An Alberta Rose in the New Curriculum?

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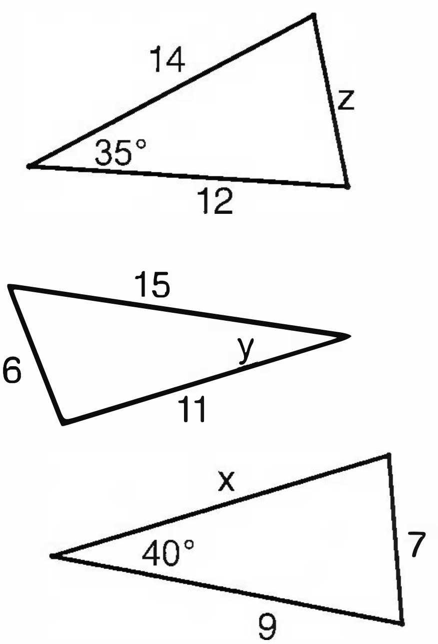
Over the years I have had occasion to look at the mathematics curricula for different provinces of Canada. It is not surprising that the expectations for student learning are often similar and the regional differences subtle. This observation is consistent with the implication of the NCTM [National Council of Teachers of Mathematics] standards document (NCTM 2000) that there is broad consensus about the key topical areas of mathematics. This article examines one small difference that may help Alberta teachers make connections between the cosine law and quadratics in Mathematics 20-1 (Alberta Education 2008) that teachers in Ontario (and, doubtless, other jurisdictions that aren't home to the author) would find difficult to make.

The cosine law has three types of questions, shown in Figure 1. Usually, only the first two shown—finding the angle given the three side lengths and finding the length of the side opposite a given angle that is between two side lengths—are used when teaching the cosine law. It is not unusual for teachers to omit having the unknown side adjacent to the given angle for two reasons: first, it can be solved using a double application of the sine law and, second, if the cosine law is used it generates a quadratic equation. The purpose of this article is to demonstrate that using the cosine law to develop the quadratic case has benefits. The equivalent method of applying the sine law twice is also explored to see what trigonometric interpretations of the quadratic formula might come to light.

Figure 1: Cosine law questions

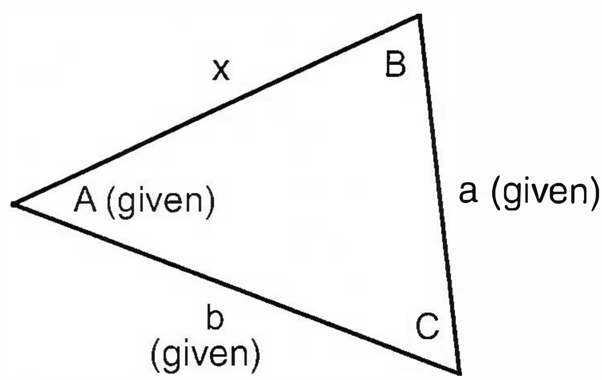


The curriculum distinction that allows this approach in Alberta, but not as easily in Ontario, arises because the quadratic equation has two solutions. The double application of the sine law generates two solutions when the ambiguous case of the sine law arises. It is this last point that is an option in Alberta, whereas in Ontario, the ambiguous case of the sine law is taught a year after the sine law for acute angles, the cosine law and quadratics are introduced. While Ontario teachers could re-examine quadratics after the ambiguous case is discussed, it would be after the horse has left the barn. In Alberta, however, the option is an interesting possibility because the ambiguous case is included when the sine law, cosine law and quadratics are introduced.

The Cosine Law

The following general analysis is based on Figure 2, in which x is the unknown and A, a, and b are specified values. The general derivation enables the teacher to choose values suitable to teaching, which will be considered later in this article. We start by examining the cosine law approach and then examine the double application of the sine law.

Figure 2: General setup



Setting up the cosine law using the given values and rearranging to highlight the quadratic in x leads to

$$x^2 - 2bCos(A)x + (b^2 - a^2) = 0$$

Using the quadratic formula, this can be solved directly as

$$x = \frac{2bCos(A) \pm \sqrt{4b^2Cos^2(A) - 4(b^2 - a^2)}}{2}$$

The common factor of 4 is extracted from the square root, leading to

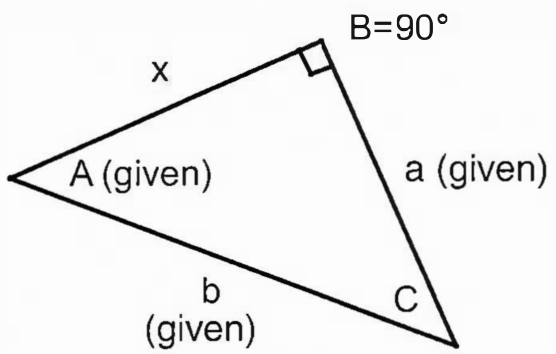
$$x = bCos(A) \pm \sqrt{b^2Cos^2(A) - b^2 + a^2}$$

Under the square root, the first two terms have a common factor that leads to a trigonometric simplification: $b^2(Cos^2(A) - 1) = -b^2Sin^2(A)$. This gives

$$x = bCos(A) \pm \sqrt{a^2 - b^2Sin^2(A)}$$

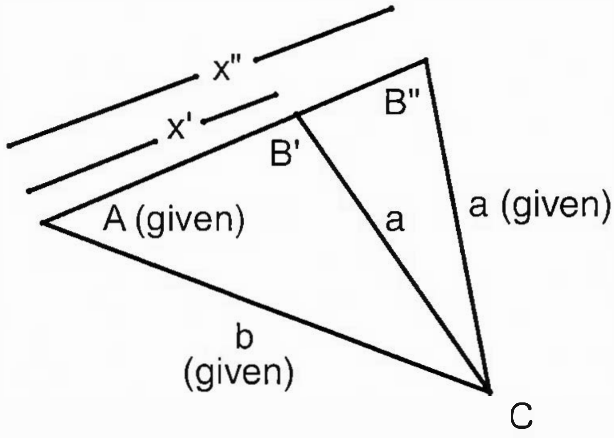
The discriminant, which appears under the square root, indicates whether the quadratic equation has two, one, or no real solutions. In the form rendered here, it has a value of zero, indicating a single solution to the quadratic, if side a has a length such that sin(A) = a / b—in other words, if the triangle shown in Figure 3 has angle B=90°. This is the solution that corresponds to the minimal value of a that allows a solution. When side a is made shorter than the right angle distance (ie, the minimal a distance), it is not long enough to connect the vertex to the other side and the triangle cannot physically be constructed.

Figure 3: The minimal *a* solution



If a is made longer, as shown in Figure 4, the discriminant is positive and there are two solutions corresponding to a being rotated about point C and intersecting at two values x = x' and x = x". These two solutions correspond to acute (ie, B") and obtuse (ie, B') values of the supplementary angles such that sin(B") = sin(B'). This is a much more satisfying interpretation of the discriminant than algebraic interpretations because it gives a geometric meaning.

Figure 4: Case of a positive discriminant



This analysis shows that the cosine law generates a quadratic where the discriminant has a geometric interpretation. The geometric interpretation is consistent with whether or not the triangle can be constructed. However, it requires an understanding of the ambiguous case of the sine law. Specifically, it requires that students recognize that sin(B) = sin(180° – B).

Double Application of the Sine Law

If the sine law is applied to Figure 2 twice, then the acute and obtuse solutions must be considered. In the following section, B is taken to be acute, thereby defining the acute case, and the obtuse case uses the supplementary angle 180° – B. (Since there are two solutions, some readers may find Figure 4 useful for visualizing both cases simultaneously, provided that one remembers that B" and B' are supplementary.)

$$\frac{Sin(A)}{a} = \frac{Sin(B)}{b} \qquad \frac{Sin(A)}{a} = \frac{Sin(180° - B)}{b}$$

Using the given A, a, and b allows one to solve for angle B. This is then used to find angle C so that a second application of the sine law can determine x. In the acute case, C = 180° – A – B, while in the obtuse case C = 180° – A – (180° – B) or C = B – A. Note that in the acute case, sin(180° – A – B) = sin(A+B) so that C = A+B can be used for the argument of the sine function, and it is this form that is used below.

Applying the sine law again with these two possible C values across from x gives

$$\frac{Sin(A)}{a} = \frac{Sin(A+B)}{x} \qquad \frac{Sin(A)}{a} = \frac{Sin(B-A)}{x}$$

Rearranging this gives

$$x = \frac{aSin(A+B)}{Sin(A)} \qquad x = \frac{aSin(B-A)}{Sin(A)}.$$

These are the two solutions to the quadratic. On one hand, this is so very different from the quadratic formula you wonder if you have lost your way; on the other hand, it is interesting to see that the ± component of the quadratic formula has been reduced to the sum and difference of angles.

A short proof that this result is consistent with the quadratic formula

$$x = bCos(A) \pm \sqrt{a^2 - b^2 Sin^2(A)}$$

is now provided. This would not be used in teaching, but is provided so that teachers can have confidence in the result. The method of proof is to take the trigonometric solution, use the sum and difference of angles formulas, and interpret the resulting algebraic form geometrically. Following this approach,

$$x = \frac{aSin(B \pm A)}{Sin(A)}$$

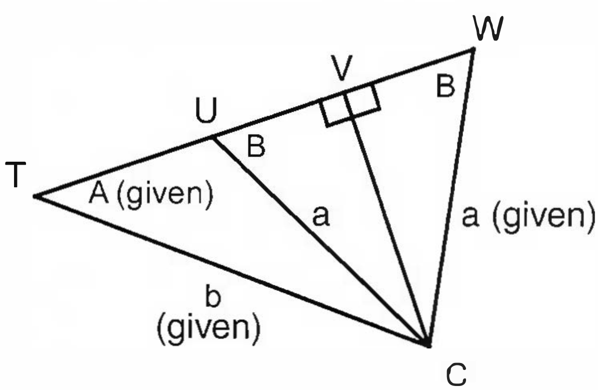
$$= \frac{aSin(B)Cos(A) \pm aCos(B)Sin(A)}{Sin(A)}$$

$$= \frac{aSin(B)Cos(A)}{Sin(A)} \pm aCos(B)$$

It is the last algebraic form that is interpreted geometrically using Figure 5. This figure is essentially the same as Figure 4, but has had letters added to the four vertices at the top as well as a perpendicular line segment VC. The two terms are considered separately by replacing the trigonometric terms with the equivalent ratios of sides. The first term is

$$\frac{aSin(B)Cos(A)}{Sin(A)} = \frac{a\left(\frac{VC}{a}\right)\left(\frac{TV}{b}\right)}{\left(\frac{VC}{b}\right)} = TV.$$

Figure 5: Geometric interpretation



The second term, aCos(B), is equal in magnitude to VU and VW. The formula as a whole shows that x = TV – VU = TU or x = TV + VW = TW.

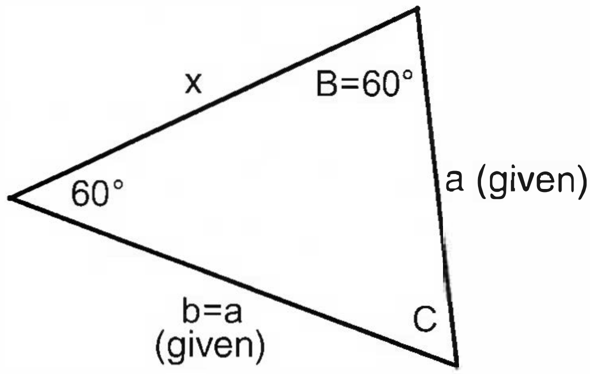
Instructional Examples

In terms of instruction, the connection highlighted here is considerable and many teachers may be hesitant to develop the connection explicitly. A useful option is to use specific instructional cases that are developed from the theory, without explicit instruction about the theoretical aspect. A few examples are provided to whet the reader's instructional appetite.

As a first example (shown in Figure 6), if $a = b$, then the constant term in the quadratic formula is zero. In addition, if A = 60° so that cos(A) = ½, the quadratic equation becomes

$$x^2 - bx = 0.$$

Figure 6: Instructional example.



This has solutions $x = 0$ and $x = b$, which correspond geometrically to a triangle with side length zero (ie, a line segment) and to the equilateral triangle.

As a second example, if A = 30°, $b = 2\sqrt{3}$ and $a = 2$, then the quadratic equation becomes

$$x^2 - 6x + (12-4) = 0 \quad \text{or} \quad x^2 - 6x + 8 = 0.$$

This can be factored, drawn as a geometric triangle and solved using the sine law. To confess, I used a pattern in which for some value of k, A = 30°, $b = k\sqrt{3}$ and $a = k$ gives

$$x^2 - 3kx + 2k^2 = 0, \quad \text{leading to} \quad (x-k)(x-2k) = 0$$

for whatever value of k one chooses.

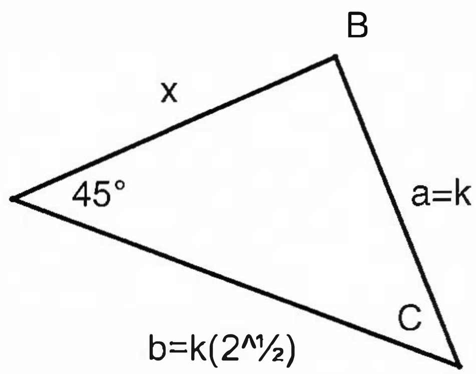
Using k again to highlight a family of solutions, one can choose A = 45°, $b = k\sqrt{2}$ (or numerical approximation), and $a = k$, as shown in Figure 7, to get

$$x^2 - 2kx + k^2 = 0 \quad \text{leading to} \quad (x-k)(x-k) = 0.$$

This has instructional value because the Pythagorean theorem can be used to show, geometrically, that there is only one solution. Specifically, in the absence of knowing angle B, a solution for x can be found under the assumption that B is 90°. The solution is then observed to be unique, which implies that this is the minimal a solution and therefore B is a right angle. There is no need to use the sine law for this particular example, although it would support student understanding of the logic used.

Last, examples with no solution can be generated by making a smaller than the minimal value provided by bSin(A). Using Figure 7, one could use A = 45°, $b = 2\sqrt{2}$ and $a = 1$.

Figure 7: A Pythagorean example



Conclusion

The utility of this overall approach is that it can be used as a demonstrative bridge that brings together the algebra of quadratics and geometry of triangles. It facilitates multiple instructional approaches—some teachers may simply want to use instructional examples that support the connection, while other teachers may explore the connection as a pedagogical approach. In either case, there is a geometric rationale for the number of solutions to a quadratic equation that rivals the algebraic completion of the square. The only caveat in the entire approach is that it is instructional and not a replacement for factoring or the quadratic formula. Rather, its value is in the introduction of the quadratic and factoring using trigonometry that may provide a much deeper connection than other instructional approaches.

References

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