What I Learned from My Student's Thinking

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Imagine that this rectangular object is made of toothpicks. How many toothpicks are there in this rectangle? How many toothpicks are needed for any m × n *rectangle? Test your generalization by considering a 10 × 15 rectangle.*

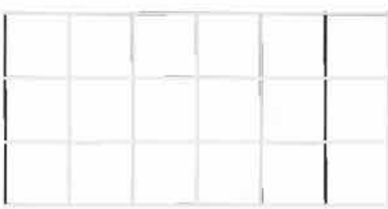
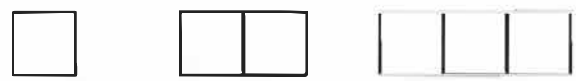


Figure 1. Toothpick array

Although classrooms are normally thought of as places for students to learn mathematics, there are ample opportunities for teachers to deepen their understanding of mathematics in those same rooms. In a mathematics class there is an abundance of ideas and knowledge from which the teacher can grow as both a mathematician and a teacher, but we have to open ourselves up to the moments that present themselves for that to happen. In this paper I offer an illustration of how my knowledge grew from a student's response to a test question, a response that was quite puzzling to me and forced me to grab a pencil and do some figuring. The purpose of this paper is not to propose what Grade 7 students should be expected to explore, but how a particular student's response stimulated my own mathematical understanding and hence my professional development.[1]

Before you read on, have you tried the problem yet? The potential for you to experience surprise, as I did, is greater if you have thought through the problem.

The problem noted above was a test item on a patterns and relations unit exam given to a Grade 7 class. Prior to the exam, the students had been working on generalizing from patterns and communicating those generalizations, both in common language using a sentence or two and through the use of mathematical expressions. For example, students had worked on toothpick trains. The students were provided with a set of images for three toothpick trains from which to work and asked "How many toothpicks does it take to build a train with 1 car, 2 cars, 3 cars, 4 cars, 5 cars, 10 cars, 100 cars?"



1-car train 2-car train 3-car train

Figure 2. Toothpick trains

As they worked on the problem they were asked the following questions:

- How are you counting the toothpicks?
- Do you count every toothpick for every train or do you have a shortcut?
- How would you describe to your friend, in a sentence or two, how you know how many toothpicks are needed for the 4th train or the 100th train?
- Write a mathematical expression that would enable someone to calculate the number of toothpicks needed for the 10th or 100th train.
- Write a mathematical expression that would enable someone to pick a train and then know how many toothpicks it would take to build it.
- If I told you the train had 49 toothpicks, could you tell me how many cars the train has?

Students expressed their thinking differently from one another but yet were able to communicate and evaluate the methods and expressions used to predict the number of toothpicks (t) it takes to build a train of size $l \times n$. Consider the following examples.

Case 1. Recursive solution $t_{(1)} = 4$, for $n \neq 1$ $t_{(n)} = t_{(n-1)} + 3$

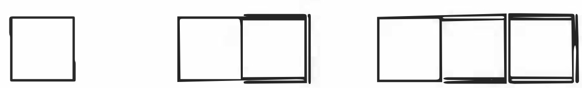


Figure 3. Toothpick train—Add three more

The student who sees the four toothpicks in the first train and then sees three more added to each consecutive train might say, "It's four plus three more every time." Indeed, this is the most common response first given by the Grade 7 students. And it is one they struggle with expressing as a mathematical statement (the notation used above is mine). At this point, the teacher can challenge the students to figure out how many toothpicks are needed for a train of length 100, knowing full well that most students have neither the time nor the patience to figure out all of the trains between the 3rd and the 99th in order to figure out the 100th. (Although we all know students who will do just this, or at least try to!) This is always a good time to point out one of the differences between humans and computers. Computers are very good at number crunching and, indeed, figuring out how many toothpicks it would take for a train of length 100 by figuring out how many it takes for 1,2,3, ... 99. It takes no longer for a computer to do all of the calculations between 1 and 100 than to figure this out for 1,000 or even 100,000. Human beings are not such good number crunchers, but we are good at seeing patterns, which in this case can help us determine a way to figure out how many toothpicks for the 100th train by calculating only the solution for the 100th case.

Case 2. Non-recursive forms of counting and expressing

Case 2.1. Add three more

With this problem, there were students who noted that you can find the 100th train without knowing the 99th train; that is, you can find a nonrecursive, or explicit, solution. This requires a shift in the student's attention from *add three more* to *how is the adding three more related to the number of train cars?* There are two ways to think about this. The first is to see that the first toothpick is present for all trains (invariant or constant) and then that for each additional car (variable), you need 3 (1:3 relationship) toothpicks, $1 + 3n$.

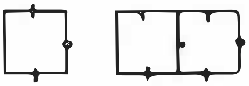


Figure 4. Toothpick train—One plus three more each time

The second way (see figure 4) to see this is to begin with the four toothpicks needed and note that the first car is already counted, so it must be removed from counting the added threes, or $4 + (n - 1)3$. It is important to note that many Grade 7-level students do not have an understanding of the conventions of algebraic notation. Hence, the expressions they write reflect how they think about the situation—in this case, 4 (first car) plus the number of cars minus the first car times 3 toothpicks to make each of those cars. This is something the teacher will work on as the students grow confident in their use of mathematical notion.

Case 2.2. Top + bottom + standing up

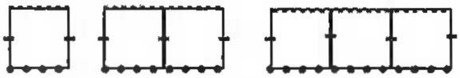


Figure 5. Toothpick train—Top + bottom + standing up

A student who counts the top, then the bottom (horizontally laid) toothpicks and finally counts the standing (vertically laid) toothpicks might say, "top plus bottom plus standing up." In other words, if you know that there are n trains, then there are n toothpicks across the top, n toothpicks across the bottom and $n + 1$ toothpicks standing up, or $n + n + (n + 1)$. Note that students are likely to initially present their mathematical expression in very literal terms $(t + b + s)$, which does not acknowledge the relationships between the number of toothpicks across the top and bottom or the relationship between that value and the number of cars in the train. Again, as students grow in their ability and confidence to count systematically, express the patterns they notice and then write mathematical expressions, the teacher is able to discuss with them the advantages of using the same variable to represent the same value.

Case 2.3. Count the top, count the bottom, count the inside plus two ends.

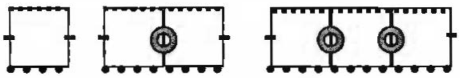


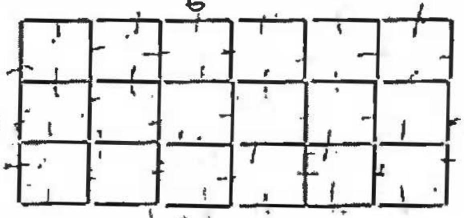
Figure 6. Toothpick train—Count the top, count the bottom, count the inside plus two ends

In this case, the students might write their mathematical expression as top + bottom + inside + ends, $t + b + i + e$. But if they notice that the ends are constant (2) and that top and bottom are the same and that they are also the same value as the number of cars (n), they might write $n + n + 2 + i$. The next step is to figure out the relationship between the number of cars and the number of inside toothpicks, $n - 1$. The expression becomes $n + n + 2 + (n - 1)$. Repeating the common variables and putting brackets around the $n - 1$ might seem cumbersome to someone who understands the conventions in algebra, but the way in which the expression is written usually reflects what the student was attending to when working through the problem. Hence, there is great value for both the teacher and the students to work on communicating generalizations. It is through such work that more robust algebraic thinking can develop.

Asking students to attend to *how* they are counting is meant to direct their attention to a regularity or a pattern. It promotes the students' capacity to communicate, visualize and reason mathematically. It was with these kinds of experiences behind the students that I created the following assessment item. One student responded as follows.

Standard of Excellence: Pattern
(10 marks)

8. Imagine this picture is made up of toothpicks.



$3 \times 3 = 9$
$6 \times 6 = 36$

$$\begin{array}{r} 36 \\ \times \ 9 \\ \hline 45 \end{array}$$

- How many toothpicks are there in this picture? Explain how you figured it out.

There are 45 toothpics. I know this because I counted the amount of toothpics on each side (3 of each side) and multiplyed the two numbers (3x3). Then I counted the amount of toothpics on the top and bottom. (6 on the top and 6 on the bottom), then multiplyed the numbers (6x6). The i got the answers 9, 36 so i multiplyed those on got 45.

- Write a rule (that does not require counting each and every toothpick) for determining how many toothpicks there would be in a rectangular shape of any size (length and width), made with toothpicks.

First count the amount of toothpics on the top and bottom, then multiply the two numbers. Then count the amount of toothpics on both sides, multiply the two numbers. then when you have the two quotents multiply them togethers and the answer is the amount toothpics used.

- Express your rule as a mathematical expression.

$t \times b \times s \times s = N$

t= top
b= bottom
s= side
N= Number of toothpics

- Test your rule by determining the number of toothpicks in a 10 x 15 rectangle.

$10 \times 10 = 100$

$$\begin{array}{r} 15 \\ \times 15 \\ \hline 75 \\ +150 \\ \hline 225 \end{array} \qquad \begin{array}{r} 225 \\ \times 100 \\ \hline 000 \\ 0000 \\ +22500 \\ \hline 22500 \end{array}$$

The rectangle had 22500 toothpics

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Figure 7. Assessment item

Before reading on, you might want to assess this student's response. How would you have marked it? What parts of the student response make sense to you? Which parts puzzle you? What do you make of the question itself? How might you improve the question?

I will begin by commenting on my assessment of this student's response.

- He counted the number of toothpicks correctly for the 3 × 6 rectangle. Unfortunately, the markings on the rectangle do not reveal to me how he counted.
- The calculations in that first part of the item are only partially correct; that is, the student claims to be multiplying ($t \times b$) by ($s \times s$), but clearly he added. $36 \times 9 \neq 45$, but $36 + 9 = 45$. There are 45 toothpicks.
- I assume that the student counted first; this would be consistent with how the students worked on these problems. But I have no way of knowing if the student added 36 and 9 by mistake or if the student knew the answer was 45 and made the two numbers 36 and 9 equal that amount by adding. I suspect it was an unconscious combination of the two—at once the student knew the answer had to be 45 and he saw that 36 and 9 equalled 45.
- The student is consistent in his response. He communicates his "rule" in a couple of sentences and then he writes an expression for it. I can see no obvious reason why the student suggests that the two values should be multiplied together. I suspect that the student is grasping at making meaning of the situation and doesn't have an image for how to calculate without counting.
- In both cases (3 × 6 and 10 × 15), the student indicates to multiply (not add). In the final question, he does calculate the number of toothpicks by multiplying. He did not draw the 10 by 15 rectangle, so I assume that he did not check his calculated response against a counted answer.
- The student uses different variables for values that are the same; top and bottom are expressed as t and b. The generality of *variable* is not demonstrated.
- The student defines his variables; he has an awareness of the need to do so in this communication with the other.
- The student does not explain how his rule is related to how he counted the toothpicks. As far as I can tell, his rule cannot describe his counting. There is no evidence that his reasoning for the expression he created is connected to the geometry of the situation he was given.

Clearly, there is a lot for the teacher to learn about this student's understanding from the test item, but *what can the teacher learn about her own teaching and her own mathematics from this student's response*? I will begin by telling how I came to this assessment item. First of all, the item was intended to take advantage of the experiences students had had in class doing toothpick trains. At the same time, I anticipated that this question would be a problem for students because two variables were changing at the same time (length and width); hence, they would have no immediate strategy for solving the problem other than to begin by counting and look for a generalization of their strategy. Finally, I wanted to have them write a written expression of the generalization, predict a value for a different rectangle and then check it against counting. Therefore, as I prepared the test I needed another example that could be counted in the time the students had. When I selected the second rectangle, I tried to be careful to select something that could be counted but that wasn't obviously related to the first example. As you may have figured out already, I was not careful enough.

When I read this student's response I was slowed down not only by his arithmetic mistakes and notation but also by the fact that the rule he wrote (if we added instead of multiplied the [$t \times b$] and [$s \times s$]) produced the correct value for the number of toothpicks for both rectangles given. This was highly bothersome to me, first of all because the algebraic expression did not look like the one I had in mind. If the student's formula were to work, I should be able to equate it to the formula I had for the problem, $t = l(w+1) + w(l+1)$. I immediately set the two expressions against each other (Figure 8, page 1 and top of page 2). They were not, in general, equivalent. I then specialized with a new case (1 × 2) (Figure 8, page 2). It did not work with the student's formula. Not trusting original calculations, I returned to the 3 × 6 case offered on the test and confirmed that I had not made an error. So, what was happening here that $l^2 + w^2 = l(w + 1) + w(l + 1)$?

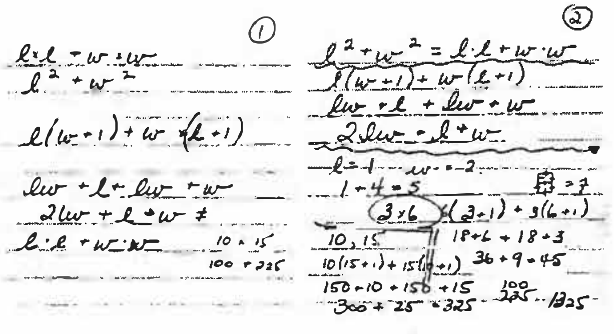

Figure 8. Teacher's working papers—looking for equivalence of expressions

I returned to the figure. I thought I should be able to uncover the student's expression from the geometry if I carried out some systematic counting, but I couldn't. I could see no reason why the sum of the squares of the length and the width had anything to do with the problem. I was forced to go back and consider some more examples. At this point, I thought to consider the set of (l,w) that worked and the set of those that do not work with the student's formula $l^2 + w^2$. The ordered pair (2,3) did not satisfy both equations (figure 9, page 4). Figure 9, page 3 is a record of the ordered pairs that worked (top) and that did not work (bottom).

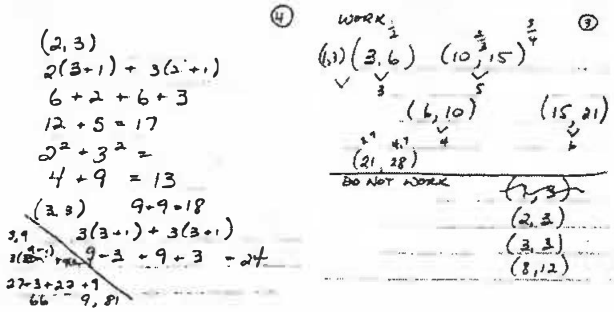

Figure 9. Teacher's working papers, pages 4 and 3.

Once I found more cases of pairs that worked (Figure 10, page 8), it seemed that there *must* be a set of pairs that worked; my challenge was to determine the properties of that set. I decided I needed to explore the conditions under which the two expressions were equivalent. After some time and twelve pages of sticky notes, I concluded that whenever the order pair is composed of two consecutive triangular numbers the expressions are equivalent (Figure 10, page 12). What a surprise! When I made the choice for the examples in the test I had not noticed this feature of the numbers I selected.

My learning opportunity arose in the context of teaching patterns and relations to Grade 7 students. I continue to be pleased with the thought that trying to understand one student's thinking led me to a better understanding of patterns and relations and enhanced my understanding of triangular numbers and how special they are. This student provided me with the experience of exploring some new mathematics. But maybe most significant, this student gave me the opportunity to experience the joy of discovery, the surprise of mathematics and the satisfaction that comes from making sense. Although I will try to be more careful when choosing examples for students, I suspect this will not be the last time that mathematics and a learner teach me something new.

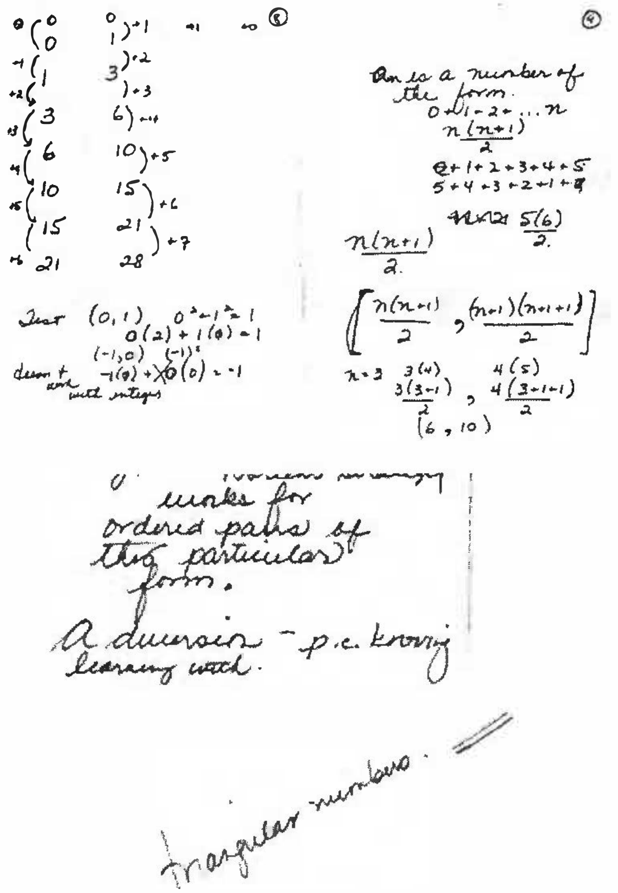

Figure 10. Teacher's working papers, pages 8, 9 and 12.

Note

1 For further discussion of this episode from my research study, please see Simmt, E, B Davis, L Gordon and J Towers. 2003. "Teachers' Mathematics: Curious Obligations." In *Proceedings of the 2003 Joint Meeting of PME and PMENA*, ed N Pateman, B Dougherty and J Zilliox, 175–82. Honolulu, Hawaii: University of Hawaii.

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