delta-k

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Mathematics Education for a Global Village

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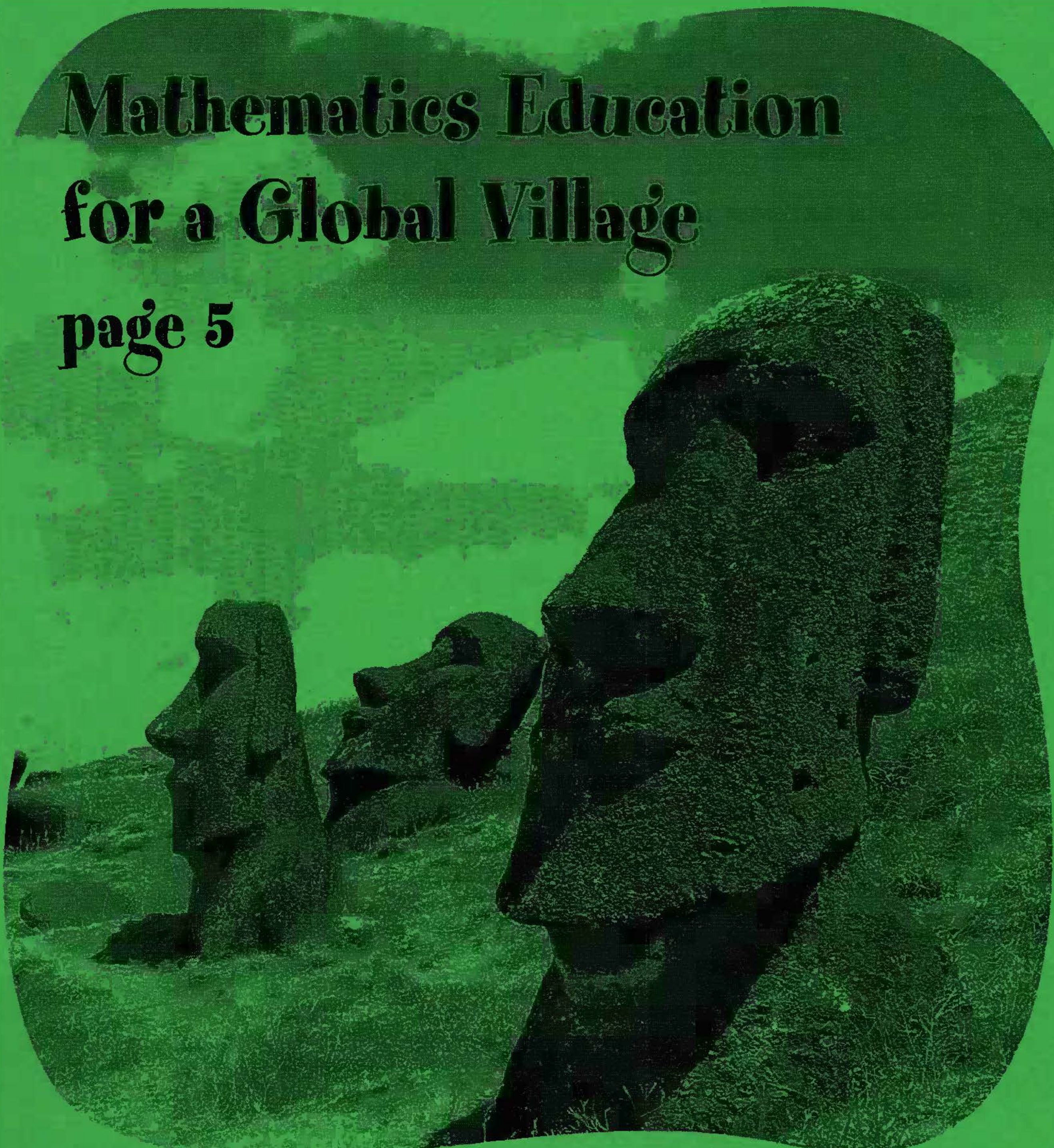

Guidelines for Manuscripts

delta-K is a professional journal for mathematics teachers in Alberta. It is published twice a year to

- promote the professional development of mathematics educators, and
- stimulate thinking, explore new ideas, and offer various viewpoints.

Submissions are requested that have a classroom as well as a scholarly focus. They may include

- personal explorations of significant classroom experiences;
- descriptions of innovative classroom and school practices;
- reviews or evaluations of instructional and curricular methods, programs or materials;
- discussions of trends, issues or policies;
- a specific focus on technology in the classroom; or
- a focus on the curriculum, professional and assessment standards of the NCTM.

Suggestions for Writers

1. *delta-K* is a refereed journal. Manuscripts submitted to *delta-K* should be original material. Articles currently under consideration by other journals will not be reviewed.
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4. The author's name and full address should be provided on a separate page. If an article has more than one author, the contact author must be clearly identified. Authors should avoid all other references that may reveal their identities to the reviewers.
5. All manuscripts should be submitted electronically, using Microsoft Word format.
6. Pictures or illustrations should be clearly labelled and placed where you want them to appear in the article. A caption and photo credit should accompany each photograph.
7. References should be formatted consistently using *The Chicago Manual of Style*'s author–date system or *The American Psychological Association (APA)* style manual.
8. If any student sample work is included, please provide a release letter from the student's parent/guardian allowing publication in the journal.
9. Articles are normally 8–10 pages in length.
10. Letters to the editor or reviews of curriculum materials are welcome.
11. Send manuscripts and inquiries to the editor: Gladys Sterenberg, 195 Sheep River Cove, Okotoks, AB T1S 2L4; e-mail gladyss@ualberta.ca.

MCATA Mission Statement

Providing leadership to encourage the continuing enhancement of teaching, learning and understanding mathematics.

delta-K

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From the Editor's Pencil

Gladys Sterenberg

This issue of *delta-K* brings ideas that I believe will both inspire and motivate you in creating learning opportunities for your students. Many of the current professional initiatives in this province direct our attention to the importance of real learning. The authors of the articles included here demonstrate some ways this can be done.

As we contemplate real learning in mathematics, one consideration might be a re-visioning of mathematics classrooms. Carrie Watt presents some reflections on her vision of a global village and how mathematics can become real through learning in context. Solving problems is paramount in her vision.

A problem-solving focus is not new. However, Jennifer Holm describes various perceptions of problem solving to help us thoughtfully consider how we incorporate it into our classrooms. Elaine Simmt reflects on her own journey of problem solving with a task offered to her students. She prompts each of us to consider our mathematical development as learners.

One of the ways of encouraging real learning involves the use of integrating stories and mathematics. In this issue, Benet Copeland investigates coordinate graphing through his story. Karen Cleveland shares her integration of problem solving and literature, an example of how teaching mathematics can be creative and authentic.

As we continue to work hard in our classrooms, I hope you find inspiration in the words of these authors. As always, we depend on the sharing of teaching ideas. If you have an innovative lesson idea or if you are excited by your students' responses to a mathematical task, please consider sharing these experiences with our readership. I hope you enjoy reading the articles in this issue.

The Right Angle: Report from Alberta Education

Jennifer Dolecki

The New Learn Alberta Website

The new Learn Alberta website, www.learnalberta.ca, incorporates digital learning and teaching resources, implementation supports, and distributed learning courses. Previous versions of LearnAlberta.ca and the online guide to implementation are no longer available.

The supports previously available through the online guide to implementation can now be accessed on the LearnAlberta.ca site through links embedded within the applicable program of studies or by using the search function. Users who have created links to resources found in the former online guide to implementation will be automatically redirected to the same resources on the new website.

Teachers in Alberta who hold an active and valid professional certificate can create a personal teacher account by selecting the Sign Up link at the top right-hand corner of the home page. To complete the sign-up process, teachers will require a jurisdictional username and password for LearnAlberta.ca, a professional certificate number, and access to an e-mail account.

A personal teacher account gives increased functionality within the My Workspace feature of the website. Within My Workspace, teacher account holders can save resources, add their own Web-based resources, create folders to organize resources, add notes to resources and folders, and publish lists of resources using either e-mail or a Web link.

Finding Learning and Teaching Resources

The new design of the homepage gives users a number of ways to find learning and teaching resources. Tabs allow users to choose whether they wish to find resources within the context of the program of study or to search for resources. The online reference centre is also available at all times to allow users easy access to the collection of multimedia encyclopedias and online databases to support inquiry and research. The Find Resources box on the homepage also allows users to perform a quick keyword search using grade and subject filters.

New Resources/Most Accessed Resources Area

The homepage offers users a snapshot of the most accessed resources, as well as any new resources available on the site. On the lower right of the homepage, a statistics area offers users a choice of information they would like displayed for their own interest. Users may choose to display resources for all subjects or for a specific subject by making a choice from the drop-down menu. Users with a personal account may also choose to display resources according to the personal preferences set in their account profile. These preferences will be automatically displayed in the resource box on the homepage whenever the user returns to the site.

Program of Study Search

The Program of Study tab offers users a choice of either French or English programs. From a drop-down menu, users can select a subject area to reveal a list of all available programs in that subject area. For example, if a user chooses English social studies as a core program, a list of all Alberta Education social studies programs of study will appear on the right side of the screen. Clicking on the appropriate program and grade will open the program of studies with a table of contents on the left and the text of the program of studies on the right.

As they scroll through the program of studies, users will see icons. These icons indicate that resources are available for a particular outcome, generalization or key understanding. By hovering over the icon users will see a pop-up information box indicating the number of resources available and whether the resources are for teachers or students. Clicking on the icon will reveal a list of those resources. The list includes information on each resource, such as the resource name, description, grade level, subject and media format. Expanding the list by clicking on the "more info" link provides the language of the resource, keywords, intended audience and learner outcomes. Clicking on the title of the resource from the list will launch the resource.

A user who has signed in to a personal account will see the words "Save to My Workspace" to the right of the resource's title. The Save to My Workspace link will be active if the user has signed in using his or her personal account information. Clicking on this link allows users who have set up a personal account to save the listed resource directly to their My Workspace area. After clicking the Save to My Workspace link, users will see a pop-up window notifying them that the resource has been successfully saved to My Workspace.

Search

The Search tab allows users to refine their search in more detail than they get using the Find Resource box on the homepage. By clicking on the Search tab, the user is taken to a page that lets them search using the options of grade, subject, audience, language of the resource, media format and learning resource type. The number of individual resources available on the website is shown in brackets beside the check boxes within each filter.

Of course, if the user knows the name of the resource, he or she can simply type it in the Find Resources box at the top of the page. However, if the user types in a broader search term, such as *Gizmo*, the search will return a list of all the resources on the website that contain the word *Gizmo* in the description or title. Refining a search by using the available filters allows users to narrow their search, thereby retrieving fewer resources and improving the likelihood of locating a useful resource.

Mathematics Education for a Global Village

Carrie Watt

Introduction

In 1897, the tormented writer and artist Paul Gauguin painted a vast painting that he titled *D'Où Venons Nous? Que Sommes Nous? Où Allons Nous?* (Where do we come from? What are we? Where are we going?). Historian Ronald Wright used Gauguin's questions to frame his book *An Illustrated Short History of Progress* (2006), in which he warned of the inherent danger in humankind's progress. Wright began with the example of our progress in weapon technology and commented, "When the bang we can make can blow up our world, we have made rather too much progress" (p 5). I share my experience in wrestling with Gauguin's questions. My vision of a global village (McLuhan, 1962) informs my decisions as a mathematics teacher and leads to where I hope we are going.

D'Où Venons Nous (Where Do We Come From)?

I am not a historian and I do not propose to look at a history of humankind or evolution. However, I did have the opportunity to experience a place "2000 years and half a world away. Dying trees still grow greener when you pray" (Cockburn 1974). This experience gave me a sense of where humankind came from and provided me with a new lens through which to look critically at my Western society.

In high school, I became involved with a nongovernmental development organization that works in Nepal. As a group of high school students, we raised the money to build a school in Nepal. We trekked to the remote mountain village of Thulo Pokhara, where the school was to be built, and met the villagers there. There was no electricity; the people in the village shared one water pump. The villagers eked out meagre food supplies by terrace farming on the steep hillside and travelled hours each day to collect firewood and dung for cooking and to attend school. At first glance, with my Western eyes, I saw Thulo Pokhara as primitive and uncivilized. However, I saw during the week we were there that, despite the hardships, daily life was full of laughter and love and joy. When we arrived in Thulo Pokhara, the villagers insisted that we partake in tea and eggs, despite [the

D'Où Venons Nous? Que Sommes Nous? Où Allons Nous? Gauguin 1897 poverty demonstrated by] their children's bloated bellies. There was a boy in the village with a physical disability who was not able to walk; different members of the community carried him to every event. In a place with no accessibility laws, that child was never excluded. Unfortunately, the only viable site on which to build the school was where an ancient and huge tree stood; the entire village deliberated for a day about whether or not to cut down that tree.

Where do we come from? Gauguin ran away from European civilization seeking what he called the "savage—primordial man" in Tahiti and other South Sea islands, perhaps hoping to find an unspoiled paradise of the past (Wright 2006, 1). For me, visiting Thulo Pokhara was like taking a step back in time; I saw a small society that acted as a collective community in harmony with its environment. Gauguin remained tormented in the South Seas, so I do not think he found his paradise. I would not consider Thulo Pokhara a paradise, given the villagers' constant struggle to meet basic needs; however, when I returned to Canada I felt a void and a longing for the holistic way of being (mind, body and spirit) and the harmony with each other and the environment that I saw in Thulo Pokhara.

Que Sommes Nous (What Are We)?

I will take some liberty with Gauguin's question and speak to *where* humankind is now rather than *what* we are. Ronald Wright (2006) said that we have an ideology of "material progress" that coincides with "the rise of science and industry and the corresponding decline of traditional beliefs" (p 3). This was what I saw so blatantly when I returned from Nepal. In Canada, I saw people with plenty who just wanted more. I recognized that I, too, was acculturated in this ideology and felt unfulfilled. As Benjamin Franklin remarked, "The more a man has, the more he wants. Instead of filling a vacuum, it makes one" (as cited in Suzuki, McConnell and Mason 2007, 42).

Mathematics has been influenced by and influences modem Western culture. Western society has promoted a mathematics curriculum that fuels our economic progress. With globalization, the spread of this mathematics curriculum is increasing. In a way, capitalism is the new face of colonialism: for countries to be globally competitive, they align with and are assimilated into the mainstream ideology. In this way, school mathematics has become "a cultural homogenizing force" (Namukasa 2004, 211) because the world is adopting a Eurocentric, "industrially oriented curriculum" (p 222).

Coupled with the ideology of material progress we have a "dominant ideology of scientific determinism" (Namukasa 2004, 221) in the Western world: a belief that we can use our logical reasoning and powers of deduction to know everything about our world. As Namukasa (2004) put forth, "It appears the modern world is filled with the spirit of Descartes, a dream about a universal method whereby all human problems could be worked out rationally, systematically, and by logical computation" (pp 210–11). With this complete faith in logic and deduction, mathematics is seen as "a pure, perfect system and an infallible tool" (p 221). This philosophy has several ramifications. First, it gives undue power to mathematics. In education, I see too much emphasis placed on standardized achievement test results—the numbers are considered the best indicator of our success in schooling. Next, it is used as "a critical filter for status" (p 211). Most university programs require success in the standardized mathematics curriculum for entrance. It is a matter of debate how many of these students will actually need to use the industrially driven mathematics they were required to learn. Finally, it does not honour different ways of knowing about our world. In schools, I see the importance placed on mathematics over physical education or the arts.

> Scientific determinism has also resulted in fragmentation of our knowledge. The scientific method has reduced the world into disciplines, and within each discipline the focus of study is further narrowed and fragmented. This objective, precise scientific methodology has gained us detailed knowledge about snippets of our world, but this system also has its limits and unimagined consequences. When our world is reduced to minutiae, it is easier to treat everything as a commodity. (Suzuki, McConnell and Mason 2007, 36)

In school, the curriculum is divided into subjects, and within subjects knowledge is further classified (eg, algebra and geometry). There *is* a need to classify and, perhaps, specialize, given our vast amount of knowledge, but it is harmful when the connections between the ways of knowing are lost.

Where are we now? In Canada, I primarily see a society that is insular, fragmented and competitive. With all our progress, we have become specialized in our knowledge and our functions, and we have lost our connections to each other and the Earth. Unlike the villagers in Thulo Pokhara, in Canada I have plenty of food but I may not know who produced it, how, where or even what it is. But I also see a growing movement to use our technology to restore some of those connections. I see power in the shared language of mathematics to come together and solve problems:

> Basic mathematics is a globally understood language; it is an approach to being in, engaging with, and relating to the world, and to perceiving and understanding the structure of our worlds. Both the social and physical worlds are being understood from a mathematical view at an increasing rate. (Namukasa 2004, 210).

Mathematics provides a shared language to examine our world and base our technology on. Technology can help us, as consumers, make informed decisions about how to live. I could go to the grocery store and Google information about the farmer who grew the apples I want to buy, thereby restoring my connection to the producer of my food, which impacts my health and the Earth.

Où Allons Nous (Where Are We Going)?

Path 1: Easter Island

If our current ideology of material and economic progress at all costs continues, I am afraid that we will repeat history, and our progress will lead to our demise. Wright (2006) provided the example of what happened on the remote Polynesian Island called Easter Island, or Rapa Nui. It was originally settled in fifth century AD, and the inhabitants developed a clan system in which they honoured their ancestors with huge stone carvings called *ahu*. Their quest for bigger and better statues led to deforestation of the island and, eventually, destruction of the population through clan warfare and cannibalism caused by their scant resources. Western society's drive for material progress is parallel to that of Easter Island, and Wright warned, "We are now at the stage when Easter Islanders could still have halted the senseless cutting and carving, could have gathered the last trees' seeds to plant out of reach of the rats" (p 190). Our current path is unsustainable, but I draw hope from Thulo Pokhara and I believe that educators can help shape our future.

Path 2: A Global Village Like Thulo Pokhara

Marshall McLuhan (1962) stunningly predicted that the new era of electronic technology would "recreate the world in the image of a global village" (p 31). With our technology and growth, we now make an impact on the entire planet—so Earth is our closed society or village. I hope to model our global village after Thulo Pokhara and not Easter Island. The villagers of Thulo Pokhara had scant resources, like the people of Easter Island. Why did they choose to work together and share their resources instead of going to war over them? I believe that it was because of their cohesive world view, which holistically integrated physical, spiritual and cognitive ways of knowing. The villagers' decision to cut down the tree to make room for a school was carefully considered because the tree was sacred and provided physical shelter and dead branches to use as a resource. In our global village, we have the challenge of creating a world view, a shared vision and a way of being for our planet, across cultures. As Suzuki, McConnell and Mason (2007) ask, "Can we combine the descriptive knowledge of modern science with the wisdom of the ancients to create a new world view, a story that includes us all?" (p 48). They suggest that we treat Earth as our larger sacred community according to ancient wisdom and hold "human needs and the needs of all our companions on the planet ... in balance with the sacred, self-renewing processes of Earth" (p 330). The electronic revolution can enable development of a shared vision for our global village and growing awareness of our place in its interconnected web.

Mathematics Education for a Global Village

Educators have immense power to shape this global village. "Daily classroom experiences constitute and perpetuate (and so are capable of transforming) pervasive mathematical ideologies" (Namukasa 2004, 216). As a mathematics teacher, I will not change the world in one class (nor would I know where to start!), but I can work towards "cultivating values" (p 209) and "the critical and mindful conscious" (p 224) of my students and myself.

Developing a sense of collective well-being starts in the community of my classroom. Instead of presenting my classroom rules, I can engage the class in a discussion of how we want our classroom to be. We can decide what our shared values are and how to respect them. This is not just a one-time discussion for the first day of school, but an ongoing evaluation of our actions (mine and my students') and how they affect others and our vision for the classroom. My vision is to build a caring and successful community that is respectful and values all diversity (cultural, ways of knowing, gender and so forth). "When community exists, learning is strengthened—everyone is smarter, more ambitious, and productive. Well-formed ideas and intentions amount to little without a community to bring them to life" (Peterson 1992, 2). I want to foster a mentality that each gives his best for the good of all instead of dog eat dog. I want to shape the future using Thulo Pokhara as a model, not Easter Island.

The inhabitants of Easter Island continued to use their diminishing wood supply to erect stone statues, likely because that was the tradition handed down to them and they did not question it. I want my students to be critical about the knowledge they are presented with or construct and ask, "Does this make sense?" In a traditional mathematics class, the teacher acts as a "broadcaster of information" (Schifter and Fosnot 1992, 13), while the students follow (usually mindlessly) the presented rules and procedures. I want my class to be a mathematics community that debates and discusses to further our understanding, rather than a teacher-directed model in which "a rigid notion of truth is reinforced" (Namukasa 2004, 220). Different ideas are valued as I guide my students towards understanding and learning the conventions of the mathematics community.

To restore the integrity of mathematical ideas, "learning mathematics should happen in context and as a social and human activity rather than a non-corporeal discipline" (Namukasa 2004, 212). I got a sense of the fragmentation of my own knowledge in learning about Pascal's triangle. I first encountered the numbers in Pascal's triangle as the coefficients in binomial expansions while teaching Pure Math 30. Then Pascal's triangle mysteriously appeared in pathways problems and combinatorics. I was curious about the connections but felt that I had to rush on to prepare my students for the diploma exam. When studying in my master of education program, I looked at geometric representations of triangular numbers and saw the patterns within Pascal's triangle in a whole new light. I also learned that "Pascal's triangle may have its origin in China 350 years before Pascal" (p 214). My path to knowledge about Pascal's triangle was fragmented and utilitarian and had no connection to the historical development of the ideas. To restore the connections severed by scientific determinism, I need to keep "things in place, nested in the deep communities of relations that make them whole, healthy and sane" (Jardine, LaGrange and Everest 2004, 324). If I were to teach Pure Math 30 again, I would try to find a problem that the Chinese mathematicians were exploring when they developed the idea and share the context. My students could struggle with the problem themselves by using different tools (geometry, algebra) and sharing ideas. This would provide a richer mathematical understanding and a historical context for the mathematical conventions. My students could learn that mathematics is an evolving, human field and that there are different ways of doing mathematics.

Conclusion

Wright (2006) concluded his book with his answers to Gauguin's questions. In the myths of humankind, we come from some sort of paradise and we are "an Ice Age hunter only half-evolved towards intelligence; clever but seldom wise" (p 189). To determine where we go, we must use the wisdom of our cultural capital to know that "... human beings drove themselves out of Eden, and they have done it again and again by fouling their own nests. If we want to live in an earthly paradise, it is up to us to shape it, share it, and look after it" (p 8).

In my mathematics classroom, I hope "to treat things that come to meet us with integrity, to heal the ways that things have become fragmented" (Jardine, LaGrange and Everest 2004, 328), to cultivate values and a "critical and mindful conscious" (Namukasa 2004, 224) to help shape my global village paradise.

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Carrie Watt is currently supply teaching with Edmonton Public Schools. She began with a career in civil engineering, followed by a bachelor of elementary education. Carrie's first teaching position was with junior and senior high students, where she wondered, "Why do so many students fear and loathe mathematics?" She is currently seeking answers by studying for a master's degree in elementary mathematics and science education. Carrie is hoping to try out some of her ideas with her own elementary class in the future. She can be reached at cwatt1@ualberta.ca.

Perceptions of Problem Solving in Elementary Curriculum

Jennifer Holm

Introduction: A Look at the Meanings of Problem Solving

Recent research with intermediate teachers[1] indicates that the phrase *problem solving* often evokes multiple meanings in mathematics teaching and learning (Kajander and Mason 2007). While some teachers support the vision of problem solving espoused by the National Council of Teachers of Mathematics (NCTM) (2000) as "engaging in a task for which the solution method is not known in advance" (p 51), others view it as something to be done after students are taught and only if there is time (Kajander and Mason 2007). The goal of this article is to examine the usefulness and intent of the various meanings of this "problematic" phrase, while shedding light on the best way to engage in effective problem solving with students.

The inclusion of problem solving in mathematical learning is not new. Polya's (1957) famous model is even included in some provincial curricula (for example, Ontario's Ministry of Education 2005), as shown in Figure 1, and is one of the best known outlines of the possible processes involved in problem solving. What is perhaps new in many classrooms is that effective problem solving should be more than having students solve a problem using formulas or methods that *the teacher has previously shown.*

As a Grade 2 teacher in the United States, I was able to experience first-hand the effects of new legislation that required us to drill mathematics facts and algorithms into the minds of our students so that they could survive testing. Each year the same concepts had to be reviewed, because retention was minimal if at all. We met each year as a school staff to discuss ways to better meet the needs of the students with information handed to us from our school board. We discussed at length using problem solving to improve our mathematics test scores. Yet what this actually entailed, according to what we were told, was handing students a sheet of word problems to solve using the algorithm the teachers had already given them to use. It was suggested that we do a similar word problem with the students so that they would follow a similar process. Similar understandings of the implementation of problem solving have also been found in some Canadian classrooms (eg, Kajander and Mason 2007; Kajander and Zuke 2008).

Elementary mathematics curricular goals may refer to problem solving without exploring what the phrase really means. In Alberta, for example, the curriculum endorses the importance of using a problem-solving approach noting that "students need to explore problem-solving strategies in order to develop personal strategies and become mathematically literate" (Alberta Education 2007, 1). Ontario, as well, mentions that "problem solving forms the basis of effective mathematics programs and should be the mainstay of mathematical instruction" (Ontario Ministry of Education 2005, 11). These statements could be interpreted in multiple ways. From these statements in the mathematics curriculum guides, teachers could assume that problem solving means having students read a problem and find a correct solution using a given, previously taught method. Research has shown that this is not the most effective way to use problem solving in mathematics classrooms (Bay-Williams and Meyer 2005; Boaler and Humphreys 2005; Buschman 2004), nor is it the most effective method for teaching mathematics (Askey 1999; NCTM 2000; Van de Walle and Lovin 2006). This article will further examine some alternatives to this common interpretation.

Problem Solving *as* Learning

I take the stance that *true* problem solving involves students *really learning* something new and not just applying a previously taught strategy to a new example or task. This position underscores the importance of problem solving *as* learning. As Bay-Williams and Meyer (2005) note, "teacher-directed instruction may help a teacher feel that more topics have been covered, but it reduces the chances that students are (1) making connections with other mathematical ideas and (2) understanding the concepts related to the skill" (p 340). In fact, students should engage in rich problem-solving tasks ***in their daily** mathematics classroom experience* in order to construct new knowledge and understanding by connecting it to their previous knowledge.

This interpretation of effective problem solving differs from the belief that students must be taught the concepts before they can engage in problem solving (Kajander and Mason 2007; Kajander and Zuke 2007, 2008). One view might be that by assigning the problem-solving questions in the textbook for homework, exercises, and on tests, students have the opportunity to problem solve. Typically, such problems are really *applications* of known formulas or methods to new examples. However, the goal with problem-solving tasks should be to allow students to figure out *how* they will solve the problem. The importance is placed on the *method for determining the solution*, as opposed to the solution itself. As McGatha and Sheffield (2006) point out, in problem-solving classrooms "students are pushed beyond simply *finding a right answer to questioning the answer*" (p 79; emphasis in original). The one single right answer is no longer the singular goal of the mathematics classroom; rather, the process taken to find an answer is where the real learning lies. Students should subsequently be given opportunities to discuss how they solved the problem so that they can learn from each other and see different ways of arriving at a possible solution. This is very different from classrooms in which the teacher tells the students how to go about solving problems so that they can arrive at the single right answer in this so-called correct way. True problem-based learning involves students constructing new ideas based on their experiences with appropriate problems, *not* applying known methods to new contexts.

How Effective Problem Solving Is Accomplished

Effective problem-solving tasks can be implemented as part of a three-part lesson plan (Van de Walle and Lovin 2006). It is important to consider that the actual lesson may take more than a single mathematics class period to finish, depending on the students. In the first part of the lesson, the teacher sets up the current problem to be worked on. The teacher acquaints the students with any previously unknown vocabulary at this time. This portion of the lesson does *not* include the teacher showing the students a similar problem and how to solve it. After the teacher sets the stage for learning, the students begin to explore the given problem.

The second stage of a problem-solving lesson requires teachers to set up an environment and procedures that are conducive to exploratory learning. While exploring the problem, students may work individually, in pairs or in groups. Students need to be arranged in a way that allows them to share their ideas with each other. During this phase of the lesson, the students work with the problem to figure out a solution method that makes sense to them. As students work with the mathematics concepts embedded in the problem, they should record their thoughts to share during the final portion of an effective problem-solving lesson—the discussion.

Discussion is an absolutely essential phase of the problem-solving method because it allows students to come together and share while explaining their thinking. As Boaler and Humphreys (2005) note,

> students are not asked to present their answers; they are asked to show representations of their ideas and to justify why they make sense. None of the audience members will have the exact same answer, and all the students have a role. (p 50)

Not only are students more engaged while discussing ideas with their peers, they also learn more from each other and discover new ways of thinking about a problem. Students need to be able to put their solutions into words and discuss how they solved the problem so that they can explain their methods to others. This forces students to get at *how* their solution was found, not just what they decided was the correct answer. It is important that students learn "to *question the answers* by posing additional questions when solving the original problem [because this] is one way that teachers and students can develop mathematical power" (McGatha and Sheffield 2006, 79; emphasis in original). It is this power that helps further students' understanding of and learning in mathematics. Boaler and Humphreys (2005) suggest using the method of "convincing a skeptic" when trying to explain the solution the students came up with (originally from Mason, Burton and Stacey 1982). Their belief is that "this strategy ... helps place responsibility on the person who is explaining to make his (sic) explanations understandable and gives permission for anyone who doesn't understand *yet* to play the role of being unconvinced rather than being just slow to catch on" (p 67; emphasis in original). Students are given the opportunity to question each other and refine their thought processes until everyone sees why the solution method works. Seeing alternative solutions is important because "if their knowledge is limited to the computational procedure without any idea why the procedure works, this is also not enough to build on. Students need both" (Askey 1999, 3). Through exploring a problem and discussing the solution, students learn how and why their method and procedures work and gain deeper mathematical understanding. At this point, teachers can help students see the generalizations or the procedures that are being developed through examining the students' solutions. Teachers play an important role in fostering this development of ideas. Since students are sharing their knowledge and understandings, or even misunderstandings, during this portion of the lesson, the teacher must create an environment where all contributions are valued and allowed to be expressed.

In order to use the problem-solving method effectively, students must be given opportunities to share their solution methods so that the teacher can see where any misunderstandings or confusions lie. These essential discussions also allow students to learn from each other. This very important aspect can be the deal breaker for the success of problem-based lessons if the teachers do not allow time for sufficient sharing of ideas. Sometimes issues or difficulties that arise during the discussion can prompt the teacher to suggest a new problem for the next class.

In a problem-solving lesson as just described, problem solving is the vehicle for knowledge and learning instead of simply the way that students showcase what they have learned. One issue with doing problem solving *after* the teacher has *taught* a concept is that students have trouble switching from a teacher-directed lesson one day to a lesson in which they control the learning path the next (Van de Walle and Folk 2007). Also, if students are to truly engage in problem solving, they need to know that the teacher is not about to step in and tell them the strategy or the answer eventually. If they feel that this will happen, my experience is that some students will simply wait for the instruction or answer to come from the teacher and therefore will not deeply engage in the task. In other words, they feel that their work will be devalued eventually when the teacher provides the right answer or method, and they have simply learned to wait for this to happen. By engaging in problem-solving lessons as the main curricular vehicle, students learn that their thoughts and ideas are important and are correct ways to solve a problem.

Teachers should choose problems that allow students to explore and construct knowledge for the big ideas or the overall expectations of the grade level. This allows the teacher to address multiple curricular expectations in one problem while, at the same time, addressing the needs of different learners. The benefit of well-chosen problems is that they "can be solved at different levels of sophistication, enabling all children to access the powerful mathematical ideas embedded in the problem" (English, Fox and Watters 2005, 156). For example, a problem like the handshake problem could be used with a class: "There are 20 students in a class. On the first day, the teacher asks each student to shake hands with each other student. How many handshakes were there?" (Small 2008, 567; similar problem in Kajander 2007). In order to make the problem accessible to all students, the teacher could make several different size classes, starting with 5 students and ranging up to 20. Students who are more advanced could begin looking for patterns in the different numbers in order to make the problem more challenging, while students who are struggling can simply tackle what would happen if five students shook hands. By having students solve the problems from their ability level, all students are engaged and learning from each mathematics lesson.

The problem-solving approach also allows all students to be included in the discussions. Students choose methods to solve the problem that make sense to them, which is more meaningful than just repeating what the teacher has said. Using problem solving in the classroom allows all students to reach mathematical understanding at a level that they are comfortable with. Since the goal is to have students use their prior knowledge, all students will be able to work with the problems using what they already know to build their own new ideas. As the Alberta curriculum asserts, "students learn by attaching meaning to what they do, and they need to construct their own meaning of mathematics" (Alberta Education 2007, 1). Teachers can also use this baseline knowledge to help students to come up with new ideas and more effective solution methods instead of teaching formulas that students apply without really understanding. For example, the solution to the handshake problem could be arrived at in many different ways, including drawing a picture, acting out the problem with children, looking for patterns or even solving algebraically. Students would be able to solve the problem with their own solution methods, but during the discussion would be exposed to all the different methods and thereby learn from the other students. The problem itself can be used to teach or review addition, multiplication, division, geometric patterning, numeric patterning, pattern rules and iterative patterns (Kajander 2007), depending on how the teacher guides the students through the discussions and what areas are highlighted as students present their solution methods.

One caution does need to be made in choosing effective problems to solve in order for students to gain the most benefits. Teachers should avoid forcing too much content into a single lesson; therefore, each lesson "focuses on investigating one rich problem, probing deeply into a different mathematical content strand each day" (McGatha and Sheffield 2006, 79). By narrowing the focus to one main concept each day, teachers can allow students to look further into the problem in order to reach a deeper understanding. For example, simply introducing a simple problem like the handshake problem with different-sized classes would allow the students to explore the solution methods; the teacher could then guide discussions to accomplish the necessary curriculum goals. By focusing on one problem, students are not overwhelmed by a worksheet full of problems and could be challenged to come up with multiple solution methods. Another important consideration is that, as one teacher said, "too much choice could be overwhelming for the children and difficult for me to manage" (Whitin 2004, 181). Putting too much into one lesson is not only hard for a teacher to organize and observe, but it can confuse students and prevent them from delving deeply into the topic being explored.

Another benefit of using problem solving extends beyond the mathematics classroom. Since the goal is not for teachers to show students a formula and the exact method to solve the problem, students use their *own* problem-solving skills to solve the problem. This can affect students' lives—not only do students learn mathematical concepts with deep understanding, they also gain skills that enable them to solve problems in their daily lives. The benefits of using problem solving and allowing students to learn how to solve a problem in their own daily lives are great. I turn now to showing how this can fit within the mathematics curriculum.

Examples of Using Problem Solving with Ontario and Alberta Curriculum

It is my experience that curriculum guides mention problem solving while not explicitly laying out how to use problem solving in a classroom. For example, in the curriculum I am most familiar with, the Ontario Ministry of Education sets out several mathematical processes that should be included in the elementary curriculum: "problem solving; reasoning and proving; reflecting; selecting tools and computational strategies; connecting; representing; communicating" (Ontario Ministry of Education 2005, 11). These processes are listed as separate entities; yet an effective problem-solving approach to teaching would encompass all of these processes and would, therefore, be the only method necessary to accomplish these curricular goals. After being given a problem, students would have to *select tools* and the *computational strategies* needed to solve the problem. Since the students would be using prior knowledge to pursue a solution, they are *connecting* the new concept to previous knowledge and skills. Students would then be required to *reason* through their solution and *prove* that it works to the class and teacher. Through the discussion of the solution, students would have to *reflect* on whether or not their method makes sense in order to solve the problem. By sharing their solution with others, students would be required to *communicate* their thought processes and *represent* the solution so that others can see how they solved the problem. Using problems with a similar focus on different days would allow students to practise their skills and create more in-depth conceptual knowledge. By using a problem-solving approach to teaching, teachers are able to simplify their planning while meeting all the goals of the curriculum.

In Alberta, the curricular goals identified are that students will "use mathematics confidently to solve problems; communicate and reason mathematically; appreciate and value mathematics; make connections between mathematics; make connections between mathematics and its applications; commit themselves to lifelong learning; [and] become mathematically literate adults, using mathematics to contribute to society" (Alberta Education 2007, 2–3). As with the Ontario curriculum, the curriculum guide provides the goal of using problem solving in the classroom as part of the routine. Effective problem solving would also help to accomplish the other goals by giving students ample opportunities to use mathematics in meaningful ways that will benefit students throughout their lives. Where Alberta's curriculum differs from Ontario is that it explicitly states that students "must realize that it is acceptable to solve problems in a variety of ways and that a variety of solutions may be acceptable" (p 1). This statement lends itself more to the problem-solving methods described within this article, but teachers should keep in mind that this should not mean giving students different solution methods but allowing them to discover multiple solution methods. The two curriculums mention the importance of problem solving, but it is my experience that teachers are often left to their own devices to locate the problems that would address these goals. Some resources that I have used and found successful with students include *50 Problem-Solving Lessons: Grades 1–6*, by Marilyn Burns, and *Big Ideas for Small Mathematicians*, by Ann

Kajander. The problems in these books mention different curriculum strands addressed by the problems, and a single problem could be used in a lesson to give students a chance to delve deeply into the topics. In order to encourage more problem solving, the Ontario Ministry of Education has also created its own lessons that relate to the curriculum for Grades 7 to 10, called *Targeted Implementation and Planning Supports* (TIPS). Textbooks might also be a helpful tool—teachers could choose or create a single problem from the lesson that students could explore on their own in order to determine their own solution methods.

In Ontario, the Ministry of Education (2005) does provide a valuable framework that can be used with students while exploring a problem. During the exploratory phase of problem solving, the Ministry of Education suggests using Polya's problem-solving model (see Figure 1) to guide the students in thinking about how to solve a problem. The belief is that teachers should guide students in Grades 1 and 2 through the model without directly teaching the steps, whereas students in Grade 3 and above should be taught the terminology of each step of the model directly. For Grades 1 and 2, a simpler way of remembering the steps can be beneficial.

Thomas (2006) has suggested the use of the THINK strategy to get students organized in their thinking, which could be a useful mnemonic for Grades 1 and 2. First, students *talk* about the problem. Second, students look at ***how*** the problem could be solved. Third, students *identify* a strategy for solving the problem. Fourth, students *notice* how the strategy helped solve the problem. Finally, they ***keep*** thinking about the problem. As students continue working on the problem, they may need to cycle through this framework several times until they arrive at a solution that makes sense to them. According to the research study, Thomas notes that "students who used THINK demonstrated greater growth in problem solving than students who did not use the framework" (p 86). The use of a model is beneficial because

> a teacher who is aware of the model and who uses it to guide his or her questioning and prompting during the problem solving process will help students internalize a valuable approach that can be generalized to other problem solving situations, not only in mathematics but in other subjects as well. (Ontario Ministry of Education 2005, 13)

Students could be coached using this model while they are exploring the problem. The Polya model should be directly taught to higher grade levels (Ontario Ministry of Education 2005). While older students are working with the problem, the first step is for them to understand the problem. According to Outhred and Sardelich (2005), "understanding the problem requires children: to be able to read the problem; to comprehend the quantities and relationships in the problem; to translate this information into mathematical form; and to check whether their answer is reasonable" (p 146). Students begin by rereading the problem and deciding what the problem is asking them to figure out. Next, students make a plan for deciding how to solve the problem through examining different strategies to solve it. As Askey (1999) discovered when working with teachers, "the teachers argued that not only should students know various ways of calculating a problem [solution] but they should also be able to evaluate these ways to determine which would be the most reasonable to use" (p 6). Third, students enact the plan that they decide

Understand the Problem (the exploratory stage)
➤ reread and restate the problem ➤ identify the information given and the information that needs to be determined *Communication:* talk about the problem to understand it better
Make a Plan
➤ relate the problem to similar problems solved in the past ➤ consider possible strategies ➤ select a strategy or a combination of strategies *Communication:* discuss ideas with others to clarify which strategy or strategies would work best
Carry Out the Plan
➤ execute the chosen strategy ➤ do the necessary calculations ➤ monitor success ➤ revise or apply different strategies as necessary *Communication:* ➤ draw pictures; use manipulatives to represent interim results ➤ use words and symbols to represent the steps in carrying out the plan or doing the calculations ➤ share results of computer or calculator operations
Look Back at the Solution
➤ check the reasonableness of the answer ➤ review the method used: Did it make sense? Is there a better way to approach the problem? ➤ consider extensions or variations *Communication:* describe how the solution was reached, using the most suitable format, and explain the solution

Figure 1. Polya's problem-solving model. (Ontario Ministry of Education 2005, 13).

to use to solve the problem. Finally, students assess whether or not the solution is reasonable through re-examining the problem. If the solution is determined to be unreasonable, students would then go back through the model. This model is important because when students are used to traditional instruction they typically do not have the skills and strategies developed to effectively problem solve (Van de Walle and Folk 2007; Kajander and Zuke 2007). Once students are given a problem, they need to be given a way of organizing their thinking in order to solve the problem.

Summary

Problem solving may have varying definitions for different teachers, but effective problem solving should allow students to explore a problem for themselves to find a solution. I have argued that problem solving does *not* involve giving students a method or formula for how to get the answer; rather, it involves giving them a framework to think through the problem and work to develop their own method. Students need a structure to develop problem-solving skills, and this must be supported by peer and teacher-facilitated discussion at certain points in the learning. Neither of these can take place when problem solving is attempted in isolation as homework or on tests. True problem solving cannot happen for most students (or even most mathematicians!) in a time-limited situation such as a test. Students need time to reflect, discuss, and try possibilities. Tests are simply *not* good places to attempt problem solving. While tests might play a role in efficiently assessing procedural skills, learning and assessment tasks are much better vehicles for learning through problem solving. Teachers in an effective problem-solving environment are no longer disseminators of knowledge but facilitators and coaches who help students create their own knowledge. In my experience, one of the most rewarding experiences can be watching students grapple with a problem and come to a solution after they have worked on the concepts within the problem. The excitement and feelings of accomplishment that accompany the final product can be empowering to their mathematical abilities, as well as foster the idea that they, too, can do mathematics! While using problem solving and discussion may be uncomfortable at first, the long-term benefits for both student learning and engagement are phenomenal (NCTM 2000). The goal of mathematics classrooms is to have students learn and understand mathematics, and engaging in effective problem-solving tasks is the best way to accomplish these goals.

Note

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Jennifer Holm is currently a doctoral student at Lakehead University, where she teaches a Math for Teaching course for junior/intermediate preservice teachers. She has a master of education from Lakehead University, as well as a bachelor of arts in elementary education from the University of West Florida. Jennifer has previously worked for elementary school boards in both Tennessee and Florida in the United States. Her research is focused on furthering mathematics education while supporting educators in utilizing reform-based methodologies for teaching mathematics.

What I Learned from My Student's Thinking

Elaine Simmt

Imagine that this rectangular object is made of toothpicks. How many toothpicks are there in this rectangle? How many toothpicks are needed for any m × n *rectangle? Test your generalization by considering a 10 × 15 rectangle.*

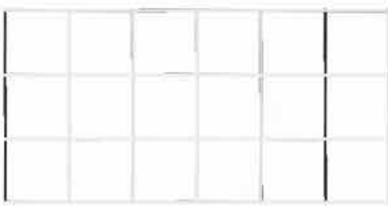
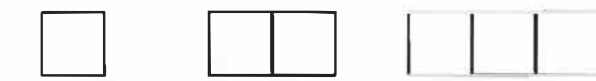


Figure 1. Toothpick array

Although classrooms are normally thought of as places for students to learn mathematics, there are ample opportunities for teachers to deepen their understanding of mathematics in those same rooms. In a mathematics class there is an abundance of ideas and knowledge from which the teacher can grow as both a mathematician and a teacher, but we have to open ourselves up to the moments that present themselves for that to happen. In this paper I offer an illustration of how my knowledge grew from a student's response to a test question, a response that was quite puzzling to me and forced me to grab a pencil and do some figuring. The purpose of this paper is not to propose what Grade 7 students should be expected to explore, but how a particular student's response stimulated my own mathematical understanding and hence my professional development.[1]

Before you read on, have you tried the problem yet? The potential for you to experience surprise, as I did, is greater if you have thought through the problem.

The problem noted above was a test item on a patterns and relations unit exam given to a Grade 7 class. Prior to the exam, the students had been working on generalizing from patterns and communicating those generalizations, both in common language using a sentence or two and through the use of mathematical expressions. For example, students had worked on toothpick trains. The students were provided with a set of images for three toothpick trains from which to work and asked "How many toothpicks does it take to build a train with 1 car, 2 cars, 3 cars, 4 cars, 5 cars, 10 cars, 100 cars?"



1-car train   2-car train   3-car train

Figure 2. Toothpick trains

As they worked on the problem they were asked the following questions:

- How are you counting the toothpicks?
- Do you count every toothpick for every train or do you have a shortcut?
- How would you describe to your friend, in a sentence or two, how you know how many toothpicks are needed for the 4th train or the 100th train?
- Write a mathematical expression that would enable someone to calculate the number of toothpicks needed for the 10th or 100th train.
- Write a mathematical expression that would enable someone to pick a train and then know how many toothpicks it would take to build it.
- If I told you the train had 49 toothpicks, could you tell me how many cars the train has?

Students expressed their thinking differently from one another but yet were able to communicate and evaluate the methods and expressions used to predict the number of toothpicks (t) it takes to build a train of size $l \times n$. Consider the following examples.

Case 1. Recursive solution $t_{(1)} = 4$, for $n \neq 1$ $t_{(n)} = t_{(n-1)} + 3$

Figure 3. Toothpick train—Add three more

The student who sees the four toothpicks in the first train and then sees three more added to each consecutive train might say, "It's four plus three more every time." Indeed, this is the most common response first given by the Grade 7 students. And it is one they struggle with expressing as a mathematical statement (the notation used above is mine). At this point, the teacher can challenge the students to figure out how many toothpicks are needed for a train of length 100, knowing full well that most students have neither the time nor the patience to figure out all of the trains between the 3rd and the 99th in order to figure out the 100th. (Although we all know students who will do just this, or at least try to!) This is always a good time to point out one of the differences between humans and computers. Computers are very good at number crunching and, indeed, figuring out how many toothpicks it would take for a train of length 100 by figuring out how many it takes for 1,2,3, ... 99. It takes no longer for a computer to do all of the calculations between 1 and 100 than to figure this out for 1,000 or even 100,000. Human beings are not such good number crunchers, but we are good at seeing patterns, which in this case can help us determine a way to figure out how many toothpicks for the 100th train by calculating only the solution for the 100th case.

Case 2. Non-recursive forms of counting and expressing

Case 2.1. Add three more

With this problem, there were students who noted that you can find the 100th train without knowing the 99th train; that is, you can find a nonrecursive, or explicit, solution. This requires a shift in the student's attention from *add three more* to *how is the adding three more related to the number of train cars?* There are two ways to think about this. The first is to see that the first toothpick is present for all trains (invariant or constant) and then that for each additional car (variable), you need 3 (1:3 relationship) toothpicks, $1 + 3n$.

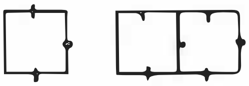


Figure 4. Toothpick train—One plus three more each time

The second way (see figure 4) to see this is to begin with the four toothpicks needed and note that the first car is already counted, so it must be removed from counting the added threes, or $4 + (n - 1)3$. It is important to note that many Grade 7-level students do not have an understanding of the conventions of algebraic notation. Hence, the expressions they write reflect how they think about the situation—in this case, 4 (first car) plus the number of cars minus the first car times 3 toothpicks to make each of those cars. This is something the teacher will work on as the students grow confident in their use of mathematical notion.

Case 2.2. Top + bottom + standing up

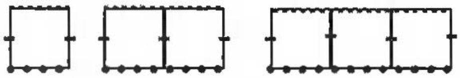


Figure 5. Toothpick train—Top + bottom + standing up

A student who counts the top, then the bottom (horizontally laid) toothpicks and finally counts the standing (vertically laid) toothpicks might say, "top plus bottom plus standing up." In other words, if you know that there are n trains, then there are n toothpicks across the top, n toothpicks across the bottom and $n + 1$ toothpicks standing up, or $n + n + (n + 1)$. Note that students are likely to initially present their mathematical expression in very literal terms $(t + b + s)$, which does not acknowledge the relationships between the number of toothpicks across the top and bottom or the relationship between that value and the number of cars in the train. Again, as students grow in their ability and confidence to count systematically, express the patterns they notice and then write mathematical expressions, the teacher is able to discuss with them the advantages of using the same variable to represent the same value.

Case 2.3. Count the top, count the bottom, count the inside plus two ends.

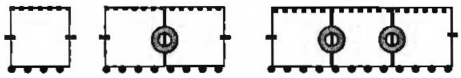


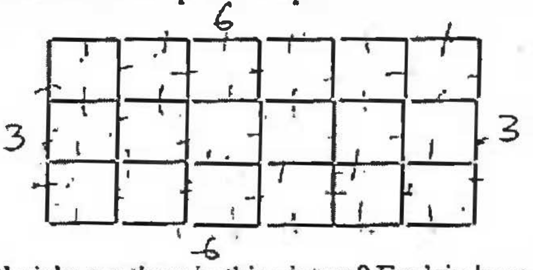
Figure 6. Toothpick train—Count the top, count the bottom, count the inside plus two ends

In this case, the students might write their mathematical expression as top + bottom + inside + ends, $t + b + i + e$. But if they notice that the ends are constant (2) and that top and bottom are the same and that they are also the same value as the number of cars (n), they might write $n + n + 2 + i$. The next step is to figure out the relationship between the number of cars and the number of inside toothpicks, $n - 1$. The expression becomes $n + n + 2 + (n - 1)$. Repeating the common variables and putting brackets around the $n - 1$ might seem cumbersome to someone who understands the conventions in algebra, but the way in which the expression is written usually reflects what the student was attending to when working through the problem. Hence, there is great value for both the teacher and the students to work on communicating generalizations. It is through such work that more robust algebraic thinking can develop.

Asking students to attend to *how* they are counting is meant to direct their attention to a regularity or a pattern. It promotes the students' capacity to communicate, visualize and reason mathematically. It was with these kinds of experiences behind the students that I created the following assessment item. One student responded as follows.

Standard of Excellence: Pattern
(10 marks)

8. Imagine this picture is made up of toothpicks.



3x3 = 9
6x6 = 36

36
x 9
45

- How many toothpicks are there in this picture? Explain how you figured it out.

There are 45 toothpics. I know this because I counted the amount of toothpics on each side (3 of each side) and multiplyed the two numbers (3x3). Then I counted the amount of toothpics on the top and bottom, (6 on the top and 6 on the bottom), then multiplyed the numbers (6x6). The i got the answers 9, 36 so i multiplyed those an got 45.

- Write a rule (that does not require counting each and every toothpick) for determining how many toothpicks there would be in a rectangular shape of any size (length and width), made with toothpicks.

First count the amount of toothpics on the top and bottom, then multiply the two numbers. Then count the amount of toothpics on both sides, Multiply the two numbers. then when you have the two quotents multiply them togethers and the answer is the amount toothpics used.

- Express your rule as a mathematical expression.

txbxsxs = N

t= top
b= bottom
s= side
N= Number of toothpics

- Test your rule by determining the number of toothpicks in a 10 x 15 rectangle.

10x10= 100

15
x15
75
+150
225

225
x 100
000
0000
+22500
22500

The rectangle had 22500 toothpics

2/27/02

Figure 7. Assessment item

Before reading on, you might want to assess this student's response. How would you have marked it? What parts of the student response make sense to you? Which parts puzzle you? What do you make of the question itself? How might you improve the question?

I will begin by commenting on my assessment of this student's response.

- He counted the number of toothpicks correctly for the 3 × 6 rectangle. Unfortunately, the markings on the rectangle do not reveal to me how he counted.
- The calculations in that first part of the item are only partially correct; that is, the student claims to be multiplying ($t \times b$) by ($s \times s$), but clearly he added. $36 \times 9 \neq 45$, but $36 + 9 = 45$. There are 45 toothpicks.
- I assume that the student counted first; this would be consistent with how the students worked on these problems. But I have no way of knowing if the student added 36 and 9 by mistake or if the student knew the answer was 45 and made the two numbers 36 and 9 equal that amount by adding. I suspect it was an unconscious combination of the two—at once the student knew the answer had to be 45 and he saw that 36 and 9 equalled 45.
- The student is consistent in his response. He communicates his "rule" in a couple of sentences and then he writes an expression for it. I can see no obvious reason why the student suggests that the two values should be multiplied together. I suspect that the student is grasping at making meaning of the situation and doesn't have an image for how to calculate without counting.
- In both cases (3 × 6 and 10 × 15), the student indicates to multiply (not add). In the final question, he does calculate the number of toothpicks by multiplying. He did not draw the 10 by 15 rectangle, so I assume that he did not check his calculated response against a counted answer.
- The student uses different variables for values that are the same; top and bottom are expressed as t and b. The generality of *variable* is not demonstrated.
- The student defines his variables; he has an awareness of the need to do so in this communication with the other.
- The student does not explain how his rule is related to how he counted the toothpicks. As far as I can tell, his rule cannot describe his counting. There is no evidence that his reasoning for the expression he created is connected to the geometry of the situation he was given.

Clearly, there is a lot for the teacher to learn about this student's understanding from the test item, but *what can the teacher learn about her own teaching and her own mathematics from this student's response*? I will begin by telling how I came to this assessment item. First of all, the item was intended to take advantage of the experiences students had had in class doing toothpick trains. At the same time, I anticipated that this question would be a problem for students because two variables were changing at the same time (length and width); hence, they would have no immediate strategy for solving the problem other than to begin by counting and look for a generalization of their strategy. Finally, I wanted to have them write a written expression of the generalization, predict a value for a different rectangle and then check it against counting. Therefore, as I prepared the test I needed another example that could be counted in the time the students had. When I selected the second rectangle, I tried to be careful to select something that could be counted but that wasn't obviously related to the first example. As you may have figured out already, I was not careful enough.

When I read this student's response I was slowed down not only by his arithmetic mistakes and notation but also by the fact that the rule he wrote (if we added instead of multiplied the [$t \times b$] and [$s \times s$]) produced the correct value for the number of toothpicks for both rectangles given. This was highly bothersome to me, first of all because the algebraic expression did not look like the one I had in mind. If the student's formula were to work, I should be able to equate it to the formula I had for the problem, $t = l(w+1) + w(l+1)$. I immediately set the two expressions against each other (Figure 8, page 1 and top of page 2). They were not, in general, equivalent. I then specialized with a new case (1 × 2) (Figure 8, page 2). It did not work with the student's formula. Not trusting original calculations, I returned to the 3 × 6 case offered on the test and confirmed that I had not made an error. So, what was happening here that $l^2 + w^2 = l(w + 1) + w(l + 1)$?

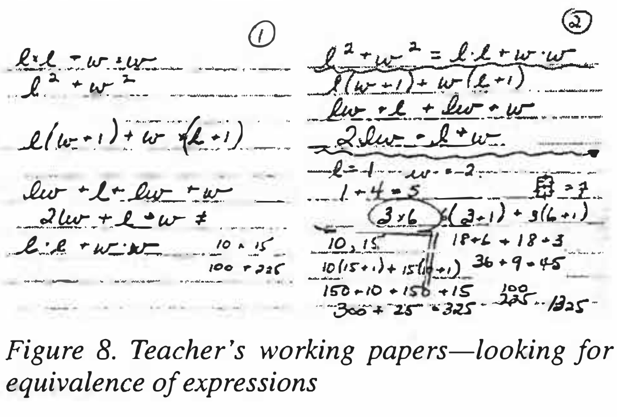


Figure 8. Teacher's working papers—looking for equivalence of expressions

I returned to the figure. I thought I should be able to uncover the student's expression from the geometry if I carried out some systematic counting, but I couldn't. I could see no reason why the sum of the squares of the length and the width had anything to do with the problem. I was forced to go back and consider some more examples. At this point, I thought to consider the set of (l,w) that worked and the set of those that do not work with the student's formula $l^2 + w^2$. The ordered pair (2,3) did not satisfy both equations (figure 9, page 4). Figure 9, page 3 is a record of the ordered pairs that worked (top) and that did not work (bottom).

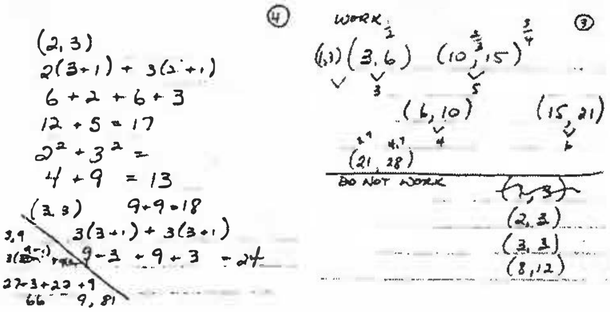


Figure 9. Teacher's working papers, pages 4 and 3.

Once I found more cases of pairs that worked (Figure 10, page 8), it seemed that there *must* be a set of pairs that worked; my challenge was to determine the properties of that set. I decided I needed to explore the conditions under which the two expressions were equivalent. After some time and twelve pages of sticky notes, I concluded that whenever the order pair is composed of two consecutive triangular numbers the expressions are equivalent (Figure 10, page 12). What a surprise! When I made the choice for the examples in the test I had not noticed this feature of the numbers I selected.

My learning opportunity arose in the context of teaching patterns and relations to Grade 7 students. I continue to be pleased with the thought that trying to understand one student's thinking led me to a better understanding of patterns and relations and enhanced my understanding of triangular numbers and how special they are. This student provided me with the experience of exploring some new mathematics. But maybe most significant, this student gave me the opportunity to experience the joy of discovery, the surprise of mathematics and the satisfaction that comes from making sense. Although I will try to be more careful when choosing examples for students, I suspect this will not be the last time that mathematics and a learner teach me something new.

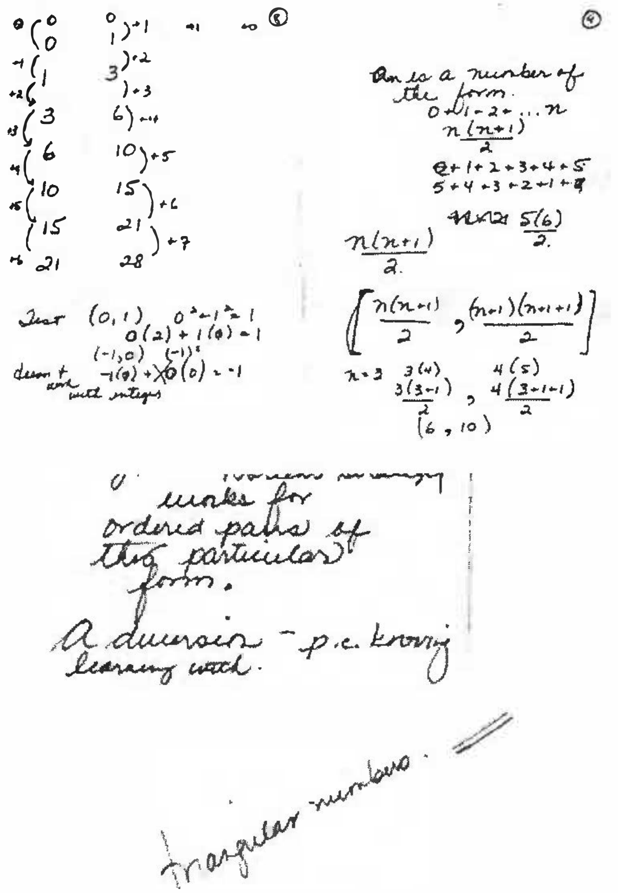


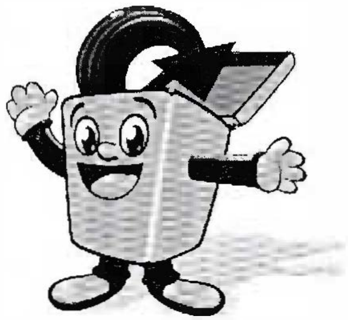
Figure 10. Teacher's working papers, pages 8, 9 and 12.

Note

1 For further discussion of this episode from my research study, please see Simmt, E, B Davis, L Gordon and J Towers. 2003. "Teachers' Mathematics: Curious Obligations." In *Proceedings of the 2003 Joint Meeting of PME and PMENA*, ed N Pateman, B Dougherty and J Zilliox, 175–82. Honolulu, Hawaii: University of Hawaii.

Elaine Simmt is an associate professor of mathematics education and the chair of the department of secondary education at the University of Alberta. Elaine's research is focused in mathematics education. In particular, she explores teaching and learning as understood through the frame of complexity theory and is interested in mathematics-for-teaching. School-based research has led to Elaine's interest in investigating research methods and designs suitable for studying the complex and contextual phenomena of teaching and learning.

BOB AND HIS INCREDIBLY AMAZING GRAPH ADVENTURES!

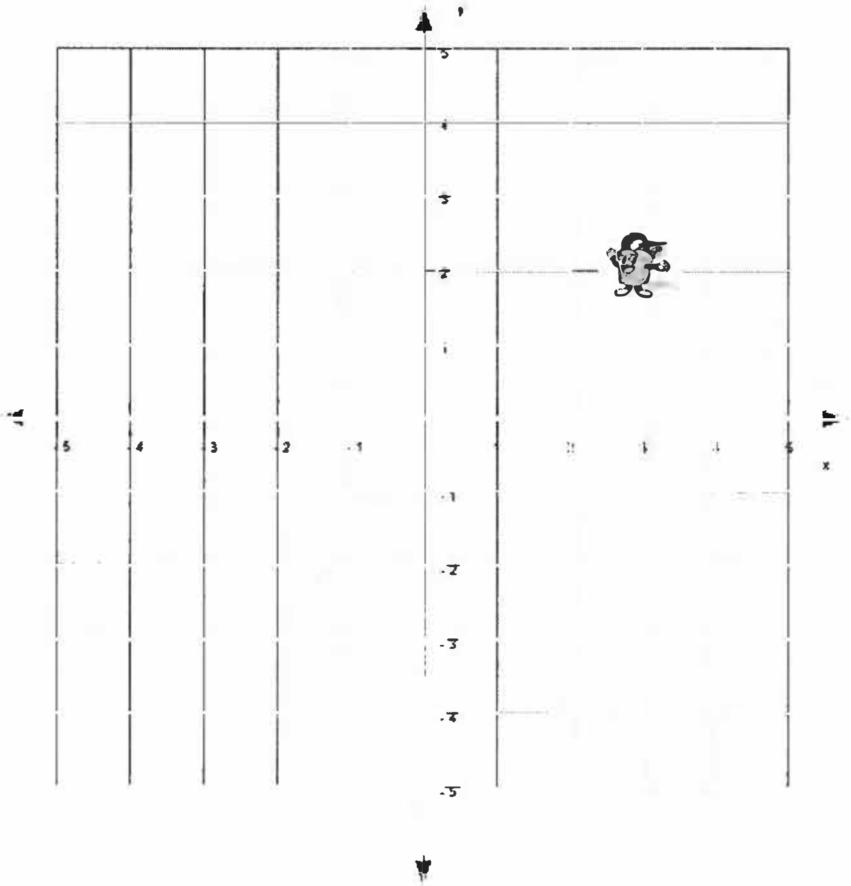


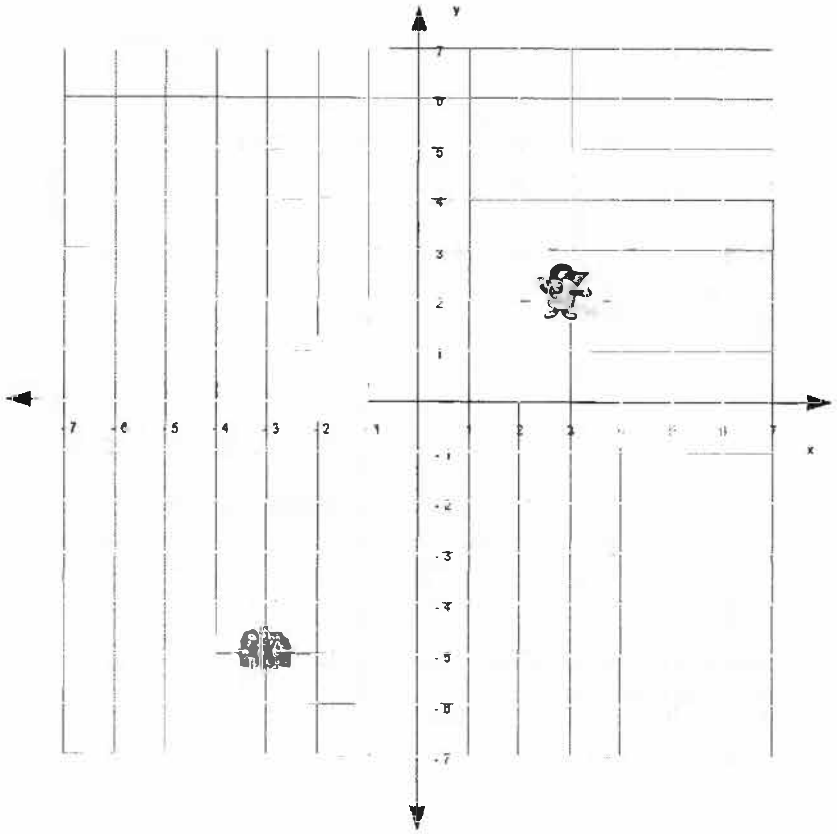
Benet Copeland

Ages 6–10

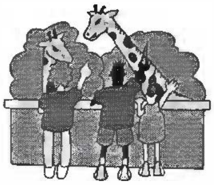
Bob was an average dot.

He lived on x and *y*.



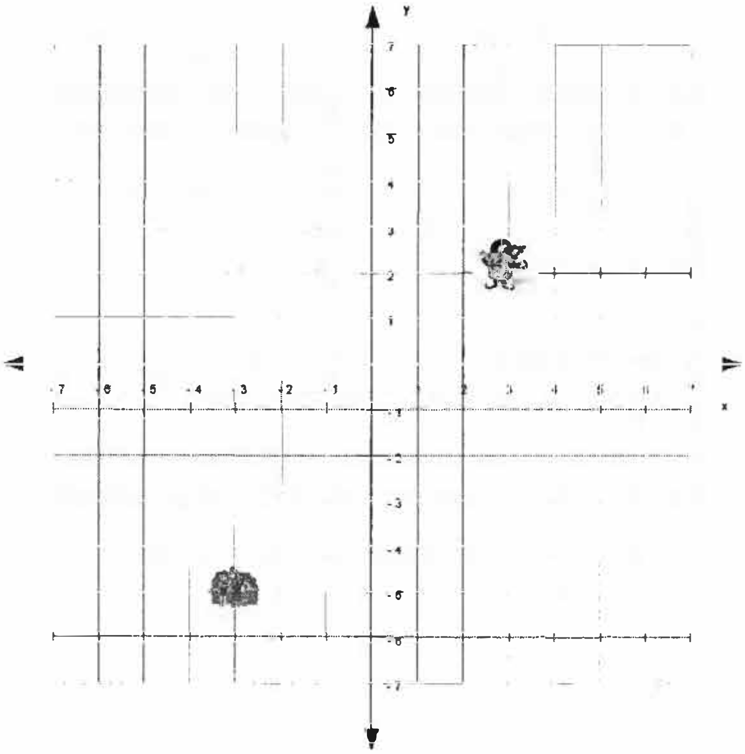


One day, Bob decided to go to the Zoo.

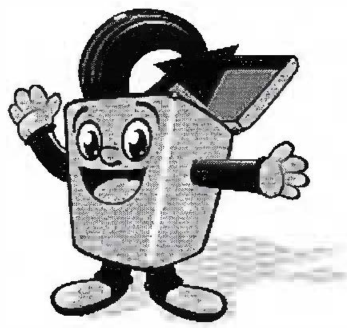


= Zoo

Map



He looked on a map and said, "Ohh noes!"

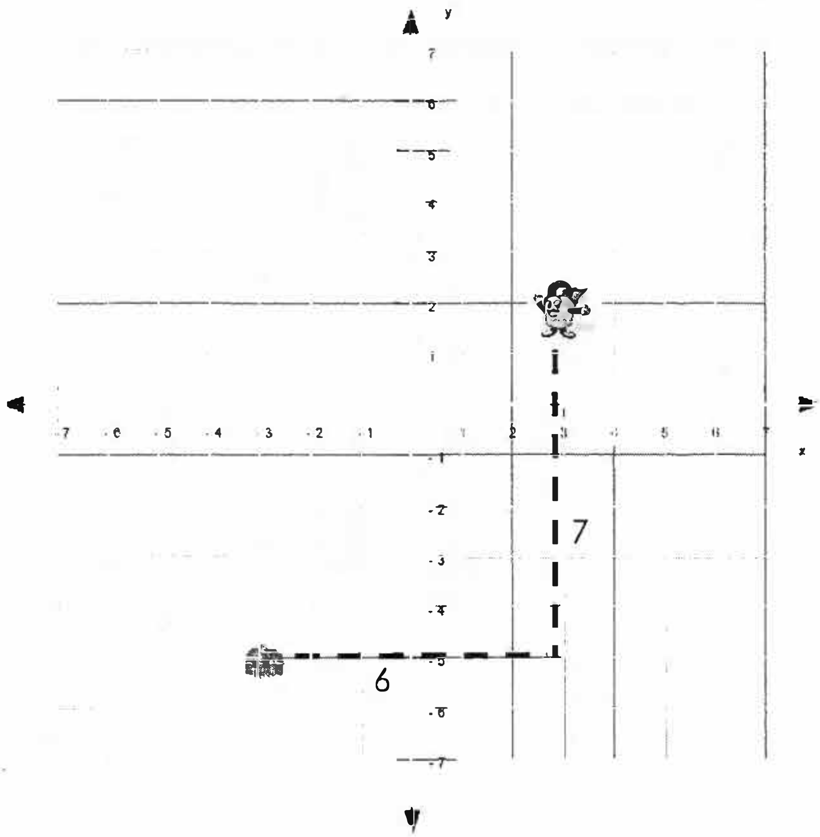


"Why is it so far?!"

But Bob wanted to know just how far.

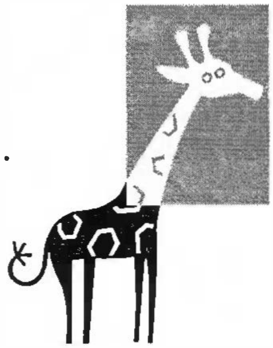
So he counted!

So Bob walked
7 blocks y-wise and
6 blocks x-wise.



Look at all the animals!

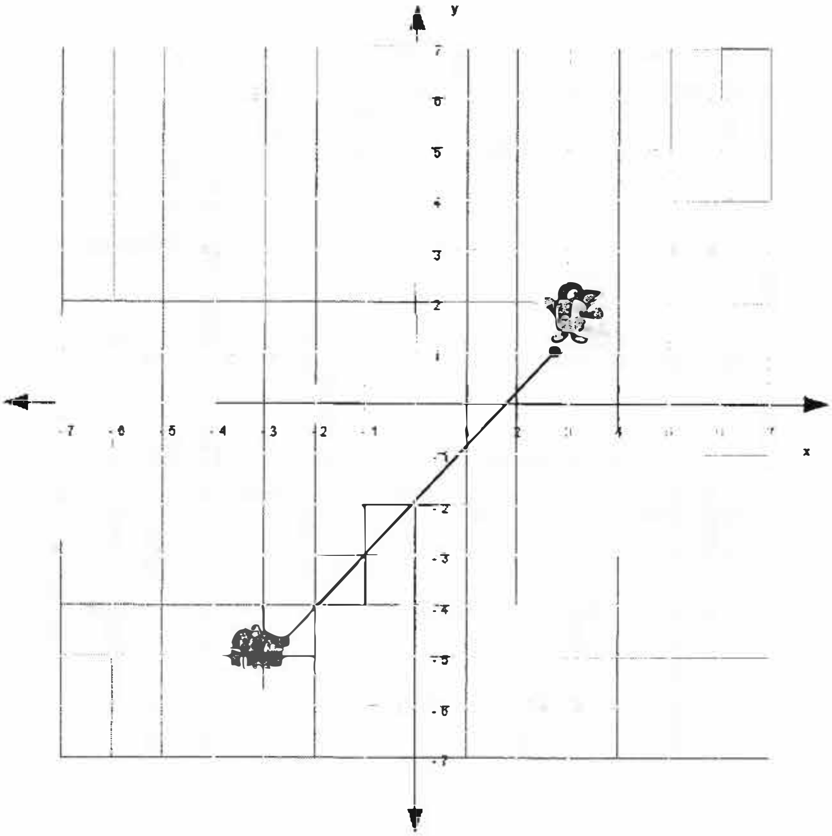
Well, there was only one animal.



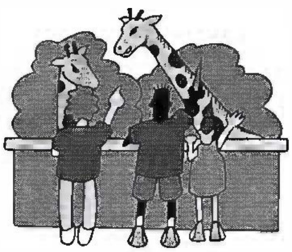
The gi-graph!

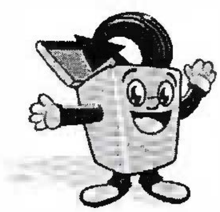
After Bob was done,
he wanted to go home.

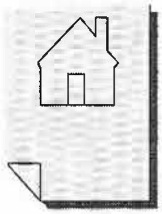
Bob was too tired to walk, so he caught the line home.



His teacher always told him a line was the shortest path between two places!







Bob was home in no time!

THE END

Benet Copeland is a Grade 12 student at Paul Kane High School, in St Albert, Alberta. He believes that encouraging children to take an early interest in mathematics is important. He wrote this story in the hope that it will foster an enduring interest in mathematics in the adults of tomorrow. Benet would like to thank Mrs C Vandermeer for the preproduction process that helped to bring his project to life.

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A Mathematical Adventure with the Number Devil

Karen J Cleveland

Introduction

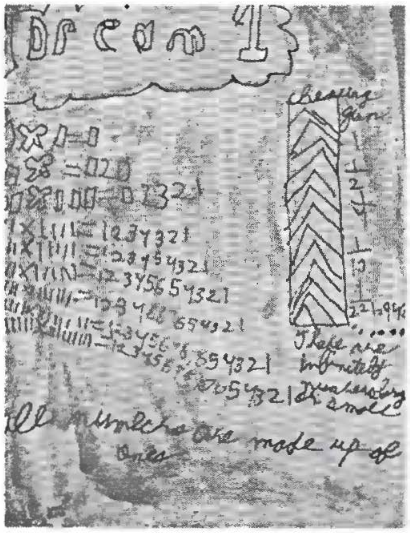
A university professor introduced me to Hans Magnus Enzensberger's *The Number Devil* and implored me to read the book at some point in my teaching career. A few years later, I picked it up as a summer read simply with the intention of honouring my professor's request. Before long—in fact, I'm sure it was within the first chapter—I knew I needed to share this book with children.

If you have read the book, you will know that some of the mathematics is insanely complicated. Knowing that I would be teaching Grade 4 students that fall, I was not sure how well the book would be received; nevertheless, whether it was an effective tool for teaching mathematics by promoting mathematical thinking or simply a way to spark students' imagination and interest, I decided to give it a try.

Mathematics aside, the book itself is a very easy read and is brilliantly illustrated. Its characters are Robert, a 12-year-old mathphobe, and the Number Devil, a lively and rather convincing little imp whom the students truly enjoyed following. A brief synopsis: Robert dislikes his math teacher, Mr Bockel, who gives his class boring math problems and does not let them use their calculators. Each night, as Robert drifts off to sleep, the Number Devil, who brings the subject magically to life for him, visits him. The Number Devil illustrates with wit and charm (and a few fun words like *nincompoop*, *prima donnas* and *rutabagas*) a world in which numbers can amaze and fascinate, and where math is nothing like the dreary, difficult process that so many of us dread.

Before I finished the first chapter, I knew that this book was to be shared with students. As I closed the book after reading the first chapter, I asked my class, "So ... there was a whole lot of math that just went on in those first few pages. Would you like me to keep reading, or do you want to explore some of this great math?" It was a unanimous decision to slow down and explore each of the concepts the chapters had to offer. And that was what we did.

Below is a description of how we explored with an open mind, and sometimes a calculator, all (well, most) of what the Number Devil shared with Robert.



Dream 1—Ones: Is Everything Really Made Up of Ones?

Students quickly realized they did not need to be afraid of large numbers, because they are simply made up of ones. They realized that if you really wanted to, you could count to "five million etcetera" starting with 1 + 1. Then 1 + 1 + 1. Then 1 + 1 + 1 + 1 ... and so on.

It was also suggested to the students that not only could they count to a number like five million, seven hundred twenty-three thousand, eight hundred twelve by ones, but the reverse is also true. Just as there are infinitely large numbers, there are infinitely small numbers.

Just as students enjoy coming up with words such as *noon* or *racecar*, known as palindromes (a word that reads the same backward as forward), the Number Devil had us creating a similar pattern by simply using ones. This time, instead of adding them to make increasingly larger numbers, we were multiplying:

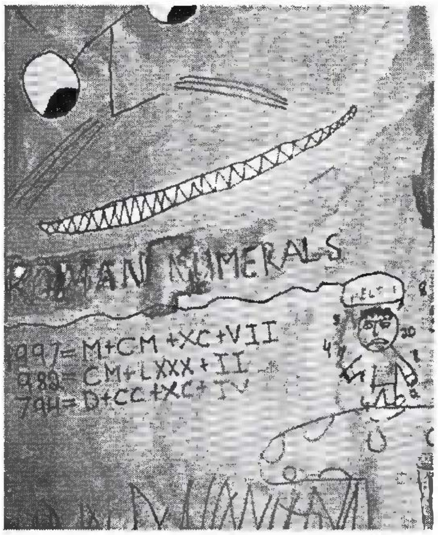
$1 \times 1 = 1$

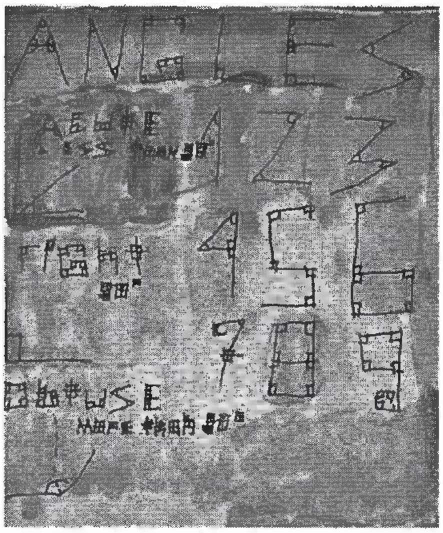
$11 \times 11 = 121$

$111 \times 111 = 12321$

Students were fascinated as they conjectured and explored how these "palindrome products" were occurring, as well as how long the pattern would continue.

Dream 2—Place Value: Thank Goodness for Zero!



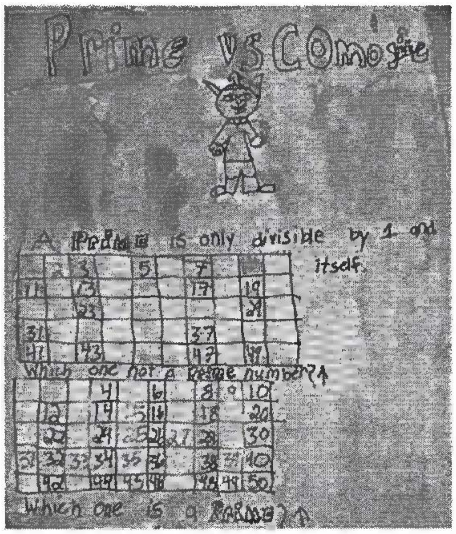


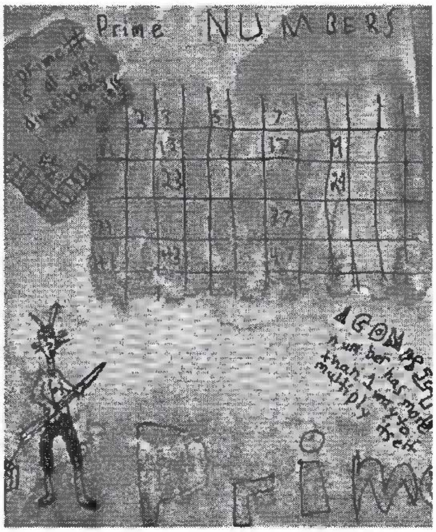
Most of my Grade 4 students were born in 1997. We know that in expanded form to be 1000 + 900 + 90 + 7, or 1997 (a great place-value teachable moment). To the Romans however, this would be much more complicated: MCMXCVII. M = 1000. C is 100, but because it comes before the second M you must subtract it to get 900. C, as we know, is 100, but because X comes before the second C you must subtract it to get 90. V is 5, which you add to two ones to get a sum of 7. So this is 1000 + (1000 – 100) + (100 – 10) + (5 + 2) = 1000 + 900 + 90 + 7 = 1997.

We explored the order of operations, which was not necessarily Grade 4 content, but fun just the same.

While we were back in time with the Romans, we also explored how, in the very early stages of our digits/number system, each digit (1 through 9) was made with the same number of angles as its value (see the artwork above). This was a great exploration, and we discovered that there are a number of ways to make a few of the digits. Also, this was an excellent opportunity to name the angles created within each of the digits.

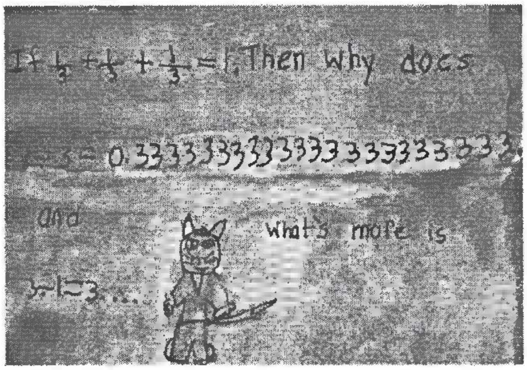
Dream 3—Prima Donna Numbers: Working with Prime Numbers





Students worked together to find all the prima donna numbers between 0 and 50. We were reminded that 0 doesn't count because when you divide any number by 0, you get 0! And that 1 also doesn't count because every number is made up of 1s!

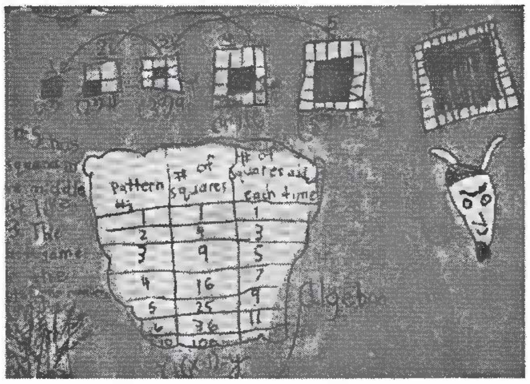
The exploration of prime numbers brought out many mathematical concepts, such as factors, composite numbers and square numbers. Students wondered why certain numbers had more factors than others, and figured out why square numbers had an odd number of factors. It also was a great way for the students to understand why 2, an even number, is a prime number and that not all odd numbers are prime.

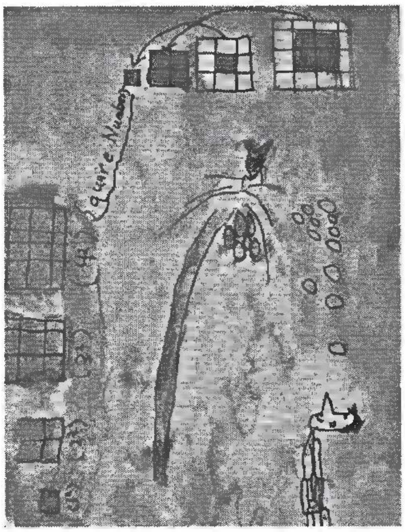


Dream 4—Powers: Unreasonable Numbers

Using the examples in the book, as well as the language introduced by the Number Devil, we took a quick look at powers ("hopping numbers") and square roots ("rutabagas"). Anyone can be confused by the unreasonableness of the following: if 1/3 + 1/3 + 1/3 = 1, then why when you divide 1 by 3 does this not come out quite so perfectly?

Dream 5—Square Numbers





Who knew that an innocent exploration of square numbers would lead to algebra? Starting with a visual representation of their hopping numbers (powers), students were asked the following questions:

- What patterns do you notice?
- How can you display these patterns?
- Can you predict the number of squares there will be in the 10th pattern?
- Can you come up with a rule that would work every time?

Also, it is no secret that if you close your classroom door and whisper to your students, "Now *don't* tell the Grade 6 teacher that you just learned about algebra! Can you believe you are doing math that some of your older brothers and sisters are doing?" you will get them into a mathematical space like never before. Bragging rights!

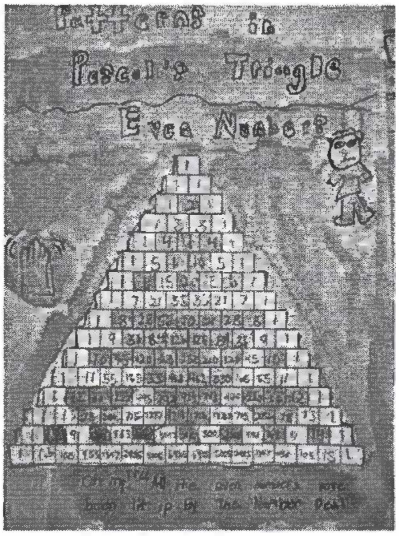
Dream 6—Fibonacci Rules!

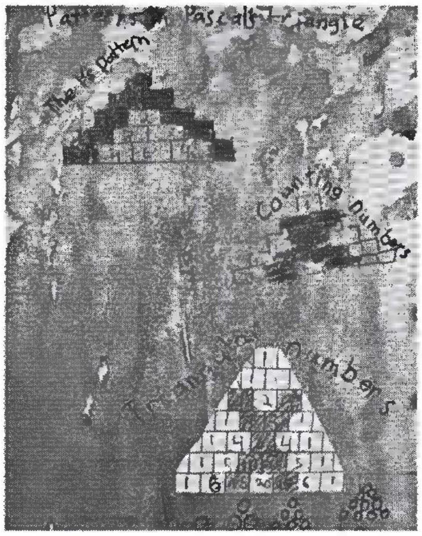
The day began with a brief discussion on patterns: What is a pattern? How might it grow? Students called out responses such as "ABABAB," "ABCABC," and "something that repeats or grows or both." Students were then asked to consider the next 8 numbers in the following sequence of numbers:

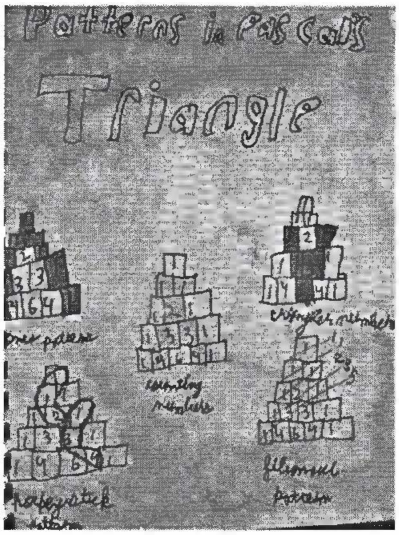
1, 1, 2, 3, 5, 8, 13, ___, ___, ___, ___, ___, ___, ___, ___.

Once the sequence was revealed and named as the Fibonacci sequence of numbers, I coupled this with the Number Devil exposing the truth about the rapid growth of rabbits. Then the students were given the opportunity to do some research on the amazing pattern. Discovering the beauty of the pattern in nature as well as in their own bodies was fascinating to them (see student work at http://projects.cbe.ab.ca/glendale/showcase/numberdevil/fibonacci.html).

Dream 7—Pascal's Triangle



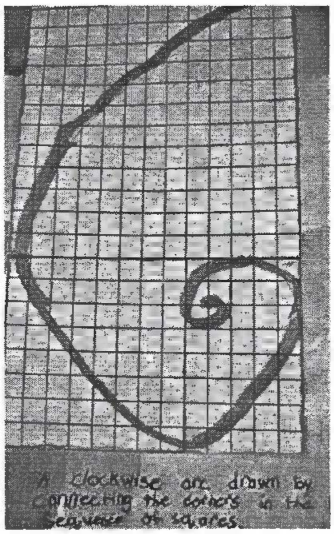




Playing with the Fibonacci sequence of numbers was only the beginning of looking at patterns. Within Pascal's Triangle we explored many other patterns, some recognizable and others simply interesting or perplexing, including

- counting numbers (1, 2, 3, 4, 5...),
- Fibonacci sequence (1, 1, 2, 3, 5, 8...),
- triangular numbers—coconuts! (1, 3, 6, 10, 15...),
- multiples of 2s (even numbers) and 5s,
- the hockey stick pattern and
- the sums of rows.

The list of patterns goes on and on, and there are excellent websites to support these explorations. The best way to begin is to start with just a few numbers on the triangle and have students complete the patterns they see.



Dream 8—Fibonacci Spiral

During the students' research on the Fibonacci sequence, the concept of the spiral came up everywhere for them—the seashells they examined, the pineapple they ate or pictures of the cochlea of an ear. This spurred on the next exploration—squaring Fibonacci numbers to create the spiral for themselves.

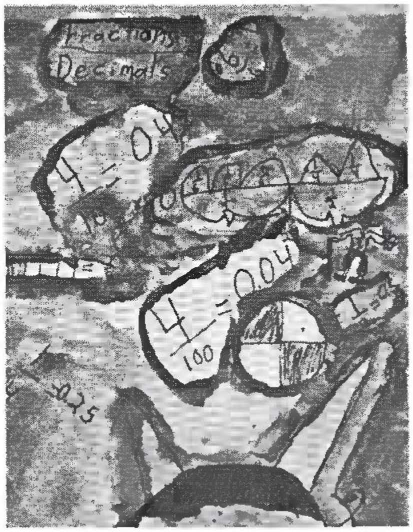
The equiangular spiral can be created by drawing an arc through a series of squares that grow in size by following the Fibonacci sequence (1×1, 1×1, 2×2, 3×3, 5×5, 8×8, 13×13). Students first used tiles to generate the spiral, and then were asked to find it again using grid paper.

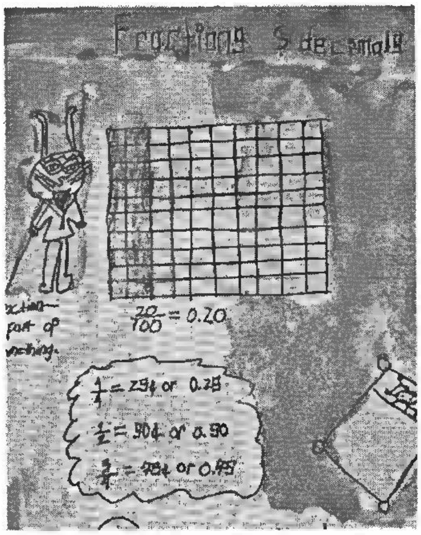
Taking a look at how the Fibonacci spiral approximates the Golden Spiral was something that we explored only briefly, but this could definitely take students down another amazing path.

Dream 9—Fractions

While Robert was dreaming about fractions, we were eating them! I find it interesting that when food is involved, the concepts of equivalence and the importance of comparing the same vs different wholes come to light. With respect to equivalence, watching a student eat, for example, two pieces of licorice from an original piece that had been cut into fourths and another eat four pieces from a piece that had been cut into eighths demonstrated that while it appears someone has more (and they do in number of pieces), if placed side by side, the two students in fact had an equivalent amount of licorice.

Very often, when comparing fractions we consider the same whole and, for the most part, it is important that we do so. However, when I asked my students, "Can 1/4 ever be greater than 1/2?" their first inclination was to say, "No." On one particular day I asked my students to bring in a variety of food items that could be shared with either the entire class or only a few students. The conversations and explanations about how their item could be cut and shared were very rich. It was also on this day that the concept of comparing the same whole came to light. For example, half of a store-bought chocolate chip cookie was, in fact, much smaller than a fourth of a pizza-pan-sized chocolate chip cookie. Aha!





Bringing It to Life: Our Number Devil Performance

I cannot remember exactly when in the reading of the novel the next unanimous decision came about, but when I asked the students, "How can we ever share all that we've been learning with others?" they excitedly replied (as if they had already been conspiring with one another), "A play! We can act it out!"

I used *The Number Devil* as well as *A Place for Zero* (another excellent piece of literature for bringing math to students) and essentially turned them into a screenplay by simply summarizing chapters or following the text verbatim (the script we created follows as Appendix A). Tryouts were conducted, sets were made and, with the help of a resident artist, songs were generated to turn our play into a musical. It was an evening many teachers, students, parents and administrators will never forget.

Conclusion

Through our reading, rereading, exploring, conjecturing, drawing, creating, designing, singing, building and performing, the mathematical understanding generated by this group of Grade 4 students far exceeded my expectations. While it may appear that these 10-year-olds were simply great at memorizing and reciting prime and triangular numbers, the Fibonacci sequence and patterns within Pascal's Triangle, I assure you that there was nothing superficial about their level of understanding. I, too, have a greater appreciation of and ongoing fascination with the numbers and patterns that surround us.

After watching and participating in the performance of *The Number Devil: A Mathematical Adventure*, many parents were quoted as saying, "Now *that's* the way to learn mathematics!"

References

Enzensberger, H M. 1997. *The Number Devil: A Mathematical Adventure.* New York: Holt.

LoPresi, A. 2003. *A Place for Zero: A Math Adventure.* Watertown, Mass: Charlesbridge.

Karen Cleveland is currently a learning leader for the Calgary Board of Education where she supports teachers in the area of mathematics, primarily with the program of studies changes. Prior to this, she taught in an inquiry-infused environment at Glendale School, in Calgary, Alberta. She believes that working collaboratively can generate growth and understanding. While she does not identify herself as a mathematics expert, she has a passion for mathematics and it is in sharing her passion that she continues to learn.

Appendix A

The Number Devil Script

Robert: Mom, I just don't get this question.
Mom: Here, let me help you.
Robert: Okay, if 2 pretzel makers can make 444 pretzels in 6 hours, how long does it take 5 pretzel makers to make 88 pretzels?
Mom: Oh my ... gee ... I was never very good in math, Robert. How about you ask your dad in the morning? Can you get ready for bed, please?
Robert: Oh man ... Mr Bockel is going to be so mad at me if I don't have my homework done.

Number Devil's (ND) first entrance ... ND song. Note that songs have not been added to the script, but simply shown where they were placed within the performance.

ND: Hey, Pretzel Boy ...

Wagon #1

Robert: What? Who are you? Am I dreaming?
ND: Where does your pretzel tale come from? School, I bet.
Robert: Where else? Mr Bockel, he's our teacher. He's always eating pretzels and always making up dumb questions about pretzels.
ND: I see. I have nothing against your Mr Bockel, but do you want to know something? Most genuine mathematicians are bad at sums. Besides, they have no time to waste on them—that's what calculators are for. I assume you have one.
Robert: Well, yes, but we're not allowed to use them in school.
ND: I see. That's alright; I guess there's nothing wrong with a little addition and subtraction. It's good to know what to do if ever you don't have your calculator. But mathematics, my boy—that's something else!
Robert: Oh, just go away. The last thing I want to do is dream about math! Shoo! Scram!
ND: That's no way to talk to a devil!
Robert: I'm sorry. I'm listening.
ND: All right then. The thing that makes numbers so devilish is precisely that they are simple. And you don't need a calculator to prove it. You need one thing and one thing only: one.
Narrator #1: The Number Devil went on to explain to Robert that really, to make big numbers such as 5,723,812, all you have to do is start with 1 + 1 and go on until you have 5 million ... etcetera.
Narrator #2: The Number Devil also convinced Robert that once you got to 5 million ... etcetera, you could keep on going if you wanted to, because there is an infinite number of numbers.
Narrator #3: Poor Robert couldn't believe he was actually dreaming math. He pulled the sheets over his head hoping the Number Devil would go away. And he did...until the next night.
ND: Hey, Pretzel Boy! Tell me what you know about hopping numbers.

Wagon #2

Robert: I ... I don't know what you mean. Numbers don't hop!
ND: Want me to tell you?
Robert: Fine.
ND: Good. Okay, let's go back to square one. Or rather, the number one.
Narrator: #1: The Number Devil wrote this on Robert's wall.

Overhead turns on ... music

$$1 \times 1 = 1$$
$$1 \times 1 \times 1 = 1$$

ND: Take as many 1s as you like and you still get 1 for the answer.
Robert: Sure, but what's your point?
ND: You'll see if you try the same thing with 2s.
Narrator #2: The Number Devil erased all the 1s and wrote this.

Overhead ... music

$2 \times 2 = 4$
$2 \times 2 \times 2 = 8$
$2 \times 2 \times 2 \times 2 = 16$
$2 \times 2 \times 2 \times 2 \times 2 = 32$

Robert: Wow, that goes fast! If you go much faster I'll need my calculator!
ND: It goes even faster if you start with 5. Watch.
Narrator #3: In a flash, the 2s were down and the 5s were up.

Overhead ... music

$5 \times 5 = 25$
$5 \times 5 \times 5 = 125$
$5 \times 5 \times 5 \times 5 = 625$
$5 \times 5 \times 5 \times 5 \times 5 = 3125$

Robert: Whoa!
ND: Why do large numbers scare you? I assure you they're perfectly harmless.
Robert: Says you! Plus that's just too many 5s to have to write.
ND: Aha. I thought you'd never ask. You see, instead of writing all those 5s, I write

Overhead ... music

$5^1 = 5$
$5^2 = 25$
$5^3 = 125$

and so on. Now do the same with 10. You can throw your calculator away. Make the 10 do one hop, and it remains exactly the same

$10^1 = 10$

Make it hop twice and you get

$10^2 = 100$

Make it hop three times and you get

$10^3 = 1{,}000$

Robert: So if I make it hop five times, I get 100,000. Once more, and I get a million!
ND: You've got it, my boy! With the help of my friend zero and a bit of hopping you can produce any number you please. I'd say that's enough for tonight.
Robert: A hopping zero ... now that's a good one.
Narrator #1: Before Robert woke up he dreamed of the concept of a hopping zero.

Zero, zero, zero, zero ...
Set change.

A Place for Zero

Narrator #1: Not long ago, Zero lay floating on the calm waters of Central Lake. He could hear the happy cries of the other numbers, 1 through 9, as they played in the meadow. Zero didn't play Add-em-up because he had nothing to add. He felt he had no place among the other digits.
Narrator #2: Zero lived in Digitaria, a curious country ruled by King Multiplus and Queen Addeleine.

Digitaria song

Narrator #3: Count Infinity, the King's trusted advisor, was the one who shaped all the numbers. When old digits retired, he replaced them with shiny new numerals. Every number knew its place.
Narrator #1: A 7 was the number of days in a week, and a 5 was the number of points on a star.
Narrator #2: A 2 was handy for counting the wheels on a bicycle.
Narrator #3: The 1s were important because Count Infinity added them together to make the other numbers.
Narrator #1: Every number had a place except Zero. Count Infinity had been experimenting when he formed the strange new digit. But Zero meant nothing, and no one was sure what his job would be.

Zero song

Narrator #2: King Multiplus declared that no more zeros would be made until they found a purpose for this one.

King Mult: I, King Multiplus, King of Digitaria, do declare that no more *zeros* be made. Until we find a purpose for this *zero* we will not waste Count Infinity's precious time!

Narrator #3: Nowadays no one mentioned Zero much. The few times someone tried to count with him, they always ended up with nothing.

Counter: Okay, let's see what we have here. Zero? No, no … let's try that again. Zero. What?! Aaahhh…

Zero: This just isn't fun anymore. I need to find my place. Count Infinity must have thought of something by now.

Narrator #1: Count Infinity made incredible things. He made knickknacks, whatsits and thingamajigs. The most impressive thing he ever made was an enormous machine—the Numberator!

Narrator #2: The Numberator had a vacuum tube on one side and a large curved spout on the other.

Narrator #3: When Count Infinity placed digits under the tube, they were sucked up with a great clanking and whirring until, finally, they emerged from the spout—with a totally new number!

Narrator #1: Count Infinity could put in two 1s to make a bouncy baby 2, or three 1s to make a roly-poly little 3.

Count Infinity (CI) song

CI: Ah, young Zero. You're just in time to help me figure out a better way to make a 1.

Zero: What do you mean?

CI: Well, most numbers are easy to make. I simply pop a handful of 1s into my trusty Numberator. Making a 1, though—that's trickier.

Zero: Why don't you just stick one 1 into the Numberator?

CI: I've tried that, but the Numberator adds. You have to put in at least two numbers. No two numbers that I can think of add up to …

Narrator #2: Count Infinity eyed Zero in a strange way.

CI: Zero, could you put this 1 into the Numberator for me? Just help the 1 stand right under the vacuum tube.

Narrator #3: The Count made sure that Zero was right where he wanted him to be.

CI: Perfect! Just stand right there.

Narrator #3: As Zero did as he was told; he was sucked up into the Numberator along with the 1.

Narrator #1: Zero barely had time to realize what was happening before he was out the other side. Sitting next to him were the old 1 and a shiny new 1.

C.I: Wonderful! We have discovered your additive identity. I bet my binomials that if we add you to any number, we will always come up with the same number. You can help replenish the supply of digits in Digitaria!

Zero: Well, gee. I suppose I should be thanking you for helping me find my purpose, but 1s can make digits, too. I want to have a job that no one else has.

Narrator #2: As Zero made his way home he had another idea.

Zero: I wonder what would happen if I were multiplied with another number? I will ask the King.

Narrator #3: Everyone knew that multiplication was a powerful thing, and only the King could use it. Often the product of the multiplied numbers would be too large for anyone to understand.

Narrator #1: At the gates of the castle, Zero was stopped by two sharp-looking 7s.

Guard #1: What kind of number are you? You look like a 9 that someone squished!

Zero: I need to ask the King about multiplication.

Guard #2: Well, good luck to you.

Narrator #2: Zero entered the castle. He saw King Multiplus and Queen Addeleine on their thrones. Crowds of numbers filled the hall, waiting to address them. The 5 bowing next to Zero was rather surprised to see him there.

Five: Why are you here?

Zero: I'm asking the King to multiply me.

Five: Are you kidding?

Narrator #3:	Suddenly, the numbers began to whisper.
Nine:	That's absolute nine-sense!
Two:	This is just two much for me to handle.
King Mult:	What is the meaning of this?
Queen Add:	It's Zero ... and something about multiplication.
King Mult:	Multiplication! That is serious business. Let him come forward and explain himself.
Zero:	I would like to be multiplied. I thought it might help me find my place and purpose.
King Mult:	Yes. I have wondered myself what would happen. Guards, fetch me my Multi-tube.
King Mult:	We need to find someone who is willing to be multiplied with Zero.
King Mult:	Will no one come forward?
Brave One:	I'll do it ...
Queen Add:	Oh what a brave little 1.
Narrator #1:	The guards led Zero and the 1 into the Factor end of the tube. All that could be heard was the clanking and whirring of the tube.
Narrator #2:	Suddenly, Zero and the 1 shot out of the other end of the tube. As Zero looked around, he saw beside him a small, round figure who looked strangely familiar.
King Mult:	Five alive! It's another zero!
Queen Add:	Why, we must try that again.
King Mult:	Yes, indeed. This time, we will multiply zero with a 7. Guard 7, step forward.
Narrator #3:	The same thing happened with Zero and the 7.
King Mult:	Well, Zero, you have your answer. No matter what number we multiply you with, we get zero!
Narrator #1:	This was big news, but Zero still didn't seem pleased.
Zero:	But King, now there are just more zeros with no real place. I want to make new numbers that no one has ever seen before.
Queen Add:	Wait! What number do we get when we add 1 to 9?
King Mult:	As everyone know, anything bigger than 9 is ... *many.* We don't think about 1 plus 9 because there is no digit for a number that big.
Zero:	Sire! Look at us! We can make a new number to represent what you get when you add 1 and 9.
Zero:	When I stand in this place, next to my friend 1, as a zero I can represent zero 1s. But he now represents 9 + 1.
King Mult:	I like your thinking, Zero. But I can't call the new number *9 + 1.* Quick, think of a new name.
Zero:	This new number will *attend* to a lot of business, Sire. We could call it Attend?
King Mult:	I proclaim that this new number will be called a *TEN*!
Narrator #2:	The numbers all loved it.
Numbers:	Hooray for Ten! Hooray for Ten!
Narrator #3:	The numbers crowded around Zero and asked if they could stand beside him. Zero gave each digit a turn and soon they were making wonderful new numbers like 20, 30, 60 and 90.
Narrator #1:	Before long they were standing next to each other in different combinations, to make all kinds of new numbers, like 32, 47 and 89. Big numbers didn't seem quite so scary anymore.
Baby Zero:	What about me? Is it my turn to stand next to you?
Zero:	I'm sorry, little guy. Two 0s standing next to each other still equal nothing.
Narrator #2:	The little 0 looked so sad that the brave 1 and Zero each put an arm around him. Suddenly, Zero had a thought.
Zero:	Look, Sire! What do we look like?
King Mult:	Why, you look like a new number, bigger than any I've ever seen. Bigger even than the new number 90.
Zero:	We can represent what you get when you have ten 10s.
King Mult:	Splendid!
Narrator #3:	Soon, all the other digits were clamouring to stand next to the two 0s. They made numbers like 500 and 700.
CI:	Well, Zero, did you find what you were looking for?
Zero:	I just wanted to mean more than nothing. And now I think I've finally found my place.

Zero song

Zero song continues during set change.

Mom: Good morning, dear.
Robert: Good morning, Mom. Hey, Mom, did you know that if you wanted to count to 5,723,812 you can do so just by using 1s? And did you know numbers hop, that if you hop a 10 six times you get a million? And did you know that Zero means WAY more than nothing!
Mom: Robert, sometimes you just say the oddest things.
Narrator #2: In time, Robert grew accustomed to dreaming of the Number Devil.
Narrator #3: He even came to look forward to it.
ND: Good evening, Pretzel Boy! Ready for some more?

Wagon #3.

Robert: You bet!
ND: Alright then, I'll tell you a secret. There are two types of numbers. The garden variety, which can be divided evenly, and the rest, which cannot. Wonderful numbers like 11, 13 or 17.
Robert: Oh, that's easy—I know this one. They're all odd numbers.
ND: Oh, no, my young man. The numbers I'm talking about are more than just odd; they are numbers that are only divisible by one and itself. They are *prima donna* numbers.
Robert: Prima donnas? Why?
ND: Because from the very first they've caused mathematicians no end of trouble. Watch this. Would my numbers from 0 to 50 please come forward.

Parents in the crowd come forward. Music.

ND: Take my cane, Pretzel Boy, and tap the first prima donna number.
Robert: Well, that's simple—zero.
ND: WRONG! Zero is forbidden. We know the importance of our zero, but if we divide anything by it, we'll end up with zero. So, no. Off with you ...
Robert: All right, then—one.
ND: One doesn't count. Everything is made up of ones. Haven't you been listening to anything I've taught you? Off with you ...
Robert: Okay, okay. Calm down. Two. Two can be only be divided by one and itself.
ND: Oh, you're on a roll now!
Robert: Well, any other even number from now on will be divisible by 2, so they can't be prima donnas.
ND: You're right. All the even numbers ... off with you.
Narrator #1: Robert had ruled out 25 numbers.
Narrator #2: The Number Devil had given him one hint.
Narrator #3: That there were exactly 15 numbers between 0 and 50 that were prima donnas.
Narrator #1: Robert knew that once he passed by number 3, all other multiples of 3 should leave.
Narrator #2: He gently tapped them on the shoulder, and off they went.
Narrator #3: He did the same with 5, leaving it in the lineup, but asking all the multiples of 5 to please leave.
Narrator #1: 7 can stay, but all other multiples of 7 can't.
Narrator #2: Robert was pretty sure that he was done.
Narrator #3: But to make sure, he counted, hoping to have only 15 remaining.

Robert counts his numbers ...

ND: Well done, Robert.

Prima Donna song

ND: Prima Donnas, please have a seat.
ND chuckles to himself.
Robert: What's so funny?
ND: It's not so hard if you stop at 50. But what if it's a number like 421,356,237,307? Is it a prima donna or isn't it?

Robert: I … I don't know.

ND: If you only knew how many mathematicians have racked their brains over this issue. Why, even I have come to grief over it.

Robert: No offence, but if *you* can't look at a number and tell if it's a prima donna or not, there's no way I can. And I'm too tired anyway. I should be sleeping. I have a math test tomorrow.

Narrator #1: After explaining to Robert that he actually was sleeping, because in order to be dreaming him, Robert had to be sleeping, he decided to leave him be.

Narrator #2: The next night Robert went to bed without a bedtime snack.

Narrator #3: "It's all right, Mom," Robert called from his bedroom. "I don't want a snack; I just want to go to bed."

Mom: Oh, my! What has gotten into my boy? No bedtime snack? I promised him it wouldn't be pretzels!

ND: I bet you've forgotten your calculator again, eh, Pretzel Boy?

Wagon #4

Robert: Look, how many times do I have to tell you? I can't take all my stuff to bed with me at night. Do *you* know what you're going to dream the night before you dream it?

ND: Of course not. But still, if you dream of me, you can just as easily dream of your calculator.

Robert: Where am I? It looks like I'm in someone's vegetable garden.

ND: Perhaps you are. You recall our hopping game, I'm sure. What we did with the 2 and the 5 and the 10. Ten times ten times ten equals a thousand, which we write

$$10^3 = 1{,}000$$

because it's faster.

Robert: Right. And when we hop with 2s we get 2, 4, 8, 16, 32 ... and so on until the cows come home.

ND: Perfect! So now, let's do the first hop in reverse. Hopping backward, so to speak. Only when you go backward you don't really hop, we call the step *taking a rutabaga*. As if we were pulling one of these fine root vegetables out of the ground.

Rutabaga song

ND: So, what's the rutabaga of 4?

Robert: Two?

ND: Right. What's the rutabaga of 25?

Robert: Five!

ND: Rutabaga of 100?

Robert: Ten!

ND: Rutabaga of 36?

Robert: Six!

ND: Rutabaga of 5,929?

Robert: Are you crazy or something? How do you expect me to do that one? Mr Bockel asks me enough dumb problems in school. I don't need to dream about them.

ND: Calm down, calm down. This is what a calculator is for.

Narrator #1: The Number Devil showed Robert how to find the rutabaga of 5,929 using a calculator.

Narrator #2: As thrilled as Robert was to have learned something new, all this backwards hopping was making him tired. Or so he thought.

Narrator #3: Tired of convincing Robert that he was actually sleeping, the Number Devil tiptoed off, careful not to awaken him.

Look out---a coconut …

Main character change and coconut catchers set up.

Narrator #1: No sooner had Robert closed his eyes the next night than he found himself wandering through a desert.

Narrator #2: There was no shade or water, and he was wearing nothing but his shorts and a T-shirt.

Narrator #3: On and on he trudged, thirsty and sweaty, until at last he made out a few trees in the distance.

ND: Hello there, Pretzel Boy!

Wagon #5

Robert: I'm dying of thirst!
ND: How about some coconut milk?
Narrator #1: The Number Devil tossed down a coconut from the tree and Robert took a long drink.
ND: Heads up!
Robert: What?
ND: I'm going to throw down a few coconuts. Just toss them on the ground.
Robert: Anywhere in particular?
ND: No, just down.
Narrator #2: Robert threw the first coconut into the sand. It looked much like a dot.
Narrator #3: Robert continued to catch and toss until there were no more coconuts.
ND: What do you see?
Robert: Triangles! Funny, they fell into such neat patterns. And I wasn't even aiming. I'd never have been able to do that if I tried.
ND: You're probably right. Now, would you count up the number of coconuts in each of the triangles?
Robert: All right. But the first is not triangle at all. It's just a dot. And there is only 1.
ND: Ah yes, our good friend one. Next?
Robert: The second triangle has 3 coconuts, the third 6, the fourth 10, and the fifth ... I'm not sure. Wait. Let me count.
ND: Why? You don't need to count. You can calculate it.
Robert: No I can't.
ND: Yes you can. Look—the first triangle consists of one coconut. The second has two more—the second row—which comes to three.
The third has exactly three more. 3 + 3 = 6.
The fourth row has another row with four more. 6 + 4 = 10.
Robert: Wait. Wait. I know. The fifth will be a row of 5 added. So 10 + 5 = 15. And the next triangle would have 21 coconuts, 15 from the last one, plus the six new ones.
ND: You've got it, Robert!
Narrator #1: Robert felt quite pleased with himself as he made his way back into his bed, thankful to be out of the heat.
Narrator #2: Before Robert drifted off to sleep, the Number Devil snuck in one last trick. If you pick a number, any number, it can be shown to be the sum of two or three triangular numbers.
Narrator #3: Don't take our word for it. Try it!

Pick a number, any number, go ahead and try it.

Narrator #1: The next night, Robert just couldn't wait for the Number Devil to visit him.
Narrator #2: There was something he had been meaning to ask him.
Narrator #3: Something you may have also been wondering about.
ND: Wakey, wakey...

Wagon #6

Robert: Oh good, you're here!
ND: Wow. Someone's ready for some numbers.
Robert: No, no. It's not that. I've been meaning to ask you about all those rabbits that keep following you. It seems as if every time I see you, there are more rabbits.
ND: Like I said. You're ready for some numbers.
Robert: Huh?
ND: You see, in order to explain the rabbits, I need to tell you about a few numbers and about an Italian named Bonacci. He's been dead for years now, poor Bonacci, but he came up with what we call Bonacci numbers. A capital idea! And like most good ideas, it begins with ...
Robert: One?
ND: And not just one, but two 1s.
Narrator #1: The Number Devil scrolled the sequence out on the wall.

Narrator #2: He explained that by adding the two previous numbers, you'd get the next number in the sequence.
Narrator #3: Robert was definitely intrigued, but he still didn't understand what this had to do with rabbits.
Robert: This is all fine and dandy, but tell me—what are these numbers good for, your Bonacci numbers?
ND: You don't think mathematics is for mathematicians only do you? Nature needs numbers too.
Robert: Come on. You don't expect me to believe that.
ND: I expect you to believe that every living thing uses numbers. Or at least behaves as if it did.
Robert: Well I don't believe it, and this still isn't explaining the rabbits.
ND: Okay then, let's talk rabbits. I bet there are rabbits all over this field.
Robert: I don't see any.
ND: Look, there are two now!
Narrator #1: Sure enough, two white rabbits hopped up to Robert and plunked themselves at his feet.
ND: A male and a female I think, or one couple. And as you know, one is all we need to get things rolling.

...talking to the rabbits...

Robert: He wants me to believe you can do arithmetic.
Narrator #1: The Number Devil explained to Robert that it would only take one month for these two rabbits to grow up, and then their fur turns brown.
Narrator #2: They are now old enough to have babies; and they do—two of them.
Narrator #3: Then their babies grow up and do the same.
ND: But don't forget about the original couple. They continue to have babies every month as well.
Robert: Whoa! I get the feeling there are going to be hordes of rabbits romping around here real soon!
Narrator #1: The Number Devil threw a chart up on the wall.
ND: Do you remember these numbers?
Robert: 1, 1, 2, 3, 5, ... Of course. It's obvious. Bonacci numbers all the way.
ND: You never cease to amaze me, Pretzel Boy! *Now* will you admit that rabbits behave as if they have learned their Bonacci numbers by heart?
Robert: Fine. Great. Anything you say. I admit it.
ND: You still don't sound convinced. Let me show you where else the Bonacci numbers are found in nature.
Narrator #2: The Number Devil took Robert over to a nearby tree.
ND: Look how the branches follow the Bonacci sequence.
Narrator #3: Robert noticed how the tree's branches grew in accordance with the sequence. As he made his way back to bed, Robert was quite intrigued by the tree and started to rethink that what the Number Devil was telling him about the Bonacci and nature connection was true.

Bonacci song

Mom is on the phone ...

Mom: I'm terribly worried. I don't know what's wrong with my boy. He used to spend all his time in the park, playing ball with his friends, but now he just shuts himself up in his room painting pictures of trees and rabbits!
And the numbers he keeps muttering to himself. Numbers, numbers and more numbers. It's not normal.
Don't worry? I've got to go ...
Robert, time for bed.
Robert: Woo-hoo!
Mom: Aaahhh ...
Narrator #1: No sooner did Robert's head hit the pillow than the Number Devil was on the scene.
ND: Today I have something extraordinary to show you.
Robert: Anything you like! Just no more rabbits
ND: What do you say we build a pyramid?
Narrator #2: The Number Devil took Robert over to a huge wall and handed him his walking stick.

Narrator #3: Following his mentor's directions, Robert outlined the shape of a very large triangle, and with a final zap ...

ND: Voila!

Robert: This isn't a pyramid. Pyramids have triangular, or rectangular or square bases. This thing is flat.

ND: Fine. Then we built a triangle. Now take a look at the numbers and tell me what you notice.

Robert: I can see that the numbers along the sides are all 1s.

ND: Good.

Robert: And that the next diagonal rows on either side of the 1s are the perfectly normal counting numbers.

ND: What about the next diagonal row.

Narrator #1: Robert read down the row from right to left.

Robert: 1, 3, 6, 10 ... Hey, they look familiar.

ND: Coconuts!

Robert: Right, right. Now I remember. The triangular numbers.

ND: You have been paying attention!

Robert: So how do you know what numbers go where?

ND: I thought you'd never ask. Take a look at any row and the row directly above it and see if you notice anything.

Narrator #2: Robert studied two rows.

Robert: Oh, I see it.

Narrator #3: Robert used the walking stick to show that numbers in each cube were the sums of the two cubes directly above it.

Robert: Fun!

ND: Oh, but there's more, Pretzel Boy!

Robert: There always is ...

ND: Would you mind adding up the sums of each row for me?

Robert: Um ... 1, 2, 4, 8, 16 ... Oh, I know this one—these are the hopping 2s.

Narrator #1: As the Number Devil walked Robert back to his room, he told him about all the other patterns he'd be guaranteed to find in their magic triangle.

Narrator #2: Patterns like magic 11s, multiples of 5, hockey sticks and even the Bonacci numbers!

Narrator #3: Robert was once again thrilled with this evening's number show.

Narrator #1: He tucked himself into bed with a smile and closed his eyes in anticipation of his friend, the Number Devil's next visit.

Parade music

All students in hallway preparing for number parade.

ND #1: Robert, we've got just one final show for you. Would my hopping numbers please come in?

ND #2: Otherwise known as *powers*.

ND #1: Would my prima donna numbers please come in.

ND #2: Otherwise known as *prime numbers*.

ND #1: Would my coconut numbers please come in.

ND #2: Otherwise known as *triangular numbers*.

ND #1: Would my Bonacci numbers please come in.

ND #2: Otherwise known as the *Fibonacci sequence*.

Robert: I don't think we'll ever look at numbers the same way!

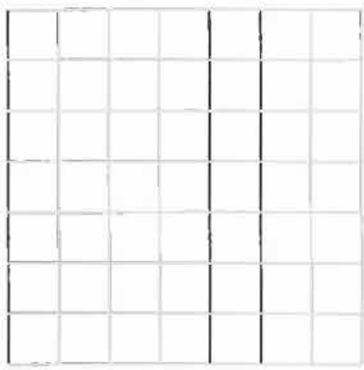
Parade music and out.

Alberta High School Mathematics Competition
Part II, 2009.

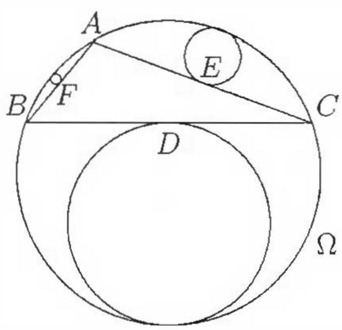
Problem 1.
Let w, x, y and z be non-negative numbers whose sum is 100. Determine the maximum possible value of $wx + xy + yz$.

Problem 2.
Determine all positive integers a and b, $a < b$, so that exactly $\frac{1}{100}$ of the consecutive integers a^2, $a^2 + 1$, $a^2 + 2$, $\ldots$, b^2 are the squares of integers.

Problem 3.
A game is played on a 7×7 board, initially blank. Betty Brown and Greta Green make alternate moves, with Betty going first. In each of her moves, Betty chooses any four blank squares which form a 2×2 block, and paints these squares brown. In each of her moves, Greta chooses any blank square and paints it green. They take alternate turns until no more moves can be made by Betty. Then Greta paints the remaining blank squares green. Which player, if either, can guarantee to be able to paint 25 or more squares in her colour, regardless of how her opponent plays?



Problem 4.
A, B and C are points on a circle Ω with radius 1. Three circles are drawn outside triangle ABC and tangent to Ω internally. These three circles are also tangent to BC, CA and AB at their respective midpoints D, E and F. If the radii of two of these three circles are $\frac{2}{3}$ and $\frac{2}{11}$, what is the radius of the third circle?



Problem 5.
Prove that there are infinitely many positive integers k such that k^k can be expressed as the sum of the cubes of two positive integers.

Alberta High School Mathematics Competition

Solutions and Comments to Part II, 2009.

Problem 1.

Since $(w+y)+(x+z)=100$, we have $w+y=50+t$ and $x+z=50-t$ for some real number t. Hence $wx+xy+yz \leq (w+y)(x+z)=(50+t)(50-t)=2500-t^2 \leq 2500$. This maximum value may be attained for instance when $w=x=50$ and $y=z=0$.

Problem 2.

Let $d=b-a$. Then there are $(a+d)^2-a^2+1=2ad+d^2+1$ integers under consideration, $d+1$ of which are the squares of integers. Hence we need $100(d+1)=2ad+d^2+1$, so that

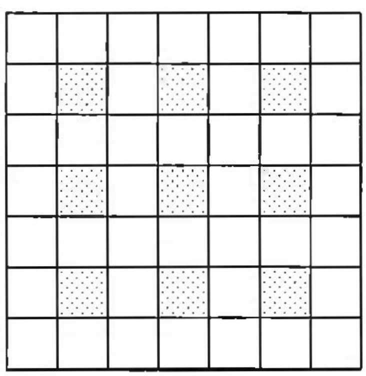
$$a=\frac{100(d+1)-d^2-1}{2d}=\frac{100-d}{2}+\frac{99}{2d}.$$

If d is even, the first term is an integer and the second is not. Hence d must be odd. Then the first term is a fraction with denominator 2, so that the second term must also be a fraction with denominator 2. This means that d must be a divisor of 99, that is, d is 1, 3, 9, 11, 33 or 99.

If $d=1$, then $a=\frac{99}{2}+\frac{99}{2}=99$ and $b=99+1=100$.
If $d=3$, then $a=\frac{97}{2}+\frac{99}{6}=65$ and $b=65+3=68$.
If $d=9$, then $a=\frac{91}{2}+\frac{99}{18}=51$ and $b=51+9=60$.
If $d=11$, then $a=\frac{89}{2}+\frac{99}{22}=49$ and $b=49+11=60$.
If $d=33$, then $a=\frac{67}{2}+\frac{99}{66}=35$ and $b=35+33=68$.
If $d=99$, then $a=\frac{1}{2}+\frac{99}{198}=1$ and $b=1+99=100$.

Problem 3.

There are 9 squares at the intersections of even-numbered rows and even-numbered columns. Any 2×2 block chosen by Betty must include one of these 9 squares. Hence Greta should play only on these squares in her first four moves. This will ensure that Betty has at most five moves, and can paint at most 20 squares brown. Hence Greta wins.



Problem 4.

Denote the circumcentre of Ω by O and note that it lies within the circle with radius $\frac{2}{3}$. We have

$$BC^2 = 4BD^2 = 4(OB^2 - OD^2) = 4\left(1 - \frac{1}{9}\right) = \frac{32}{9}$$

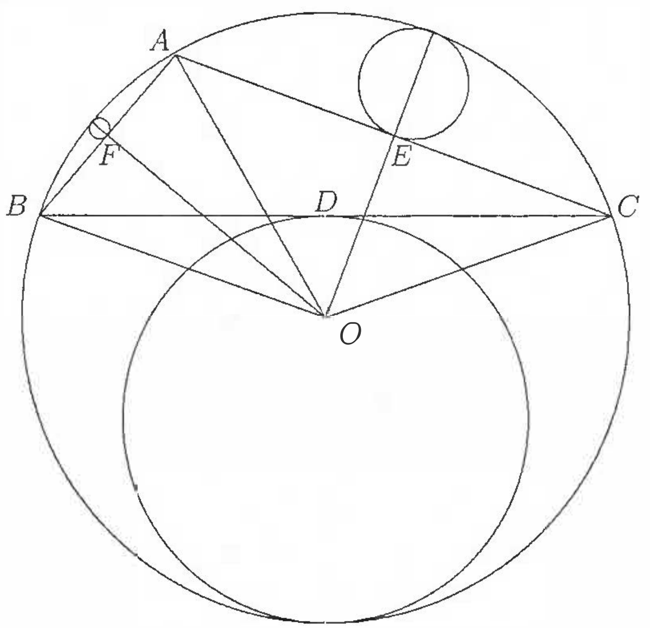
and

$$CA^2 = 4CE^2 = 4(OC^2 - OE^2) = 4\left(1 - \frac{49}{121}\right) = \frac{288}{121}.$$

By the Cosine Law, $\cos BOC = \frac{OB^2+OC^2-BC^2}{2OB\cdot OC} = -\frac{7}{9}$ and $\cos COA = \frac{OC^2+OA^2-CA^2}{2OC\cdot OA} = -\frac{23}{121}$. Hence $\sin BOC = \frac{4\sqrt{2}}{9}$ and $\sin COA = \frac{84\sqrt{2}}{121}$. It follows that

$$\begin{aligned}\cos AOB &= \cos(\angle BOC - \angle COA))\\ &= (\cos BOC)(\cos COA) + \sin BOC \sin COA\\ &= \frac{833}{1089}.\end{aligned}$$

By the Cosine Law again, $AB^2 = OA^2 + OB^2 - 2OA \cdot OB \cos AOB = \frac{512}{1089}$. Hence we have $OF^2 = OA^2 - AF^2 = \frac{961}{1089}$ and $OF = \frac{31}{33}$. It follows that the radius of the third circle is $\frac{1}{2}(1-\frac{31}{33}) = \frac{1}{33}$.



Problem 5.

We have $(a+1)^{a+1} = (a+1)^a(a+1) = a(a+1)^a + (a+1)^a$. Choose $a = 3^{3t}$ for an arbitrary positive integer t. Then $a = (3^t)^3$ and $(a+1)^a = ((3^{3t}+1)^{3^{3t-1}})^3$ are both cubes. If we take $k = 3^{3t}+1$, $m = 3^t(3^{3t}+1)^{3^{3t-1}}$ and $n = (3^{3t}+1)^{3^{3t-1}}$, then $k^k = m^3 + n^3$. Since t is an arbitrary positive integer, the number of possible choices for k is infinite.

Print ID # ________ /90

School Name ________________ Student Name ____________________
(Print First, Last)

2009 Edmonton Junior High Math Contest

Part I: Multiple Choice (PRINT neatly, use CAPITAL letters, 4 points each)

1.	6.	11.
2.	7.	12.
3.	8.	13.
4.	9.	14.
5.	10.	15.

Part II: Short Answers (PRINT small but legible, 6 points each)

16. ________ 17. ________ 18. ________ 19. 20. ________

Part I: ________ x 4 + ________ x 2 = ________ (be sure blanks ≤ 3)
Correct Blank

Part II: ______ x 6 =
Correct

Total = ________ (enter total score on top)

MARKER ONLY

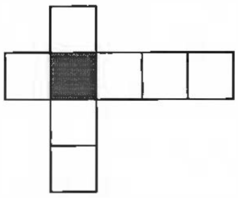
Instruction:

1. Calculator, grid paper and scrap paper are permitted. You may write on the booklet.
2. Programmable calculator and Cell phones are not allowed to be brought into class.
3. Don't write your answers too LARGE to avoid others seeing your answers. COVER your answers at all time.
4. All fractions must be proper and reduced to lowest terms.
5. Each correct answer is worth 4 points for multiple choices and 6 points for short answers.
6. Each incorrect answer is worth 0 point.
7. Each unanswered question in Part I is worth 2 points up to a maximum of 6 points.
8. Unanswered questions in Part II is worth 0 point.
9. You have 90 minutes of writing time.
10. When done, carefully REMOVE and HAND IN only page 1.

Part I: Multiple Choices

1. Each of the eight squares in the figure at the right must contain one digit. The sum of the eight digits is 38. Reading from top to bottom, the four digits in the vertical column form a number that is a power with 3 as its base. Reading from left to right, the five digits in the horizontal row form a number that is a power with 4 as its base.

   What digit will be in the square of the shaded box?



   a. 6
   b. 5
   c. 3
   d. 1

2. How many whole numbers from 10 to 60, inclusive, are divisible by both its units digit as well as its tens digit?

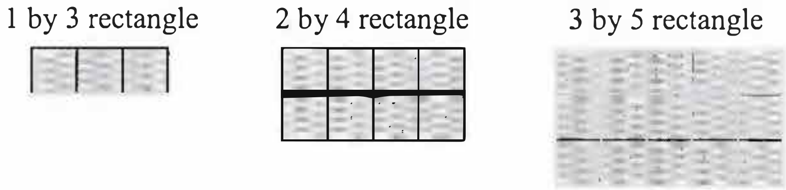
   (E.g., Although 39 is divisible by 3, it is not divisible by 9. Therefore, 39 is not one of the numbers in the solution.)

   a. ten numbers
   b. eleven numbers
   c. seventeen numbers
   d. eighteen numbers

3. Twenty people at Snow Lodge came either to ski or to snowboard. (*Nobody does both activities.*) The ratio of skiers to snowboarders is 3 to 2. The ratio of male skiers to female skiers is 5 to 1. The ratio of children snowboarders to adult snowboarders is 1 to 3. What is the sum of the number of male skiers and child snowboarders at Snow Lodge?

   a. 6
   b. 8
   c. 10
   d. 12

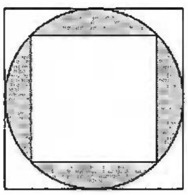
4. Rectangles can be formed by joining unit squares. Some of the line segments are part of their perimeters and some are not. For the three examples shown below, their perimeters are, respectively, 8 units, 12 units, and 16 units. The number of line segments that are not part of their perimeters are, respectively, 2, 10, and 22.



If a 50 by 100 rectangle is formed with unit squares, how many line segments would **not** be part of the perimeter?

a. 4900
b. 4950
c. 9050
d. 9850

5. In the figure at the right, the smaller square is inscribed in a circle, which is inscribed in a larger square with a side of 10 cm.
Approximately what area of the entire figure is **not** shaded?



a. 21.5 cm^2
b. 28.5 cm^2
c. 71.5 cm^2
d. 78.5 cm^2

6. The front wheel of a tricycle has a diameter of 25 cm; the rear wheels have a diameter of 15 cm. If the tricycle travels 100 m, about how many more rotations will the rear wheels make than the front wheel?

a. 212
b. 127
c. 85
d. 31

7. The year 1961 has 180° rotational symmetry because it reads the same when it is turned upside down. How many years since 1000 have 180° rotational symmetry?

   a. 3
   b. 5
   c. 8
   d. 10

8. Which of the numbers in the set { -50 , -1.5 , -1 , -0.2 , 0 , $\sqrt{2}$, $\frac{4}{5}$, π , 40 } are greater than their reciprocals?

   a. -0.2 , $\sqrt{2}$, π , 40

   b. -50 , -1.5 , -1 , 0 , $\frac{4}{5}$

   c. -0.2 , 0 , $\sqrt{2}$, π , 40

   d. $\sqrt{2}$, $\frac{4}{5}$, π , 40

9. On her last test, Crystal scored 92%, and in doing so, raised her average by 2%, to 82%. What percent must she get on her next test to raise her average one more percent, to 83%?

   a. 89%
   b. 86%
   c. 83%
   d. 82%

10. Erhan has a pair of special dice. The six faces of each die are labelled 1, 4, 5, 7, 11, and 14. He rolls the pair of dice and finds the sum of the two numbers. What is the probability that the sum will be both even and less than 20?

a. $\frac{5}{12}$

b. $\frac{1}{2}$

c. $\frac{5}{9}$

d. $\frac{5}{6}$

11. The tens digit of the square of an integer is 5. The ones digit

a. must be 6

b. must be 4

c. can be 4 or 6

d. none of the above

12. In the quadrilateral ABCD, $\angle ABC = \angle BCD = 90°$, $\angle CAB = 45°$ and $\angle CBD = 60°$. The diagonals AC and BD intersect at point E. The ratio of the areas of the triangles CDE and ABE is

a. $\sqrt{2}:1$

b. $\sqrt{3}:1$

c. 2:1

d. 3:1

13. A small box of chocolate costs \$21 while a large box costs \$41. How many different combinations of small and large boxes cost exactly \$2009?

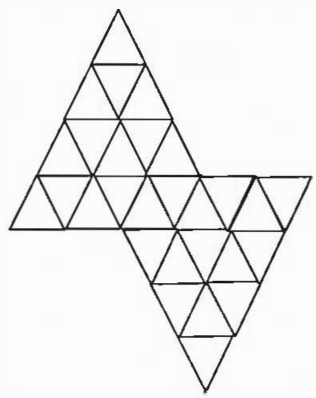
a. 1

b. 2

c. 3

d. 4

14. What is the total number of downward-pointing triangles in the diagram below? (Consider all sizes of triangles.)



a. 14

b. 25

c. 28

d. 32

15. A square has its vertices at (1, 1), (1, 3), (3, 1) and (3, 3). It is reflected by a mirror line that runs through (3, 1) and (3, 3). Then, anchored at vertex (3, 3), the first image is magnified to form a second image whose area is quadruple that of the original. The second image is then rotated 90° clockwise using (1, 3) as the turn centre. What are the coordinates of the final image?

a. (1, 1), (-1, 3), (1, 7) and (-3, -1)

b. (1, 5), (5, 5), (1,9) and (5,9)

c. (1, 1), (-3, -3), (1, -3), and (-3,1)

d. (1,1), (1, -7), (-7,1) and (-7, -7)

Part II: Short Answers. Enter answers on answer sheet.

16. Find a positive integer n so that $\frac{6n+2013}{3n+2}$ is an integer.

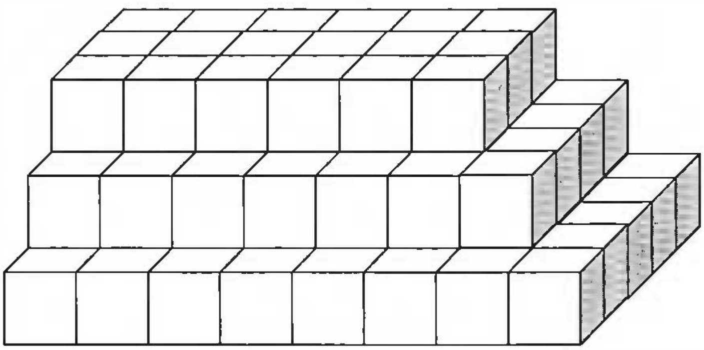
17. Emily earned some spending money by running a lemonade stand. She paid her mother 5% of what she earned for the supplies that she used. She spent $\frac{3}{5}$ of what was left on entertainment and then saved the remaining \$114. How much money did Emily pay her mom for supplies?

18. Laura, Svitlana and Robert deliver flyers. In one week, they delivered a total of 270 flyers. If Laura delivered 50% less flyers than Svitlana, and if Robert delivered 50% more flyers than Laura, how many flyers did Robert deliver?

19. The table at the right shows input and output values for a function machine. Find the missing output value.

Input (x)	Output (y)
-1	0
0	1.5
1	?
5	9
10	16.5
15	24

20. The solid 3-D object shown below is composed of layers of unit cubes. The top layer has 18 cubes, the middle layer has 28 cubes, and the bottom layer has 40 cubes. If the object is completely dipped in paint, how many unit cubes will have exactly two faces painted?



Print ID # __________ /90

School Name ____________________ Student Name

(Print First, Last)

2009 Edmonton Junior High Math Contest - Solution

Part I: Multiple Choice (PRINT neatly, use CAPITAL letters, 4 points each)

1. B	6. C	11. A
2. B	7. B	12. D
3. D	8. A	13. C
4. D	9. A	14. B
5. C	10. B	15. C

Part II: Short Answers (PRINT small but legible, 6 points each)

16. **13, 95 or 669** 17. **15** 18. **90** 19. **3** 20. **27**

Part I:	________ x 4 + (Correct)	________ x 2 = (Blank)	________ (be sure blanks ≤ 3)	MARKER ONLY
Part II:		______ x 6 = (Correct)	________	
		Total =	________ (enter total score on top)	

Instruction:

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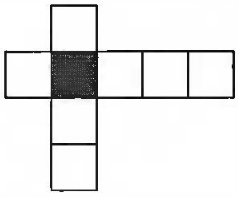
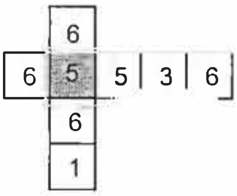
1. Each of the eight squares in the figure at the right must contain one digit. The sum of the eight digits is 38. Reading from top to bottom, the four digits in the vertical column form a number that is a power with 3 as its base. Reading from left to right, the five digits in the horizontal row form a number that is a power with 4 as its base.



What digit will be in the square of the shaded box?

a. 6
*b. 5
c. 3
d. 1

Solution:



Powers of 3: 3, 9, 27, 81, 243, 729, 2187, **6561**, 19683, …

Powers of 4: 4, 16, 64, 256, 1024, 4096, 16 384, **65 536**, 262 144, …

2. How many whole numbers from 10 to 60, inclusive, are divisible by both its units digit as well as its tens digit?

(E.g., Although 39 is divisible by 3, it is not divisible by 9. Therefore, 39 is not one of the numbers in the solution.)

a. ten numbers
*b. eleven numbers
c. seventeen numbers
d. eighteen numbers

Solution:

Solve by testing. The eleven numbers are circled below:

10	(11)	(12)	13	14	(15)	16	17	18	19	20
21	(22)	23	(24)	25	26	27	28	29	30	31
32	(33)	34	35	(36)	37	38	39	40	41	(42)
43	(44)	45	46	47	(48)	49	50	51	52	53
54	(55)	56	57	58	59	60				

3. Twenty people at Snow Lodge came either to ski or to snowboard. (*Nobody does both activities.*) The ratio of skiers to snowboarders is 3 to 2. The ratio of male skiers to female skiers is 5 to 1. The ratio of children snowboarders to adult snowboarders is 1 to 3. What is the sum of the number of male skiers and child snowboarders at Snow Lodge?

a. 6
b. 8
c. 10
* d. 12

Solution: 10 male skiers + 2 child snowboarders = 12 people

Test possibilities where the ratio of skiers to snowboarders = 3 to 2, until the number of people is 20.

Skiers	3	6	9	12
Snowboarders	2	4	6	8
Total people	5	10	15	20

There must be 12 skiers and 8 snowboarders.

Test possibilities where the ratio of male skiers to female skiers = 5 to 1, until the number of skiers is 12.

Male skiers	5	10
Female skiers	1	2
Total skiers	6	12

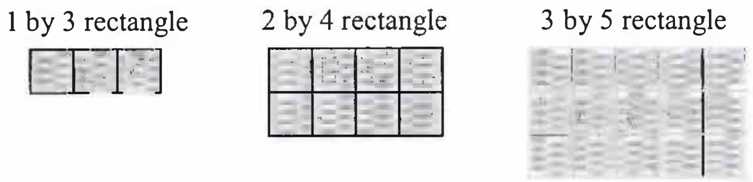
There must be 10 male skiers and 2 female skiers.

Test possibilities where the ratio of children to adult snowboarders is 1 to 3, until the number of snowboarders is 8.

child snowboarders	1	2
adult snowboarders	3	6
Total snowboarders	4	8

There must be 2 child snowboarders and 6 adult snowboarders.

4. Rectangles can be formed by joining unit squares. Some of the line segments are part of their perimeters and some are not. For the three examples shown below, their perimeters are, respectively, 8 units, 12 units, and 16 units. The number of line segments that are not part of their perimeters are, respectively, 2, 10, and 22.



If a 50 by 100 rectangle is formed with unit squares, how many line segments would **not** be part of the perimeter?

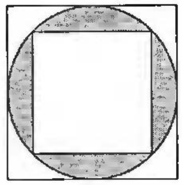
a. 4900
b. 4950

c. 9050
*d. 9850

Solution: Number of vertical segments = width × (length – 1)
Number of horizontal segments = (width – 1) × length

For a 50 by 100 rectangle:
- the number of vertical segments = 50 × 99 = 4950, and
- the number of horizontal segments = 49 × 100 = 4900, so
- the total number of unit segments = 4950 + 4900 = 9850.

5. In the figure at the right, the smaller square is inscribed in a circle, which is inscribed in a larger square with a side of 10 cm.
Approximately what area of the entire figure is **not** shaded?



a. 21.5 cm^2
b. 50 cm^2
*c. 71.5 cm^2
d. 78.5 cm^2

Solution: Unshaded area = Area of small square + (Area of large square – Area of circle)

Diameter of circle = side of large square = diagonal of small square = 10 cm

If side of small square = s, then $2s^2 = 10 \text{ cm} \times 10 \text{ cm} = 100 \text{ cm}^2$

Area of small square = $s^2 = 100 \text{ cm}^2 \div 2 = 50 \text{ cm}^2$

Area of large square = $10 \text{ cm} \times 10 \text{ cm} = 100 \text{ cm}^2$

If diameter of circle = 10 cm, then its radius = 5 cm

Area of circle = $25\pi \approx 25 \times 3.14 \approx 78.5 \text{ cm}^2$

Area of large square – Area of circle $\approx 100 \text{ cm}^2 - 25\pi$
$\approx 100 \text{ cm}^2 - 78.5 \text{ cm}^2$
$\approx 21.5 \text{ cm}^2$

Unshaded area $\approx 50 \text{ cm}^2 + 21.5 \text{ cm}^2 \approx 71.5 \text{ cm}^2$

6. The front wheel of a tricycle has a diameter of 25 cm; the rear wheels have a diameter of 15 cm. If the tricycle travels 100 m, about how many more rotations will the rear wheels make than the front wheel?

a. 212
b. 127
* c. 85
d. 31

Solution: Diameter of front wheel = 25 cm × $\pi \approx$ 78.5 cm
Diameter of rear wheel = 15 cm × $\pi \approx$ 47.1 cm
100 m = 10 000 cm

10 000 ÷ 78.5 ≈ 127 and 10 000 ÷ 47.1 ≈ 212

212 rotations – 127 rotations = 85 rotations

7. The year 1961 has 180° rotational symmetry because it reads the same when it is turned upside down. How many years since 1000 have 180° rotational symmetry?

a. 3
*b. 5
c. 8
d. 10

Solution: Numbers that have 180° rotational symmetry are 0, 1 and 8; also 6 becomes 9 and 9 becomes 6.

The first and last digits must be 1:
1001, 1111, 1691, 1881, 1961

8. Which of the numbers in the set { -50 , -1.5 , -1 , -0.2 , 0 , $\sqrt{2}$, $\frac{4}{5}$, π , 40 } are greater than their reciprocals?

* a. -0.2 , $\sqrt{2}$, π , 40
b. -50 , -1.5 , -1 , 0 , $\frac{4}{5}$
c. -0.2 , 0 , $\sqrt{2}$, π , 40
d. $\sqrt{2}$, $\frac{4}{5}$, π , 40

Solution:

Number (x)	-50	-1.5	-1	-0.2	0	$\sqrt{2}$	$\frac{4}{5}$	π	40
Reciprocal ($\frac{1}{x}$)	$-\frac{1}{50}$	$-\frac{2}{3}$	-1	-5	Undefined	≈ 0.7	$\frac{5}{4}$	≈ 0.3	$\frac{1}{40}$
Is $x > (\frac{1}{x})$?	no	no	no	yes	no	yes	no	yes	yes

9. On her last test, Crystal scored 92%, and in doing so, raised her average by 2%, to 82%. What percent must she get on her next test to raise her average one more percent, to 83%?

* a. 89%
b. 86%
c. 83%
d. 82%

Solution: If Josh raised his average by 2 points to 82%, then his previous average was 80%. If he previously wrote n tests, then the sum of his scores = 80n + 92.

$$\frac{(80n + 92)}{n + 1} = 82$$

$80n + 92 = 82 \times (n + 1)$
$80n + 92 = 82n + 82$
$80n + 10 = 82n$
$10 = 2n$
$5 = n$
His total scores on the 5 tests = 5 × 80 = 400

To raise his average to 83%, $(400 + 92 + m) \div 7 = 83$

$$\frac{492 + m}{7} = 83$$

$492 + m = 7 \times 83$
$492 + m = 581$
$m = 89$

10. Erhan has a pair of special dice. The six faces of each die are labelled 1, 4, 5, 7, 11, and 14. He rolls the pair of dice and finds the sum of the two numbers. What is the probability that the sum will be both even and less than 20?

a. $\frac{5}{12}$

* b. $\frac{1}{2}$

c. $\frac{5}{9}$

d. $\frac{5}{6}$

Solution: There are 36 combinations (possible outcomes). Of those sums, 18 are both even and less than 20.

+	1	4	5	7	11	14
1	E & <20		E & <20	E & <20	E & <20	
4		E & <20				E <20
5	E & <20		E & <20	E & <20	E & <20	
7	E & <20		E & <20	E & <20	E & <20	
11	E & <20		E & <20	E & <20		
14		E & <20				

Therefore, the probability = $\frac{18}{36} = \frac{1}{2}$

11. The tens digit of the square of an integer is 5. The ones digit

*a. must be 6

b. must be 4

c. can be 4 or 6

d. none of the above

Solution: The square of an integer can only end in 0, 1, 4, 5, 6 or 9. If the tens digit is 5 and the units digit is 1, 5 or 9, this would leave a remainder of 3 when divided by 4. If the tens digit is 5 and the units digit is 0 or 4, this would leave a remainder of 2 when divided by 4. This leaves the unit digit to be a 6, the square of 34 is one such number, 1156.

12. In the quadrilateral ABCD, $\angle ABC = \angle BCD = 90°$, $\angle CAB = 45°$ and $\angle CBD = 60°$. The diagonals AC and BD intersect at point E. The ratio of the areas of the triangles CDE and ABE is

a. $\sqrt{2}:1$

b. $\sqrt{3}:1$

c. 2:1

* d. 3:1

Solution:
Let AB = 1. Then BC = 1 and CD = $\sqrt{3}$. Since triangles CDE and ABE are similar, the desired ratio is $(\sqrt{3})^2\ to\ (1)^2$ or 3:1.

13. A small box of chocolate costs \$21 while a large box costs \$41. How many different combinations of small and large boxes cost exactly \$2009?

a. 1

b. 2

* c. 3

d. 4

Solution:

Let n be the number of small boxes and m be the number of large boxes. We know that m, n are nonnegative integers and

$$21n + 41m = 2009$$

Since 7 divides both 21 and 2009, 7 must divide 41m. Since 41 is prime, we get that 7 divides m. Thus m = 7a for some nonnegative integer a.

Since 41 divides both 41 and 2009, 41 must divide 21n. Since 41 is prime, we get that 41 divides n. Thus n = 41b for some nonnegative integer b. Replacing in the equation we get:

$$21(41b) + 41(7a) = 2009$$

Dividing by 287 = 41(7) we get:

$$3b + a = 7$$

Since a, b ≥ 0 and are integers, we get three solutions

a = 7, b = 0 implies n = 0 and m = 49
a = 4, b = 1 implies n = 41 and m = 28
a = 1, b = 2 implies n = 82 and m = 7.

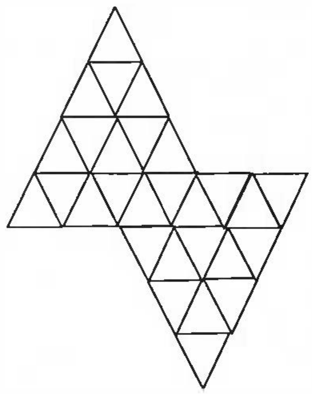
14. What is the total number of downward-pointing triangles in the diagram below? (Consider all sizes of triangles.)

a. 14

* b. 25

c. 28

d. 32



Solution: Triangles with 1-unit sides: 14
Triangles with 2-unit sides: 7
Triangles with 3-unit sides: 3
Triangles with 4-unit sides: 1
Total = 14 + 7 + 3 + 1 = 25

15. A square has its vertices at (1, 1), (1, 3), (3, 1) and (3, 3). It is reflected by a mirror line that runs through (3, 1) and (3, 3). Then, anchored at vertex (3, 3), the first image is magnified to form a second image whose area is quadruple that of the original. The second image is then rotated 90° clockwise using (1, 3) as the turn centre. What are the coordinates of the final image?

a. (1, 1), (-1, 3), (1, 7) and (-3, -1)

b. (1, 5), (5, 5), (1,9) and (5,9)

* c. (1, 1), (-3, -3), (1, -3), and (-3,1)

d. (1,1), (1, -7), (-7,1) and (-7, -7)

Solution:

When the square is first reflected, its new coordinates are (3, 1), (3, 3), (5, 1) and (5, 3).

When the new square is quadrupled, its coordinates are (3, 3), (3, -1), (7, 3) and (7, -1).

When the enlarged square is rotated, its coordinates are (1, 1), (-3,-3), (1, -3) and (-3,-1).

Short Answers:

16. Find a positive integer n so that $\frac{6n+2013}{3n+2}$ is an integer.

Answers: 13, 95 or 669

Solution:

$$\frac{6n+2013}{3n+2}=\frac{6n+4+2009}{3n+2}=2+\frac{2009}{3n+2}$$

$\frac{6n+2013}{3n+2}$ is an integer if and only if $\frac{2009}{3n+2}$ is an integer. Thus 3n+2 must divide 2009. The divisors of 2009 are 1, 7, 41, 49, 287 and 2009. Only 41, 287 and 2009 are of type 3n+2. Thus the values of n are 13, 95 and 669.

17. Emily earned some spending money by running a lemonade stand. She paid her mother 5% of what she earned for the supplies that she used. She spent $\frac{3}{5}$ of what was left on entertainment and then saved the remaining \$114. How much money did Emily pay her mom for supplies?

Answer: \$15.00

Solution:

$100\% - 5\% = 95\%$
$\frac{3}{5}$ of $95\% = 95\% \times \frac{3}{5} = 57\%$
$95\% - 57\% = 38\%$

c = cost of supplies (paid to her mom)
$$\frac{\$114}{38\%}=\frac{\$c}{5\%}$$

$c = (114)(5) \div 38$
$c = \$15$

Alternate Solution:

e = amount she earned
Cost of Supplies = $0.05e$
What was left = $e - 0.05e = 0.95e$
Spent on entertainment = $(0.6)(0.95)e = 0.57e$
Remaining profits (amount saved) = \$114

$e = 0.05e + 0.57e + 114$
$0.38e = 114$
$e = 114 \div 0.38 = \$300$
Cost of Supplies = $0.05e = (0.05)(300) = \$15$

18. Laura, Svitlana and Robert deliver flyers. In one week, they delivered a total of 270 flyers. If Laura delivered 50% less flyers than Svitlana, and if Robert delivered 50% more flyers than Laura, how many flyers did Robert deliver?

Answer: Robert delivered 90 flyers.

Solution: Let number of flyers delivered by Laura = n
Then Robert delivered $1.5n$ flyers, and Svitlana delivered $2n$ flyers

$n + 1.5n + 2n = 270$
$4.5n = 270$
$n = 270 \div 4.5 = 60$
Therefore, $1.5n = (1.5)(60) = 90$
Robert delivered 90 flyers.

19. The table at the right shows input and output values for a function machine. Find the missing output value.

Input (x)	Output (y)
-1	0
0	1.5
1	?
5	9
10	16.5
15	24

Answer: 3

Solution:

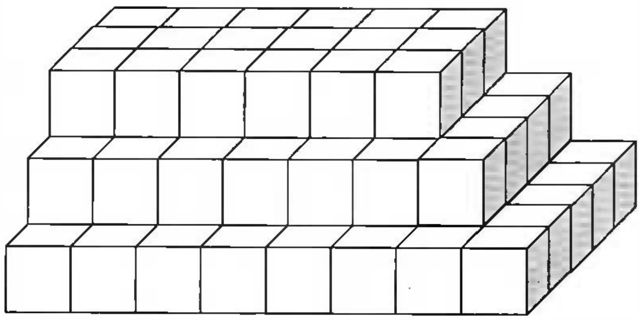
Input (x)	Output ($y - 1.5$)
-1	-1.5
0	0
1	?
5	7.5
10	15
15	22.5

Subtract 1.5 from each output value, and by inspection see that each new output value is 1.5 times the input value.

The equation is $y = 1.5\,x + 1.5$, or $y = (1.5)\,(x + 1)$

- So, the missing value (y) = $1.5 \times 1 + 1.5 = 1.5 + 1.5 = 3$ (or $y = (1.5)\,(x + 1) = (1.5)\,(1 + 1) = (1.5)\,(2) = 3$

20. The solid 3-D object shown below is composed of layers of unit cubes. The top layer has 18 cubes, the middle layer has 28 cubes, and the bottom layer has 40 cubes. If the object is completely dipped in paint, how many unit cubes will have exactly two faces painted?



Answer: 27

Solution:

Keeping in mind that the entire base of the 3-D object is painted:

- Cubes on bottom level with 2 painted faces: 9
- Cubes on middle level with 2 painted faces: 8
- Cubes on top level with 2 painted faces: 10
- Total = 9 + 8 + 10 = 27

THE CALGARY MATHEMATICAL ASSOCIATION

33rd JUNIOR HIGH SCHOOL MATHEMATICS CONTEST

April 22, 2009

NAME: SOLUTIONS
PLEASE PRINT (First name Last name)

GENDER: ☐ M ☐ F

SCHOOL: ______________________

GRADE: ______ (7,8,9)

- You have 90 minutes for the examination. The test has two parts: PART A — short answer; and PART B — long answer. The exam has 9 pages including this one.
- Each correct answer to PART A will score 5 points. You must put the answer in the space provided. No part marks are given.
- Each problem in PART B carries 9 points. You should show all your work. Some credit for each problem is based on the clarity and completeness of your answer. You should make it clear why the answer is correct. PART A has a total possible score of 45 points. PART B has a total possible score of 54 points.
- You are permitted the use of rough paper. Geometry instruments are not necessary. References including mathematical tables and formula sheets are **not** permitted. Simple calculators without programming or graphic capabilities **are** allowed. Diagrams are not drawn to scale. They are intended as visual hints only.
- When the teacher tells you to start work you should read all the problems and select those you have the best chance to do first. You should answer as many problems as possible, but you may not have time to answer all the problems.

MARKERS' USE ONLY	
PART A ×5	
B1	
B2	
B3	
B4	
B5	
B6	
TOTAL (max: 99)	

BE SURE TO MARK YOUR NAME AND SCHOOL AT THE TOP OF THIS PAGE.
THE EXAM HAS 9 PAGES INCLUDING THIS COVER PAGE.

Please return the entire exam to your supervising teacher at the end of 90 minutes.

PART A: SHORT ANSWER QUESTIONS

A1 What is the largest number of integers that can be chosen from $\{1,2,3,4,5,6,7,8,9\}$ such that no two integers are consecutive?

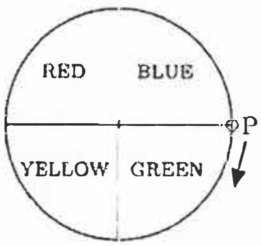
5

A2 Elves and ogres live in the land of Pixie. The average height of the elves is 80 cm, the average height of the ogres is 200 cm and the average height of the elves and the ogres together is 140 cm. There are 36 elves that live in Pixie. How many ogres live in Pixie?

36

A3 A circle with circumference 12 cm is divided into four equal sections and coloured as shown. A mouse is at point P and runs along the circumference in a clockwise direction for 100 cm and stops at a point Q. What is the colour of the section containing the point Q?

YELLOW



A4 What is the longest possible length (in cm) of a side of a triangle which has positive integer side lengths and perimeter 17 cm?

8

A5 A and B are whole numbers so that the ratio $A:B$ is equal to $2:3$. If you add 100 to each of A and B, the new ratio becomes equal to $3:4$. What is A?

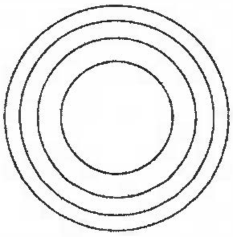
200

A6 You are given a two-digit positive integer. If you reverse the digits of your number, the result is a number which is 20% larger than your number. What is your number?

45

A7 In the picture there are four circles one inside the other, so that the four parts (three rings and one disk) each have the same area. The diameter of the largest circle is 20 cm. What is the diameter (in cm) of the smallest circle?

10

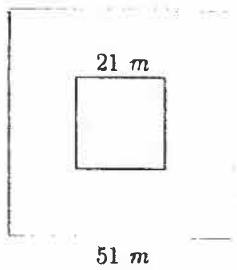


A8 Carol's job is to feed four elephants at the circus. She receives a bag of peanuts every day and feeds each elephant as many peanuts as she can so that each elephant receives the same number of peanuts. She then eats the remaining peanuts (if any) at the end of the day. On the first day Carol receives 200 peanuts. On every day after, she receives one more peanut than she did the previous day. This was done over 30 days. How many peanuts did Carol eat over the 30 days?

43

A9 Richard lives in a square house whose base has dimension 21 m by 21 m and is located in the centre of a square yard with dimension 51 m by 51 m as in the diagram. Richard is to tie one end of a leash to his puppy and the other end of the leash to a corner of his house so that the puppy can reach all parts of the yard. What is the smallest length (in m) of a leash so that this can be done?

60



PART B: LONG ANSWER QUESTIONS

B1 Ella and Bella each have an integer number of dollars. If Ella gave Bella enough dollars to double Bella's money, Ella would still have \$100 more than Bella. In fact, if Ella instead gave Bella enough dollars to triple Bella's money, Ella would still have \$40 more than Bella. How much money does Ella have?

Solution:

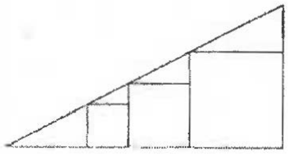
Let y be the number of dollars Bella has. The key observation is that Ella gave Bella y more dollars in the second scenario than in the first scenario. This accounts for the difference of \$60 between how much more Ella has than Bella in the two scenarios. In the second scenario, Ella has y dollars less and Bella has y dollars more than in the first scenario. Hence, the difference between what Ella and Bella have decreases by $2y$. Therefore, $2y = 60$, which means $y = 30$. Therefore, Bella has \$30 originally.

In the first scenario, Ella gave Bella \$30, resulting in Bella having \$60. Since Ella has \$100 more than Bella after this exchange, Ella has \$160 after this exchange. Before the exchange, Ella had \$160 + \$30 = \$190.

The answer is \$190.

Comment: This problem can also be solved by guess and check.

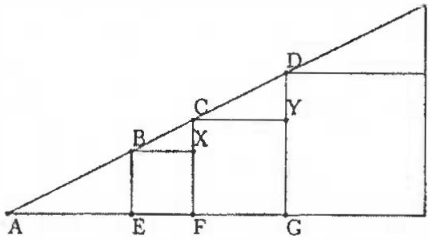
B2 Three squares are placed side-by-side inside a right-angled triangle as shown in the diagram.



The side length of the smallest of the three squares is 16 cm. The side length of the largest of the three squares is 36 cm. What is the side length (in cm) of the middle square?

Solution:

Let $A, B, C, D, E, F, G, X, Y$ be the points labeled in the diagram.



Then $\triangle ACF$ is similar to $\triangle ADG$. This means the ratio of the sides of $\triangle ACF$ to the sides of $\triangle ADG$ is equal to the ratio of a side of the smallest square and the middle square, since the smallest square is inscribed in $\triangle ACF$ and the middle square is inscribed in $\triangle ADG$. Therefore,

$$\frac{CF}{DG} = \frac{\text{Side length of smallest square}}{\text{Side length of middle square}} = \frac{16}{CF}.$$

We know that $DG = 36$, since it is the side of the largest square. We now have the equation

$$\frac{CF}{36} = \frac{16}{CF}.$$

By cross-multiplying, we have that $CF^2 = 36 \times 16 = 576$. By square rooting both sides (and noting that we only need the positive solution), we get that $CF = \sqrt{576} = 24$.

Alternate Solution:

Let x be the side length of the middle square. Note that $\triangle BCX$ and $\triangle CDY$ are similar. BX has length 16 since it is a side of the smallest square. CX has length $x - 16$ since it is the difference between a side of the middle square and a side of the smallest square. CY has length x and DY has length $36 - x$ since it is the difference between a side of the largest square and a side of the middle square. By similar triangles,

$$\frac{BX}{CX} = \frac{CY}{DY}, \text{ which means } \frac{16}{x-16} = \frac{x}{36-x}.$$

By cross-multiplying, we get that $16(36-x) = x(x-16)$, which simplies to $576-16x = x^2 - 16x$. The $16x$ term on both sides cancel and we get $576 = x^2$. Therefore, $x = \sqrt{576} = 24$.

The answer is 24 cm.

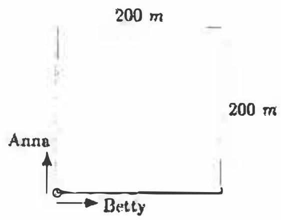
B3 Friends Maya and Naya ordered finger food in a restaurant, Maya ordering chicken wings and Naya ordering bite-size ribs. Each wing cost the same amount, and each rib cost the same amount, but one wing was more expensive than one rib. Maya received 20% more pieces that Naya did, and Maya paid 50% more in total than Naya did. The price of one wing was what percentage higher than the price of one rib?

Solution:

Suppose Naya ordered n ribs. Then Maya ordered $1.2n$ wings. Suppose Naya paid N dollars altogether. Then Maya paid $1.5N$ dollars altogether. The cost per rib was N/n dollars. The cost per wing was $1.5N/(1.2n) = (5/4)(N/n) = 1.25(N/n)$ dollars. Therefore Maya's cost per wing was 25% more than Naya's cost per rib.

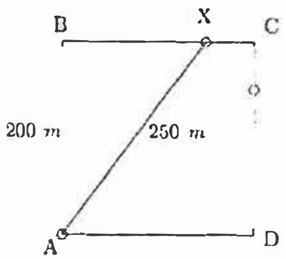
The answer is 25%.

B4 There is a running track in the shape of a square with dimensions 200 metres by 200 metres. Anna and Betty run around the track starting from the same corner of the track at the same time, each at a constant speed, but in different directions on the track. Anna runs at 6.3 kilometres per hour. Anna and Betty meet on the track for the first time after starting, at a point whose straight-line distance from the starting point is 250 metres. What are Betty's possible speeds in kilometres per hour?



Solution:

We label the corners of the track A, B, C, D in a clock-wise direction with A being the starting corner. Suppose Anna is running in the clockwise direction and Betty is running in the counter-clockwise direction. There are two possible locations on the track that are located 250 m from A. One of these locations is on segment BC and the other is on segment DC.



In the case where this meeting location is on segment BC, let X be this meeting location. Then $AX = 250$ m and $AB = 200$ m. By the Pythagorean Theorem, $BX = \sqrt{250^2 - 200^2} = \sqrt{50^2(5^2 - 4^2)} = 50\sqrt{9} = 150$ m. This means Anna ran from A to B to X before meeting Betty, which means Anna ran 200 m+150 m= 350 m. Betty ran the remainder of the track, which is $AD + DC + CX = 200$ m+200 m+50 m= 450 m. Therefore, the ratio of Anna's speed to Betty's speed is $350 : 450 = 7 : 9$. Since Anna runs at 6.3 km/hr, if we let x be Betty's speed in km/hr, we get that

$$\frac{7}{9} = \frac{6.3}{x}.$$

By cross-multiplying, we get $7x = 6.3 \times 9$. This means $x = 0.9 \times 9 = 8.1$.

The other case is where the meeting location is on segment DC. This is when Anna runs faster than Betty. By symmetry, the ratio of Anna's speed to Betty's speed is $9 : 7$. Since Anna runs at 6.3 km/hr, we get that

$$\frac{9}{7} = \frac{6.3}{x}.$$

By cross-multiplying, we get that $9x = 6.3 \times 7$. This means $x = 0.7 \times 7 = 4.9$.

Therefore, Betty's possible speeds are 4.9 km/hr and 8.1 km/hr.

The answers are 4.9 km/hr and 8.1 km/hr.

B5 Adrian owns 6 black chopsticks, 6 white chopsticks, 6 red chopsticks and 6 blue chopsticks. They are all mixed up in a drawer in a dark room.

(a) (4 points) He wants to get four chopsticks of the same colour. How many chopsticks must he grab to be guaranteed of this? Show that fewer chopsticks than your answer might not be enough.

Solution:

If Adrian picks 12 chopsticks, it is possible that Adrian has 3 chopsticks of each colour and therefore does not have 4 chopsticks of any colour.

If Adrian does not have four chopsticks of any one colour, then the maximum number of chopsticks that Adrian can pick is (3 chopsticks per colour) $\times$ (4 colours) $=$ 12. Therefore, if Adrian picks 13 chopsticks, Adrian must have 4 chopsticks of one colour.

The answer is 13 chopsticks.

(b) (5 points) Suppose instead Adrian wants to get two chopsticks of one colour and two chopsticks of another colour. How many chopsticks must he grab to be guaranteed of this? Show that fewer chopsticks than your answer might not be enough.

Solution:

If Adrian picks 9 chopsticks, it is possible that Adrian does not have two chopsticks of one colour and two chopsticks of another colour. An example of this is if Adrian picks up 6 black chopsticks and 1 chopstick of each of the other three colours.

If Adrian does not have two chopsticks of one colour and two chopsticks of another colour, then there are three colours such that Adrian has at most one chopstick of cach of these three colours. Since there are six chopsticks of the fourth colour, the maximum number of chopsticks that Adrian can pick is $6 + 1 + 1 + 1 = 9$. Therefore, if Adrian picks 10 chopsticks, Adrian must have two chopsticks of one colour and two chopsticks of another colour.

The answer is 10 chopsticks.

B6 The numbers 2 to 100 are assigned to ninety-nine people, one number to each person. Each person multiplies together the largest prime number less than or equal to the number assigned and the smallest prime number strictly greater than the number assigned. Then the person writes the reciprocal of this result on a sheet of paper.

For example, consider the person who is assigned number 9. The largest prime less than or equal to 9 is 7. The smallest prime strictly greater than 9 is 11. So this person multiplies 7 and 11 together to get 77. The person assigned number 9 then writes down the reciprocal of this answer, which is $\frac{1}{77}$.

(a) (3 points) Which people write down the number $\frac{1}{77}$ (one of these people is person #9)? Show that the sum of the numbers written down by these people is equal to $\frac{1}{7} - \frac{1}{11}$.

Solution:
Since $77 = 7 \cdot 11$ and 7 and 11 are two consecutive primes, the people that wrote down the number $\frac{1}{77}$ are those who are assigned numbers $7, 8, 9$ and 10.

Hence, there are four people that wrote down the number $\frac{1}{77}$. Therefore, the sum of the numbers written down by these four people is $\frac{4}{77}$. Note that

$$\frac{1}{7} - \frac{1}{11} = \frac{11}{77} - \frac{7}{77} = \frac{4}{77}.$$

Therefore, the sum of the numbers written down by these four people is indeed $\frac{1}{7} - \frac{1}{11}$.

(b) (6 points) What is the sum of all 99 numbers written down? Express your answer as a fraction in lowest terms.

Solution:
Note that everyone writes down a number of the form $\frac{1}{pq}$ where p, q are two consecutive prime numbers. Given a pair of consecutive prime numbers p, q, with $p < q$, the people that write down the number $\frac{1}{pq}$ are the people assigned numbers $p, p+1, p+2, \cdots, q-1$. Therefore, there are $q - p$ people that write down $\frac{1}{pq}$. The sum of the numbers written by these people is $\frac{q-p}{pq}$.

Using the same idea as in (a), we can show that the sum of all of these numbers is indeed $\frac{1}{p} - \frac{1}{q}$, since

$$\frac{1}{p} - \frac{1}{q} = \frac{q-p}{pq}$$

which is the sum of the numbers written down by the people who write down $\frac{1}{pq}$.

Since 2 is the first prime and the two largest consecutive prime numbers used in the problem are 97 and 101, the set of all pairs of consecutive primes from 2 to 101 are $(2,3), (3,5), (5,7), \cdots, (89,97), (97,101)$. Therefore, the sum of all 99 numbers is

$$\left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + \cdots + \left(\frac{1}{89} - \frac{1}{97}\right) + \left(\frac{1}{97} - \frac{1}{101}\right).$$

The intermediate terms cancel. This sum is then equal to

$$\frac{1}{2} - \frac{1}{101} = \frac{101}{202} - \frac{2}{202} = \frac{99}{202}.$$

The answer is $\frac{99}{202}$.

BOOK REVIEW

Literature-Based Teaching: Prompting New Mathematical Experiences

Gladys Sterenberg

In her new book, *Math Memories You Can Count On,* Jo-Anne Lake describes a literature-based approach to teaching mathematics in primary classrooms. She begins by presenting stories of preservice teachers' experiences with mathematics. Not surprisingly, these memories (as she calls them) demonstrate the correlation between mathematics attitude and achievement. In Chapter Two, she emphasizes the importance of creating "optimal mathematics-learning environments" (p 13) through problem solving and communication and claims that using children's literature helps create inquiry-style instruction. Literature suggestions for developing problem-solving and communication skills are offered.

The next three chapters describe how new mathematics memories can be built through a literature-based approach to teaching mathematics. Here, Lake uses specific literature examples to show ten benefits of using mathematics-related books, to explain her nine criteria for selecting and organizing the literature, and to link literature with manipulatives. She focuses on reinforcing big ideas of mathematics using literature through problem-solving contexts.

The notion of planning from big ideas is not new. However, Lake presents big ideas as key concepts made up of one or two words. For example, *place value* is listed as a big idea. This is problematic, because place value is a topic. One big idea related to place value would be that "the place value system we use is built on patterns to make our work with numbers more efficient" (Small 2009, 15). It might be more helpful to teachers if the big ideas were listed as complete concepts, not topics.

Also in this section, I noticed that the format of book descriptions is not consistent with those of the first two chapters and those listed in Chapter Seven. I'm not sure why Lake chose a variety of formats, and I feel that a more systematic listing consistent with that of Chapter Seven might have increased the book's readability.

Chapter Six contains an extensive description of assessment and evaluation strategies that include observation, performance assessment, peer and self-assessment/checklists, conference/interview, rubrics, portfolios, and math journals. While valuable information is given in this chapter, it seems disconnected from the rest of the book, and no direct link is made to using literature.

Chapter Seven is perhaps the most useful chapter for teachers. Here, Lake presents seven steps to implementing a literature-based approach. Organizing her material around five strands of mathematics (number sense and numeration, measurement, geometry and spatial sense, patterning and algebra, and data management and probability), the author provides examples of how topics (or *big ideas*, as she calls them) can be addressed through literature. For each strand, she describes fifteen books by including a summary, the connecting strand, a read-and-discuss section that lists the book and related prompt for discussion, related questions, and a sample activity. In addition, each book is correlated to mathematical processes and manipulative ideas. Some of the books listed are more appropriate for intermediate classrooms, but the inclusion of these is valuable as teachers differentiate their instruction.

Overall, this is a very good book for teachers. It is rare for a book focused on using literature in mathematics classrooms to include a Canadian focus, which is especially evident in the measurement strand. Appendices for children's and professional literature are extensive and relevant for primary classroom teachers. Lake has included templates for graphic organizers used in planning to implement a literature-based approach to teaching mathematics.

For me, the title of the book remains ambiguous. I would like to have had evidence that the experience of using literature can prompt new mathematical memories. The inclusion of teachers' or students' stories would have provided more clarity on the effect of using literature in classrooms.

This is not a teaching resource that provides a quick reference of books and teaching strategies. By embedding the book suggestions within the context of the various chapters, Lake demonstrates how she thinks literature can inform the teaching of mathematics. This book may provide guidance for the beginning teacher in planning for the use of literature in mathematics classrooms. However, for the more experienced teacher, a resource that expands on the examples presented in Chapter Seven might be more useful.

References

Lake, J. 2009. *Math Memories You Can Count On.* Markham, Ont: Pembroke.

Small, M. 2009. *Big Ideas from Dr. Small: Creating a Comfort Zone for Teaching Mathematics, Grades 4–8.* Scarborough, Ont: Nelson.

Gladys Sterenberg, the editor of delta-K, *is an assistant professor in mathematics education at the University of Alberta. She is interested in alternative ways of knowing mathematics, particularly through story.*

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Barnett House
11010 142 Street NW
Edmonton AB T5N 2R1