THE CALGARY MATHEMATICAL ASSOCIATION

33rd JUNIOR HIGH SCHOOL MATHEMATICS CONTEST

April 22, 2009

NAME: SOLUTIONS
PLEASE PRINT (First name Last name)

GENDER: ☐ M ☐ F

SCHOOL: ____________________

GRADE: ______ (7,8,9)

- You have 90 minutes for the examination. The test has two parts: PART A — short answer; and PART B — long answer. The exam has 9 pages including this one.
- Each correct answer to PART A will score 5 points. You must put the answer in the space provided. No part marks are given.
- Each problem in PART B carries 9 points. You should show all your work. Some credit for each problem is based on the clarity and completeness of your answer. You should make it clear why the answer is correct. PART A has a total possible score of 45 points. PART B has a total possible score of 54 points.
- You are permitted the use of rough paper. Geometry instruments are not necessary. References including mathematical tables and formula sheets are **not** permitted. Simple calculators without programming or graphic capabilities **are** allowed. Diagrams are not drawn to scale. They are intended as visual hints only.
- When the teacher tells you to start work you should read all the problems and select those you have the best chance to do first. You should answer as many problems as possible, but you may not have time to answer all the problems.

MARKERS' USE ONLY	
PART A ×5	
B1	
B2	
B3	
B4	
B5	
B6	
TOTAL (max: 99)	

BE SURE TO MARK YOUR NAME AND SCHOOL AT THE TOP OF THIS PAGE.
THE EXAM HAS 9 PAGES INCLUDING THIS COVER PAGE.

Please return the entire exam to your supervising teacher at the end of 90 minutes.

PART A: SHORT ANSWER QUESTIONS

A1 What is the largest number of integers that can be chosen from $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ such that no two integers are consecutive?

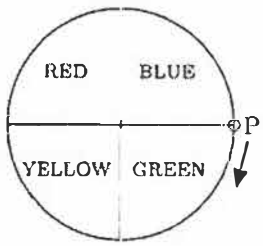
5

A2 Elves and ogres live in the land of Pixie. The average height of the elves is 80 cm, the average height of the ogres is 200 cm and the average height of the elves and the ogres together is 140 cm. There are 36 elves that live in Pixie. How many ogres live in Pixie?

36

A3 A circle with circumference 12 cm is divided into four equal sections and coloured as shown. A mouse is at point P and runs along the circumference in a clockwise direction for 100 cm and stops at a point Q. What is the colour of the section containing the point Q?

YELLOW



A4 What is the longest possible length (in cm) of a side of a triangle which has positive integer side lengths and perimeter 17 cm?

8

A5 A and B are whole numbers so that the ratio $A : B$ is equal to $2 : 3$. If you add 100 to each of A and B, the new ratio becomes equal to $3 : 4$. What is A?

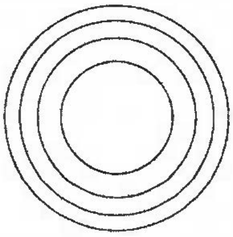
200

A6 You are given a two-digit positive integer. If you reverse the digits of your number, the result is a number which is 20% larger than your number. What is your number?

45

A7 In the picture there are four circles one inside the other, so that the four parts (three rings and one disk) each have the same area. The diameter of the largest circle is 20 cm. What is the diameter (in cm) of the smallest circle?

10

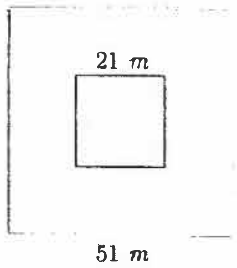


A8 Carol's job is to feed four elephants at the circus. She receives a bag of peanuts every day and feeds each elephant as many peanuts as she can so that each elephant receives the same number of peanuts. She then eats the remaining peanuts (if any) at the end of the day. On the first day Carol receives 200 peanuts. On every day after, she receives one more peanut than she did the previous day. This was done over 30 days. How many peanuts did Carol eat over the 30 days?

43

A9 Richard lives in a square house whose base has dimension 21 m by 21 m and is located in the centre of a square yard with dimension 51 m by 51 m as in the diagram. Richard is to tie one end of a leash to his puppy and the other end of the leash to a corner of his house so that the puppy can reach all parts of the yard. What is the smallest length (in m) of a leash so that this can be done?

60



PART B: LONG ANSWER QUESTIONS

B1 Ella and Bella each have an integer number of dollars. If Ella gave Bella enough dollars to double Bella's money, Ella would still have \$100 more than Bella. In fact, if Ella instead gave Bella enough dollars to triple Bella's money, Ella would still have \$40 more than Bella. How much money does Ella have?

Solution:

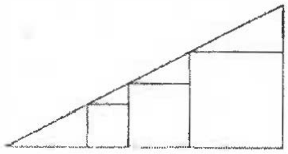
Let y be the number of dollars Bella has. The key observation is that Ella gave Bella y more dollars in the second scenario than in the first scenario. This accounts for the difference of \$60 between how much more Ella has than Bella in the two scenarios. In the second scenario, Ella has y dollars less and Bella has y dollars more than in the first scenario. Hence, the difference between what Ella and Bella have decreases by $2y$. Therefore, $2y = 60$, which means $y = 30$. Therefore, Bella has \$30 originally.

In the first scenario, Ella gave Bella \$30, resulting in Bella having \$60. Since Ella has \$100 more than Bella after this exchange, Ella has \$160 after this exchange. Before the exchange, Ella had \$160 + \$30 = \$190.

The answer is \$190.

Comment: This problem can also be solved by guess and check.

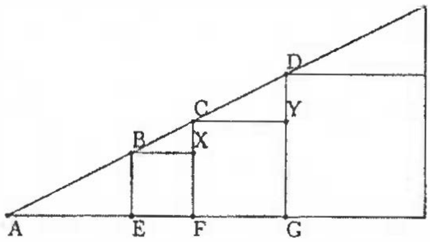
B2 Three squares are placed side-by-side inside a right-angled triangle as shown in the diagram.



The side length of the smallest of the three squares is 16 cm. The side length of the largest of the three squares is 36 cm. What is the side length (in cm) of the middle square?

Solution:

Let $A, B, C, D, E, F, G, X, Y$ be the points labeled in the diagram.



Then $\triangle ACF$ is similar to $\triangle ADG$. This means the ratio of the sides of $\triangle ACF$ to the sides of $\triangle ADG$ is equal to the ratio of a side of the smallest square and the middle square, since the smallest square is inscribed in $\triangle ACF$ and the middle square is inscribed in $\triangle ADG$. Therefore,

$$\frac{CF}{DG} = \frac{\text{Side length of smallest square}}{\text{Side length of middle square}} = \frac{16}{CF}.$$

We know that $DG = 36$, since it is the side of the largest square. We now have the equation

$$\frac{CF}{36} = \frac{16}{CF}.$$

By cross-multiplying, we have that $CF^2 = 36 \times 16 = 576$. By square rooting both sides (and noting that we only need the positive solution), we get that $CF = \sqrt{576} = 24$.

Alternate Solution:

Let x be the side length of the middle square. Note that $\triangle BCX$ and $\triangle CDY$ are similar. BX has length 16 since it is a side of the smallest square. CX has length $x - 16$ since it is the difference between a side of the middle square and a side of the smallest square. CY has length x and DY has length $36 - x$ since it is the difference between a side of the largest square and a side of the middle square. By similar triangles,

$$\frac{BX}{CX} = \frac{CY}{DY}, \text{ which means } \frac{16}{x-16} = \frac{x}{36-x}.$$

By cross-multiplying, we get that $16(36-x) = x(x-16)$, which simplies to $576 - 16x = x^2 - 16x$. The $16x$ term on both sides cancel and we get $576 = x^2$. Therefore, $x = \sqrt{576} = 24$.

The answer is 24 cm.

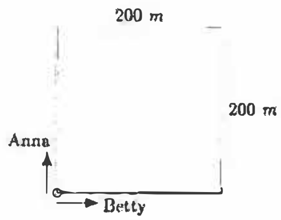
B3 Friends Maya and Naya ordered finger food in a restaurant, Maya ordering chicken wings and Naya ordering bite-size ribs. Each wing cost the same amount, and each rib cost the same amount, but one wing was more expensive than one rib. Maya received 20% more pieces that Naya did, and Maya paid 50% more in total than Naya did. The price of one wing was what percentage higher than the price of one rib?

Solution:

Suppose Naya ordered n ribs. Then Maya ordered $1.2n$ wings. Suppose Naya paid N dollars altogether. Then Maya paid $1.5N$ dollars altogether. The cost per rib was N/n dollars. The cost per wing was $1.5N/(1.2n) = (5/4)(N/n) = 1.25(N/n)$ dollars. Therefore Maya's cost per wing was 25% more than Naya's cost per rib.

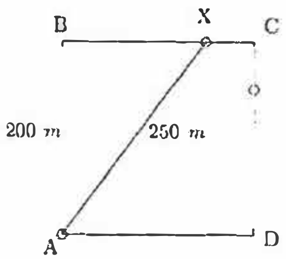
The answer is 25%.

B4 There is a running track in the shape of a square with dimensions 200 metres by 200 metres. Anna and Betty run around the track starting from the same corner of the track at the same time, each at a constant speed, but in different directions on the track. Anna runs at 6.3 kilometres per hour. Anna and Betty meet on the track for the first time after starting, at a point whose straight-line distance from the starting point is 250 metres. What are Betty's possible speeds in kilometres per hour?



Solution:

We label the corners of the track A, B, C, D in a clock-wise direction with A being the starting corner. Suppose Anna is running in the clockwise direction and Betty is running in the counter-clockwise direction. There are two possible locations on the track that are located 250 m from A. One of these locations is on segment BC and the other is on segment DC.



In the case where this meeting location is on segment BC, let X be this meeting location. Then $AX = 250$ m and $AB = 200$ m. By the Pythagorean Theorem, $BX = \sqrt{250^2 - 200^2} = \sqrt{50^2(5^2 - 4^2)} = 50\sqrt{9} = 150$ m. This means Anna ran from A to B to X before meeting Betty, which means Anna ran 200 m+150 m= 350 m. Betty ran the remainder of the track, which is $AD + DC + CX = 200$ m+200 m+50 m= 450 m. Therefore, the ratio of Anna's speed to Betty's speed is $350 : 450 = 7 : 9$. Since Anna runs at 6.3 km/hr, if we let x be Betty's speed in km/hr, we get that

$$\frac{7}{9} = \frac{6.3}{x}.$$

By cross-multiplying, we get $7x = 6.3 \times 9$. This means $x = 0.9 \times 9 = 8.1$.

The other case is where the meeting location is on segment DC. This is when Anna runs faster than Betty. By symmetry, the ratio of Anna's speed to Betty's speed is $9 : 7$. Since Anna runs at 6.3 km/hr, we get that

$$\frac{9}{7} = \frac{6.3}{x}.$$

By cross-multiplying, we get that $9x = 6.3 \times 7$. This means $x = 0.7 \times 7 = 4.9$.

Therefore, Betty's possible speeds are 4.9 km/hr and 8.1 km/hr.

The answers are 4.9 km/hr and 8.1 km/hr.

B5 Adrian owns 6 black chopsticks, 6 white chopsticks, 6 red chopsticks and 6 blue chopsticks. They are all mixed up in a drawer in a dark room.

(a) (4 points) He wants to get four chopsticks of the same colour. How many chopsticks must he grab to be guaranteed of this? Show that fewer chopsticks than your answer might not be enough.

Solution:

If Adrian picks 12 chopsticks, it is possible that Adrian has 3 chopsticks of each colour and therefore does not have 4 chopsticks of any colour.

If Adrian does not have four chopsticks of any one colour, then the maximum number of chopsticks that Adrian can pick is (3 chopsticks per colour) $\times$ (4 colours) $= 12$. Therefore, if Adrian picks 13 chopsticks, Adrian must have 4 chopsticks of one colour.

The answer is 13 chopsticks.

(b) (5 points) Suppose instead Adrian wants to get two chopsticks of one colour and two chopsticks of another colour. How many chopsticks must he grab to be guaranteed of this? Show that fewer chopsticks than your answer might not be enough.

Solution:

If Adrian picks 9 chopsticks, it is possible that Adrian does not have two chopsticks of one colour and two chopsticks of another colour. An example of this is if Adrian picks up 6 black chopsticks and 1 chopstick of each of the other three colours.

If Adrian does not have two chopsticks of one colour and two chopsticks of another colour, then there are three colours such that Adrian has at most one chopstick of cach of these three colours. Since there are six chopsticks of the fourth colour, the maximum number of chopsticks that Adrian can pick is $6 + 1 + 1 + 1 = 9$. Therefore, if Adrian picks 10 chopsticks, Adrian must have two chopsticks of one colour and two chopsticks of another colour.

The answer is 10 chopsticks.

B6 The numbers 2 to 100 are assigned to ninety-nine people, one number to each person. Each person multiplies together the largest prime number less than or equal to the number assigned and the smallest prime number strictly greater than the number assigned. Then the person writes the reciprocal of this result on a sheet of paper.

For example, consider the person who is assigned number 9. The largest prime less than or equal to 9 is 7. The smallest prime strictly greater than 9 is 11. So this person multiplies 7 and 11 together to get 77. The person assigned number 9 then writes down the reciprocal of this answer, which is $\frac{1}{77}$.

(a) (3 points) Which people write down the number $\frac{1}{77}$ (one of these people is person #9)? Show that the sum of the numbers written down by these people is equal to $\frac{1}{7} - \frac{1}{11}$.

Solution:
Since $77 = 7 \cdot 11$ and 7 and 11 are two consecutive primes, the people that wrote down the number $\frac{1}{77}$ are those who are assigned numbers $7, 8, 9$ and 10.

Hence, there are four people that wrote down the number $\frac{1}{77}$. Therefore, the sum of the numbers written down by these four people is $\frac{4}{77}$. Note that

$$\frac{1}{7} - \frac{1}{11} = \frac{11}{77} - \frac{7}{77} = \frac{4}{77}.$$

Therefore, the sum of the numbers written down by these four people is indeed $\frac{1}{7} - \frac{1}{11}$.

(b) (6 points) What is the sum of all 99 numbers written down? Express your answer as a fraction in lowest terms.

Solution:
Note that everyone writes down a number of the form $\frac{1}{pq}$ where p, q are two consecutive prime numbers. Given a pair of consecutive prime numbers p, q, with $p < q$, the people that write down the number $\frac{1}{pq}$ are the people assigned numbers $p, p+1, p+2, \cdots, q-1$. Therefore, there are $q - p$ people that write down $\frac{1}{pq}$. The sum of the numbers written by these people is $\frac{q-p}{pq}$.

Using the same idea as in (a), we can show that the sum of all of these numbers is indeed $\frac{1}{p} - \frac{1}{q}$, since

$$\frac{1}{p} - \frac{1}{q} = \frac{q-p}{pq}$$

which is the sum of the numbers written down by the people who write down $\frac{1}{pq}$.

Since 2 is the first prime and the two largest consecutive prime numbers used in the problem are 97 and 101, the set of all pairs of consecutive primes from 2 to 101 are $(2,3), (3,5), (5,7), \cdots, (89,97), (97,101)$. Therefore, the sum of all 99 numbers is

$$\left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + \cdots + \left(\frac{1}{89} - \frac{1}{97}\right) + \left(\frac{1}{97} - \frac{1}{101}\right).$$

The intermediate terms cancel. This sum is then equal to

$$\frac{1}{2} - \frac{1}{101} = \frac{101}{202} - \frac{2}{202} = \frac{99}{202}.$$

The answer is $\frac{99}{202}$.