is played with six regular six-sided dice and any number of players. (Note: It can also be played as a solitaire game testing to see how many turns are necessary to reach the final score).

1. On a turn a player starts by rolling all six dice.
2. The player now chooses to keep the points rolled (if any) or remove at least one die or combination of dice that have value according to the chart given, and then reroll the remaining dice.
3. The player continues to roll the dice, always removing one or more dice and rolling the remaining dice until one of the following happens:
   - The player chooses to stop and take the points rolled on this turn. Play passes to the left once the points are scored.
   - A combination of values is rolled that scores no points. If this happens, the player loses all the points accumulated on that turn, and play passes to the left.
   - All of the dice have been successfully rolled; that is, value combinations have been made using all six of the dice. If this happens, the player picks up all six dice and begins rolling again, adding to the total accumulated thus far on that turn.
4. The game ends when a player has reached 5000 points; however, all the other players get one final turn.
5. The player with the highest score wins.

An Example Turn

Assume a player rolls the following values with her first roll:

⚀⚀⚁⚂⚃⚃

The player must now remove at least one die. She would likely choose to remove the two ⚀s because they have a value of 200 points. The player must choose to either take the points or to risk them by rolling the remaining four dice. Assume the player chooses to risk the points and thus rerolls only the four dice that do not contribute to the point total (the ⚁⚂⚃⚃). Now let's assume the player gets the following results from the new roll: ⚀⚂⚃⚄. This second roll has produced a further 150 points (100 points for the ⚀, and 50 points for the ⚄). The player has three options:

- Stop rolling and take the 350 points accumulated over the two rolls.
- Take the ⚀ from the second roll (adding it to the two ⚀s from the first roll—a score of 300 points) and reroll the remaining three dice (the ⚂, ⚃ and ⚄) hoping for a higher score.
- Take the ⚀ and the ⚄ from the second roll (adding it to the two ⚀s from the first roll—a score of 350 points) and reroll the remaining two dice (the ⚂ and ⚃).

A turn does not end until a player chooses to stop and score the points collected, or rolls a combination of dice with no value. Remember, if at any time the player rolls a combination of dice that does not produce any points, she loses what she has collected on the turn thus far and play passes to her opponent.

A Few Questions

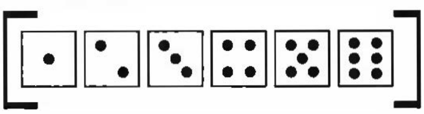
We were having a good time simply playing the game until my children started asking some interesting and rather difficult questions.

- What are my chances of rolling a ⚀ or a ⚄? At what point am I better off to stop than to roll again?
- How hard is it to roll a straight (⚀ through ⚅) in a single roll? Why is grandma the only one who seems to be able to roll a straight?
- Should I keep a ⚄ if I don't have to?

  The last question was actually mine:
- What are the chances of rolling all six dice and getting nothing? And, why does this always happen to me?

A Few Answers

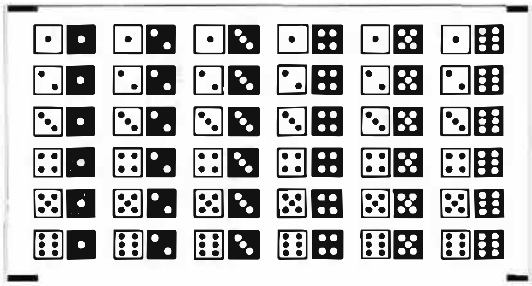
A sample space is a list of all of the possible outcomes from a chance event. When you roll a single die, the sample space is quite easy to determine. It looks like this:



From this sample space we can determine a few things: any value ⚀ through ⚅ may possibly appear, and (if the die is fair) there is an equal chance of each value appearing. That chance is determined by counting the number of possibilities for any specific outcome (any given number), divided by the number of possible outcomes in the sample space. So, the chance of rolling a ⚃ with a single is 1/6 as there are six possible outcomes. The equation looks like this:

$$P(⚃) = \frac{\text{Number of outcomes containing ⚃}}{\text{Size of the sample space}} = \frac{1}{6}$$

It is more difficult to show the sample space for the rolling of two six-sided dice, but there are 36 possible outcomes, six possible outcomes on each of the two dice.



When you are rolling two dice, the chances of rolling a ⚃ on at least one die go way up. As it turns out, 11 different combinations of two dice contain at least one ⚃, and there are 36 possible outcomes in our sample space. It follows then that the probability of rolling at least one four is 11/36 or slightly less than 1/3. Likewise, we can calculate that there would be 216 possible outcomes with three dice ($6 \times 6 \times 6 = 6^3 = 216$), and there would be 1,296 (or 6^4) outcomes with four dice, 7,776 (or 6^5) outcomes with five dice, and 46,656 (or 6^6) outcomes with six dice.

We are now ready to tackle some of those important questions that should help us play this game more cleverly (and perhaps win).

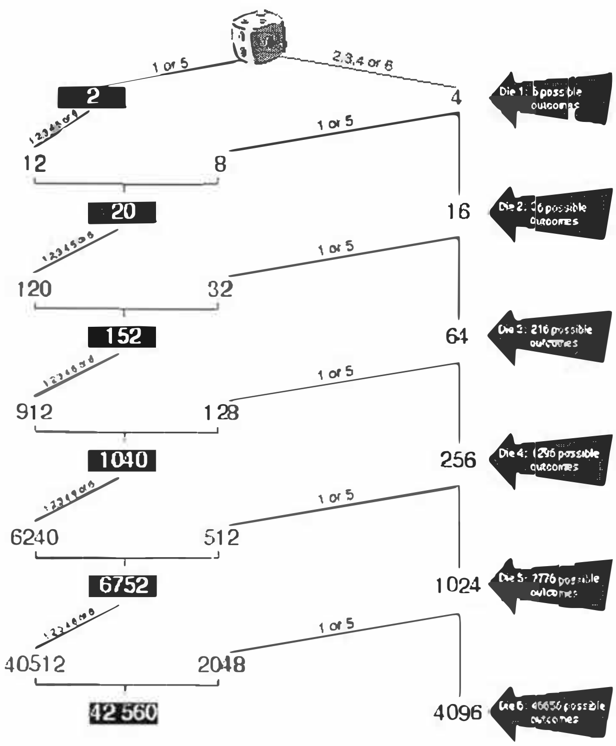
What are my chances of rolling a ⚀ or a ⚄? At what point am I better off to stop than to roll again?

The answers to these questions will obviously depend on how many dice you are rolling at the time. Let's start first with a simpler case; that is, where we are considering only a single outcome such as rolling a ⚃. As before, if the probability of rolling a ⚃ is 1/6 with one die and 11/36 with two dice, then presumably the more dice you have, the better are your chances of rolling a ⚃. The following table shows the chances of rolling at least one ⚃ with up to six dice.

No of Dice	Possible Outcomes	Positive Outcomes	Probability
1	6	1	0.167
2	36	11	0.306
3	216	91	0.421
4	1,296	671	0.517
5	7,776	4,651	0.598
6	46,656	31,031	0.665

We should be able to make similar computations with respect to rolling a ⚀ or a ⚄. This time we need to consider two outcomes, but the chance of rolling a ⚀ or ⚄ would be 2/6 (or 1/3) with a single die, and the chance of rolling at least one ⚀ or ⚄ with two dice would be 20/36. The chance has gone up from ⅓ to over ½ simply by adding the second die! This table shows the probability of rolling at least one ⚀ or one ⚄ with up to 6 dice.

No of Dice	Possible Outcomes	Positive Outcomes	Probability
1	6	2	0.333
2	36	20	0.556
3	216	152	0.704
4	1,296	1,040	0.802
5	7,776	6,752	0.868
6	46,656	42,560	0.912



The values in our chart can be verified using a tree diagram. Rolling six dice and hoping to get at least one ⚀ or one ⚄ is identical to the task of rolling a single die six times and hoping for at least one ⚀ or one ⚄ before your rolls are used up. The tasks are identical because each roll is independent from the others whether or not the dice are rolled simultaneously. The tree diagram shows the likelihood of success (rolling at least one ⚀ or one ⚄) if you are rolling a single die six times. In the diagram the number of possible successful rolls is shown with a black background. With a single die you have a 2 out of 6 chance of rolling at least one ⚀ or one ⚄, a 20 out of 36 chance with two dice, 152 out of 216 chance with three dice and so on.

We can apply what we know to build a strategy in playing this game: as long as we have two or more dice, the probabilities are on our side if we choose to roll again. But don't forget: having the probabilities on your side does not mean that you are guaranteed to roll what you want; there is still always the chance that you may "scratch."

Challenge: What is the probability of scratching if you are rolling only one die? Two dice?

Challenge: What is the probability of flipping a fair coin six times in a row and getting heads each time? Try to modify the tree diagram to calculate the probability.

How hard is it to roll a straight (a ⚀ through ⚅) in a single roll? Why is grandma the only one who seems to be able to roll a straight?

As it turns out, it is very difficult to roll a straight. We can use the same formula to compute our probability.

$$\text{Probability} = \frac{\text{Number of positive outcomes}}{\text{Size of the sample space}}$$

$$\text{P (straight)} = \frac{\text{Number of outcomes containing one of each}}{\text{Size of the sample space}}$$

Calculating the size of the sample space is relatively easy and has already been done above ($6^6 = 46{,}656$). Calculating the number of positive outcomes (that is, the outcomes that contain exactly one ⚀, one ⚁, one ⚂, one ⚃, one ⚄ and one ⚅) is more difficult. We can simplify this though by solving a similar and simpler problem: without repeating, how many numbers could you write using just two different digits; for example, just using a 1 and a 2? We can write only 12 and 21: there are two such numbers. How many could you write using three different digits (1, 2 and 3, no repeated digits)? We could write 123, 132, 231, 213, 312 and 321, a total of six in all. Keep going:

without repeating, how many ways are there of using four different digits (a 1, 2, 3 and 4) to build four-digit numbers? As it turns out, there are 24. Do you see a pattern?

$1 \times 2 = 2$
$1 \times 2 \times 3 = 6$
$1 \times 2 \times 3 \times 4 = 24$

If we continued our pattern, we would see that 120 numbers could be built using the digits 1 through 5 in various combinations, and 720 ways to use each of the numbers 1 through 6 without repeating a digit while writing unique 6-digit numbers. In other words, there are 720 different ways to arrange the digits 1 through 6. So, how is this like our dice experiment?

It is tempting to think there is only one way to roll a straight, but this would be a mistake. You see, we do not care which die gives us the ⚀, or which die gives us the ⚃ or ⚅. This means there are many different ways (720, in fact) to get one each of the values ⚀ through ⚅. We can put these numbers in our equation to calculate the probability of rolling a straight.

$$P\text{ (straight)} = \frac{\text{Number of outcomes containing one of each}}{\text{Size of the sample space}}$$

$$P\text{ (straight)} = \frac{720}{46\,656}$$

This value is obviously very small: it should be very difficult to roll a straight.

Here is a second way to think about this problem: we can imagine this situation by pretending that we are only rolling a single die, but rolling it six times. Each time we need to roll a value different from previous rolls. Any value will work for the first roll, but the second roll only has five possible values, and the third roll has four possible values and so on. The probability of rolling six consecutive usable values is calculated as follows:

$$P\text{ (straight)} = \frac{6}{6} \times \frac{5}{6} \times \frac{4}{6} \times \frac{3}{6} \times \frac{2}{6} \times \frac{1}{6} = \frac{720}{46656}$$

This computation produces a probability of approximately 0.015. It is fair to say that a player shouldn't be able to roll a straight very often. While I have no proof of cheating, perhaps grandma should explain why she is so uniquely fortunate at rolling straights when she plays this game!

Challenge: What is the chance of rolling an "almost straight;" that is, rolling six dice and getting exactly one pair?

Challenge: What is the chance of rolling a short straight (a ⚀ through ⚄, or a ⚁ through ⚅) with a single roll of six dice?

Should I keep a ⚄ if I don't have to?

This is an interesting question, and the answer may depend on a number of variables, such as a player's tolerance for risk, the number of dice this would leave for rerolling, the score in the game and so on. It is helpful, though, to consider some common scenarios in the game and to use our probabilities to help determine what the best choice may be in each case.

Scenario 1: Four dice have been successfully set aside, and rolling the remaining two dice produces a ⚀ and a ⚄. In this situation you do not need to take the ⚄, so you are faced with a decision: take the ⚀ and reroll the remaining die, or take both the ⚀ and ⚄ and roll all six dice. In each case you must consider the possible consequences of the choice you make.

If you choose to take the ⚀ and reroll the ⚄, what are the possible outcomes of this subsequent roll? If you roll a ⚀, your gamble worked. If you roll a ⚄, you are in exactly the same situation as if you had not chosen to roll (that is, you are no further ahead). If you roll a ⚁, ⚂, ⚃ or ⚅, the results are catastrophic. Obviously only one of the possible six outcomes improves your situation, and four of the possible six outcomes are disastrous. Clearly, the odds are not in your favour, and you should consider instead taking both the ⚀ and ⚄. Further, if you take the ⚀ and ⚄ (and thus have successfully scored all six dice), you may now pick up the dice and start over, and we have already seen that your chances of rolling further points in this situation exceed 90 per cent. This is the best option.

Scenario 2: A more interesting situation emerges if you have rolled your final three dice and received one ⚀, one ⚄ and another value you cannot use. In this case, are you better off to take the ⚀, or to take both the ⚀ and ⚄ even though you are not obligated to do so?

In this case the player's tolerance for risk is very important. Taking both the ⚀ and ⚄ leaves you in the unfavourable position of rerolling a single die. Taking only the ⚀ leaves you in a better position because your odds of succeeding with two dice are much better than with one; keep in mind, though, that the odds of succeeding with two dice are only slightly in your favour (55 per cent). The conservative player will probably opt to take both the ⚀ and ⚄, and pass the dice to his or her opponent. Of course, if you are not risking many points, you may just choose to roll anyway. In this game, as in life, risk can be tolerable or not, depending on both the odds and what is to be gained or lost.

Challenge: Are you better off to roll again or to take three ⚁s if you don't have to?

What are the chances of rolling all six dice and getting nothing? And, why does it always happen to me?

This question is probably the most difficult to answer. We are essentially looking for the number of possible combinations that have no ⚀, no ⚄ and no triplets (or 4, 5 or 6 of a kind). Of the 46,656 possible outcomes, there are 45,216 possible safe rolls with six dice, and thus only 1,440 ways to scratch while rolling six dice. Using our formula we can calculate that there is only about a 3 per cent chance of scratching while rolling all six dice! There is therefore no explanation as to why it happens to me so often. I'm just (un)lucky I guess!

Challenge: How do you calculate the number of safe rolls with six dice according to the rules of this game?

Variations

It is always fun to change the rules of a game just to see how it affects the game's outcome. The following variations simply increase the challenge or introduce other ways to score points.

A player must have a score of 450 to start the game. If the player does not get at least that many points he or she simply scratches the turn and play passes to the left.

Rolling four of a kind "doubles the triple;" that is, if a player rolls four ⚄s all at the same time, she or he has rolled a value of 1,000. Put another way, the first three ⚄s are worth 500 points, and the fourth ⚄ doubles that score for a final value of 1,000 points.

Play with five rather than six dice. A straight can be either a ⚁ through ⚅, or a ⚀ through ⚄ in a single roll.

A Word About Problem Solving

The Banff Game is genuinely enjoyable and worth playing for that reason alone; however, what it does have in common with other games is that it is a type of extended problem.

All problems have a list of conditions and restrictions as well as a purpose or goal. Solving the problem always requires inventing or selecting and applying a strategy. In comparison, games have conditions (the rules, how to roll the dice, how to score points), goals (reaching 5,000 points) and strategies (considering the probabilities): games qualify as problems in every way.

Anyone can play a game without being particularly strategic, but by asking a few questions a game can also become an opportunity to create an interesting problem-solving context. By applying problem-solving knowledge, we build new skills and develop mathematical understanding. The Banff game provides one context in which we may explore concepts related to probability and chance.

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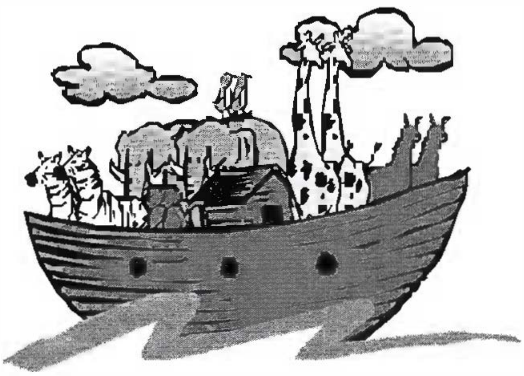
A Page of Problems ... There's always a trick!

A Craig Loewen

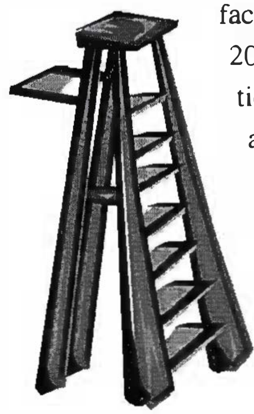
Middle School

The animals boarded the ark two-by-two. How many types of dogs did Moses bring aboard the ark?



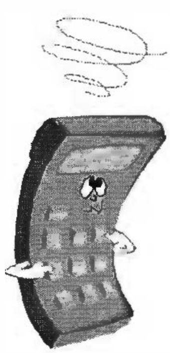
High School

A ladder on a boat is hanging over the side with the bottom two rungs below the surface. The rungs are 20 cm apart. If the tide is coming in and the water is rising at a rate of 15 cm per hour, how many rungs will be submerged in three hours?



Junior High

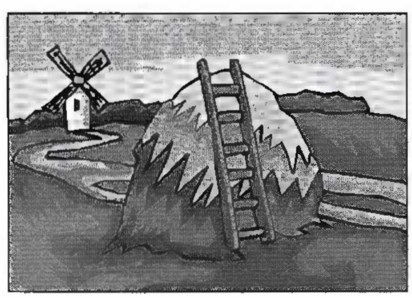
Divide 40 by a half and then add half of your result again.



What do you get?

Elementary

A farmer had four haystacks in one field and nine in another. If he combined them, how many stacks would he have?



Source: www.doe.virginia.gov/Div/Winchester/jhhs/math/puzzles/mathapt2.html

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