Alberta High School Mathematics Competition

Part II, 2009.

Problem 1.

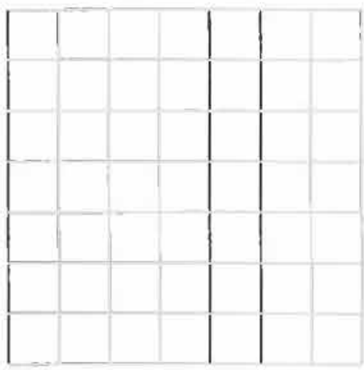
Let w, x, y and z be non-negative numbers whose sum is 100. Determine the maximum possible value of $wx + xy + yz$.

Problem 2.

Determine all positive integers a and b, $a < b$, so that exactly $\frac{1}{100}$ of the consecutive integers $a^2,\ a^2+1,\ a^2+2,\ \ldots,\ b^2$ are the squares of integers.

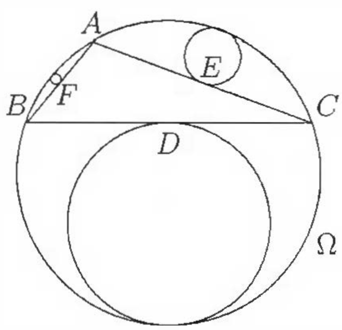
Problem 3.

A game is played on a 7×7 board, initially blank. Betty Brown and Greta Green make alternate moves, with Betty going first. In each of her moves, Betty chooses any four blank squares which form a 2×2 block, and paints these squares brown. In each of her moves, Greta chooses any blank square and paints it green. They take alternate turns until no more moves can be made by Betty. Then Greta paints the remaining blank squares green. Which player, if either, can guarantee to be able to paint 25 or more squares in her colour, regardless of how her opponent plays?



Problem 4.

A, B and C are points on a circle Ω with radius 1. Three circles are drawn outside triangle ABC and tangent to Ω internally. These three circles are also tangent to BC, CA and AB at their respective midpoints D, E and F. If the radii of two of these three circles are $\frac{2}{3}$ and $\frac{2}{11}$, what is the radius of the third circle?



Problem 5.

Prove that there are infinitely many positive integers k such that k^k can be expressed as the sum of the cubes of two positive integers.

Alberta High School Mathematics Competition
Solutions and Comments to Part II, 2009.

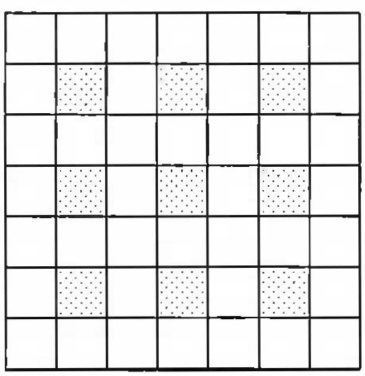
Problem 1.
Since $(w+y)+(x+z)=100$, we have $w+y=50+t$ and $x+z=50-t$ for some real number t. Hence $wx+xy+yz \leq (w+y)(x+z)=(50+t)(50-t)=2500-t^2 \leq 2500$. This maximum value may be attained for instance when $w=x=50$ and $y=z=0$.

Problem 2.
Let $d=b-a$. Then there are $(a+d)^2-a^2+1=2ad+d^2+1$ integers under consideration, $d+1$ of which are the squares of integers. Hence we need $100(d+1)=2ad+d^2+1$, so that

$$a=\frac{100(d+1)-d^2-1}{2d}=\frac{100-d}{2}+\frac{99}{2d}.$$

If d is even, the first term is an integer and the second is not. Hence d must be odd. Then the first term is a fraction with denominator 2, so that the second term must also be a fraction with denominator 2. This means that d must be a divisor of 99, that is, d is 1, 3, 9, 11, 33 or 99.
If $d=1$, then $a=\frac{99}{2}+\frac{99}{2}=99$ and $b=99+1=100$.
If $d=3$, then $a=\frac{97}{2}+\frac{99}{6}=65$ and $b=65+3=68$.
If $d=9$, then $a=\frac{91}{2}+\frac{99}{18}=51$ and $b=51+9=60$.
If $d=11$, then $a=\frac{89}{2}+\frac{99}{22}=49$ and $b=49+11=60$.
If $d=33$, then $a=\frac{67}{2}+\frac{99}{66}=35$ and $b=35+33=68$.
If $d=99$, then $a=\frac{1}{2}+\frac{99}{198}=1$ and $b=1+99=100$.

Problem 3.
There are 9 squares at the intersections of even-numbered rows and even-numbered columns. Any 2×2 block chosen by Betty must include one of these 9 squares. Hence Greta should play only on these squares in her first four moves. This will ensure that Betty has at most five moves, and can paint at most 20 squares brown. Hence Greta wins.



Problem 4.

Denote the circumcentre of Ω by O and note that it lies within the circle with radius $\frac{2}{3}$. We have

$$BC^2 = 4BD^2 = 4(OB^2 - OD^2) = 4\left(1 - \frac{1}{9}\right) = \frac{32}{9}$$

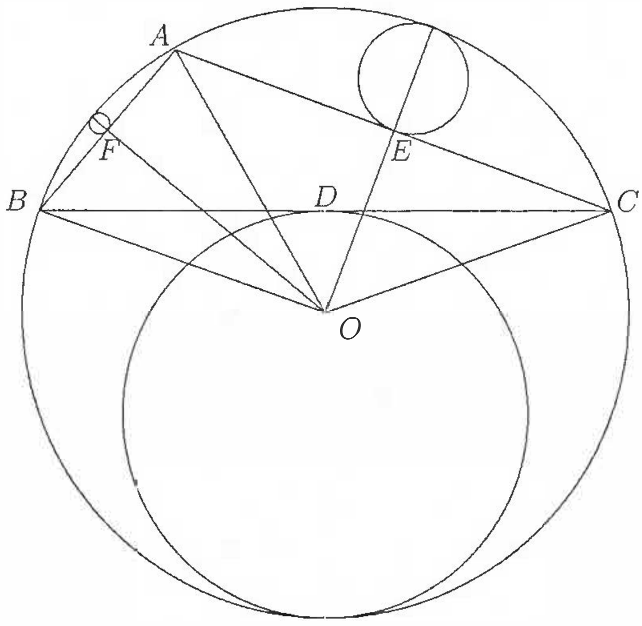
and

$$CA^2 = 4CE^2 = 4(OC^2 - OE^2) = 4\left(1 - \frac{49}{121}\right) = \frac{288}{121}.$$

By the Cosine Law, $\cos BOC = \frac{OB^2+OC^2-BC^2}{2OB\cdot OC} = -\frac{7}{9}$ and $\cos COA = \frac{OC^2+OA^2-CA^2}{2OC\cdot OA} = -\frac{23}{121}$. Hence $\sin BOC = \frac{4\sqrt{2}}{9}$ and $\sin COA = \frac{84\sqrt{2}}{121}$. It follows that

$$\begin{aligned}\cos AOB &= \cos(\angle BOC - \angle COA))\\ &= (\cos BOC)(\cos COA) + \sin BOC \sin COA\\ &= \frac{833}{1089}.\end{aligned}$$

By the Cosine Law again, $AB^2 = OA^2 + OB^2 - 2OA \cdot OB \cos AOB = \frac{512}{1089}$. Hence we have $OF^2 = OA^2 - AF^2 = \frac{961}{1089}$ and $OF = \frac{31}{33}$. It follows that the radius of the third circle is $\frac{1}{2}(1 - \frac{31}{33}) = \frac{1}{33}$.



Problem 5.

We have $(a+1)^{a+1} = (a+1)^a(a+1) = a(a+1)^a + (a+1)^a$. Choose $a = 3^{3t}$ for an arbitrary positive integer t. Then $a = (3^t)^3$ and $(a+1)^a = ((3^{3t}+1)^{3^{3t-1}})^3$ are both cubes. If we take $k = 3^{3t}+1$, $m = 3^t(3^{3t}+1)^{3^{3t-1}}$ and $n = (3^{3t}+1)^{3^{3t-1}}$, then $k^k = m^3 + n^3$. Since t is an arbitrary positive integer, the number of possible choices for k is infinite.