Digging Deeper into Fraction Addition

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Teaching mathematics methods courses to K–8 teacher candidates over a period of 20 years has led me to many observations. One of them concerns fractions. K–8 teacher candidates' conceptual understanding of fractions and their fraction algorithmic skills are weak. These observations are in line with formal research on the matter (Reeder and Utley 2007; Weller, Arnon and Dubinsky 2009; Ma, 1999; Ball 1990).

Reeder and Utley (2007) summarize the situation well.

> This study revealed that this group of prospective elementary teachers brought with them a limited understanding of fractions to their mathematics methods courses. Although these prospective teachers had many years of school mathematics ... their reasoning about simple fraction concepts was often incorrect and based heavily on misconceptions they had previously developed or, at best, understandings of fractions as part of a whole. (p 249)

The research literature also indicates that middle-years students have similar difficulties understanding fractions and doing fraction arithmetic (Clarke, Roche and Mitchell 2007). One difficulty common to teacher candidates and middle-years students concerns fraction addition. There seems to be some tendency to add numerators to numerators and denominators to denominators. The edited version of a question submitted by a middle-years student to the website Ask Dr. Math illustrates this (the author of this article is a "math doctor" for that site).

The student's question:

> How do you solve: $2/5 + 3/8 + 1/4 + 140 = x$.
> My thoughts: $6/17 + 140 = x$.

Why might some middle-years students and teacher candidates add fractions in the way indicated in the question submission? There seem to be two overall explanations for this: (1) fractions are not understood and thus are treated as whole numbers and (2) fractions are partially understood and particular life situations seem to support an incorrect addition algorithm. This article concerns the second reason in the context of working with middle-years teacher candidates.

The article also concerns teaching for conceptual understanding. One of the goals of my middle-years mathematics methods course is to improve the teacher candidates' conceptual understanding (of fractions, in this case) while at the same time encouraging the development of question/problem posing as an important teaching strategy. Teaching for conceptual understanding relies on posing questions and problems, and using instructional materials (eg, diagrams and concrete materials). The National Council of Teachers of Mathematics (NCTM) (2000) notes that the mathematical discussion involved in question/problem posing is important for developing conceptual understanding because students become active participants in the development of mathematical ideas (Barlow and Cates 2006).

Fraction Addition with Middle-Years Teacher Candidates

I begin the session on fraction addition by telling the following anecdote from an age gone by (when there still was abundant hair on my head). While I was teaching fraction arithmetic to Grade 7 students, one of them challenged me with this situation.

> You taught us to add fractions by the common denominator method. But that doesn't work for my hockey team. We won 5 out of 6 home games and 1 out of 4 away games. Altogether we won 6 out of 10 games. Saying this with fractions, my team won 5/6 of our home games and 1/4 of our away games. Altogether, we won 6/10 of our games. So, $5/6 + 1/4$ is the total fraction of games won. The correct answer of 6/10 is figured out by doing 5 add 1 over 6 add 4. If you use the common denominator method to add, you don't get 6/10. The common denominator method is wrong.

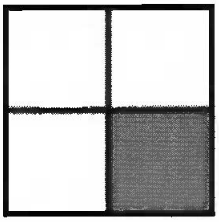
The incident points out to the teacher candidates why some Grades 7 and 8 students might be resistant to the common denominator method for adding fractions. My central purpose, though, is to probe deeply into fraction addition, an experience that inevitably generates controversy and thinking that goes beyond the conventional and relatively comforting world of using models (eg, a pie model) to develop fraction addition.

Developing the Part-of-a-Whole and Part-of-a-Set Meanings of Fraction

Before we consider the anecdote, two core meanings of fraction (part of a whole and part of a set) are developed in depth. That development proceeds by presenting the teacher candidates with a situation for each meaning (see Figures 1 and 2) and then, by question posing, extracting the critical features of each meaning.

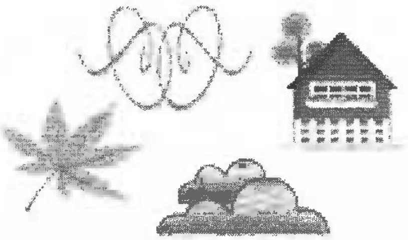
The questions concern wholes, cutting into parts, equality of parts, collections of things and attributes. My opening question for situation 1 is "Why can we say that the shaded rectangle is ¼ of the outside rectangle?"

Figure 1: Situation for part-of-a-whole meaning



The opening question for situation 2 is "There is no cutting into equal parts. Why can we say that ¼ of the pictures shows a house?"

Figure 2: Situation for part-of-a-set meaning

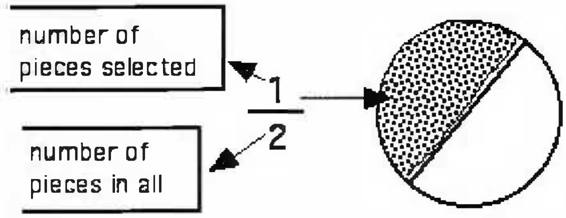


What follow are the critical features of the two fraction meanings that are uncovered by question posing and subsequent discussion.

Part-of-a-Whole Meaning of Fraction

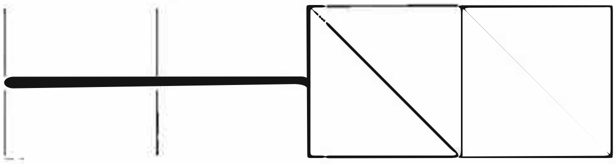
When a 5-year-old child says "I ate half the cookie," he/she is expressing a part–whole relationship. The child uses *half* not in the sense of a number but in the sense of an actual or imagined action that involves cutting a whole physical object in the middle. The imagined or actual action of cutting a whole object into n equal parts (according to a measurement concept such as length, area, volume and so on) underlies the part-of-a-whole meaning of fraction. We represent each part symbolically by the fraction notation $1/n$ (refer to Figure 3). The whole is typically a naturally existing thing. In other words, the whole is a conventional object such as an apple, rectangle, pie, loaf of bread and so on. The whole is not perceived as a collection of discrete objects.

Figure 3: Part of a whole (parts looking the same)



In Figure 3, the two pieces of the circle are equal in area and happen to look the same. This does not need to be the case. Consider a rectangle cut in the way shown in Figure 4. The eight pieces do not all look the same. Yet each piece is 1/8 of the rectangle in size because the pieces have the same area.

Figure 4: Part of a whole (parts not looking the same)



The two critical features of the part-of-a-whole meaning of fraction are that

- a natural whole exists, such as a piece of rope; and
- the whole is cut into equal parts according to size where *equal according to size* involves measurement (eg, length, area, volume, mass and so on). In the case of the piece of rope, it would be cut into sections of equal length.

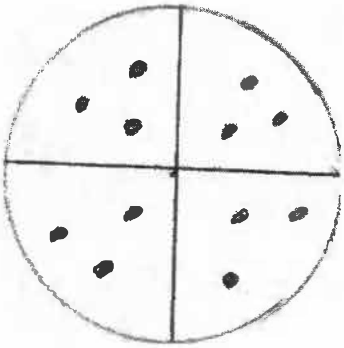
Part-of-a-Set Meaning of Fraction

The part-of-a-set meaning does not involve cutting a natural whole into equal parts—it involves selecting objects from a collection of discrete objects according to some attribute (eg, colour, being a student, made of glass). A set (or group) is not a naturally occurring whole as is a pie.

Suppose there are 23 books of varying size and content on a shelf and 14 of them are novels. We can represent this situation by the fraction 14/23. For this situation, we mean that 14 out of the 23 books are novels. The part-of-a-set meaning involves placing discrete things into categories for which the requirement is *belonging to*, not *equality of size*. This is a different enterprise than cutting up a whole object into parts of equal size based on a measurement concept.

Equal sharing is a special case of the part-of-a-set meaning. Suppose Mary receives a share of 12 candies shared equally among 4 people (including Mary). Mary's share is 3/12 (or 1/4). To some, equal sharing may seem like part of a whole (especially when a circle cut into 4 equal parts is used as the underlying prop for the sharing—refer to Figure 5), but it is not. The attribute is *equality of count* of discrete things (not equal parts of a whole). Counting is not equivalent to measuring. The things being counted do not have a stricture on them about being identical in length, mass and so on.

Figure 5: Part of a set (equal sharing using a fraction circle prop)

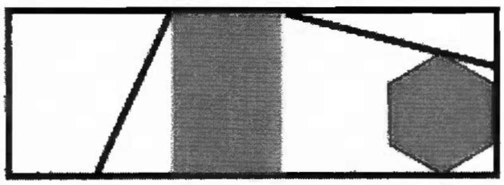


The three critical features of the part-of-a-set meaning of fraction are that

- there exists a collection of discrete things. This collection is seen as acting as a whole;
- equality of size is not required (although it could be present); and
- characteristics (attributes of interest) of the things in the collection are used to determine the fraction.

Following the development of the two core meanings of fraction, an example that compares and contrasts them further clarifies the distinction between them. Consider the large (outside) rectangle in Figure 6. The large rectangle (the whole) has been cut into eight parts, but the parts are not equal in area. We cannot use the part-of-a-whole meaning to say that the shaded area is 2/8 of the large rectangle.

Figure 6: A large rectangle cut into eight parts



However, from the perspective of the part-of-a-set meaning of fraction, equality of size is not required. What *is* required is a collection of things and an attribute of interest. There are eight shapes that comprise the large rectangle. They can be considered as the collection of things. If *shaded* is the attribute of interest, then, in that collection, two shapes are shaded. Thus, 2/8 of the collection is shaded.

When we use shading in a part-of-a-whole situation, we are not interested in shading as an attribute. Rather, shading is a way of identifying the number of equal parts to consider. Using shading for identifying the equal parts has become a tradition in educational circles. We could use other ways of identifying the equal parts—for example, we could put a checkmark in each of them.

The Anecdote Considered Through Question Posing and Discussion

Once the part-of-a-whole and the part-of-a-set meaning of fraction are understood, I retell the anecdote and shift to a question-posing and response session to delve into fraction addition. (For the sake of brevity, the teacher candidates' responses provided here are distillations of the actual ones. Also, not all of the questions and responses are included.)

Author: For the anecdote, what is the answer to the addition if you use the common denominator method that you were taught in junior high?

Teacher candidates: 5/6 + 1/4 is 10/12 + 3/12 = 13/12 or 1 1/12.

Author: Does this answer make sense?

Teacher candidates: No. It means that they won more games than they played. This is nonsense. Winning 6/10 of the games played is not nonsense.

Doubt about the correctness of the addition algorithm they were taught circulates around the room.

Author: Does the situation involve the part-of-a-whole or the part-of-a-set meaning of fraction?

There is no consensus. While the majority indicates part of a set, some see it as part of a whole. Further questioning about what is the whole and whether there is equality of size convinces a minority that the part-of-a-set meaning of fraction is involved. I return to the main thread.

Author: What is the set?

There is disagreement. Some say the number of home games is the set; others say away games. Some say home and away games. (To quote Sherlock Holmes, the game is now afoot.)

Author: If only home games or away games is the set, this creates a circumstance where one choice leaves out the other. Does it make sense to add a fraction that is not from the set? For example, suppose the set is home games. Five-sixths of the home games are won. Does it make sense to add ¼ to 5/6, if the away games are not included in the set?

There is reluctant agreement that it does not make sense. I follow up with another question.

Author: Suppose home games and away games together are the set. How many games belong to the set?

Teacher candidates: Ten games in all are in the set.

Author: What fraction of those ten games are home games won?

Teacher candidates: Five tenths.

Author: What fraction of those ten games are away games won?

Teacher candidates: One tenth.

Author: What fraction of the games played has the team won?

Teacher candidates: Six tenths.

Author: What fractions did you add to get 6/10 as the answer?

Teacher candidates: 1/10 and 5/10.

Author: Are you using the common denominator method you learned in junior high?

Teacher candidates: Yes. Both fractions already have the same denominator so we just added the numerators.

Author: What would you say to the Grade 7 student in response to his comment that the common denominator method is wrong?

Eventually, the conceptual quagmire pointed to by the question is sorted out through discussion. The teacher candidates come to realize that home and away games form a set of 10 games and that the attribute of interest is games won. While it may have appeared in the anecdote that the fraction of games won was obtained by adding numerators to numerators and denominators to denominators, what was actually happening was that the home and away games had to be combined into one set before fraction addition could take place. The combining gave the illusion of denominators being added to denominators. We could not have added the fractions had each one come from a different set.

We dig deeper into fraction addition. I pose the following problem.

There are two identical pies on the table. Joe eats 1/2 of pie #1. Hank eats ¼ of pie #2. How much pie is eaten in all?

Most of the teacher candidates add ½ and ¼, obtaining ¾ as the result. A few add ⅔ and ⅛, obtaining ⅜ as the result. I ask what meaning of fraction is involved. There is consensus that it is part of a whole. I ask the obvious question: "What is the whole?"

Those who obtained ¾ as an answer respond that it is a pie, and that ¾ of one was eaten. Those who obtained ⅜ as a result respond that it is both pies combined and that there are 8 pieces of equal size on the table, 4 quarter-sections in pie #1 and 4 in pie #2—a total of 8 quarter-sections. We discuss the matter and conclude that the answer to the addition depends on what we view as the whole. The question "How much pie is eaten in all?" is ambiguous about what is the whole. A sharper question might have been "What fraction of the pie on the table was eaten?" This question suggests both pies together as the whole and thus the answer would be ⅜.

Conclusion

The teacher candidates have experienced the murky world of fraction addition. They are starting to realize that fraction addition depends on what we consider to be the whole or the set. Without clarity about that, the answers we obtain do not necessarily make sense. If we had just used models such as circles cut into equal parts, they would not have realized that. Question posing about fuzzy situations was needed.

The question that emerges from some at the end of the session is: Why did you make us go through all of this painful thinking?

After reminding them of the anecdote, I talk about how they may have a couple of students who see the adding-numerators-and-adding-denominators method of adding fractions as making sense. This is especially likely with students who play team sports. One reason, therefore, for putting them through the painful thinking was to prepare them for such a possibility and to help them be able to address the matter in an appropriate way.

I talk about another reason. I wanted to model a question-posing method of teaching, one that they will hopefully be comfortable in using. One of the current labels for this approach is *teacher as facilitator*. I point out that it is not really a new method of teaching. In the 1960s, the label was *Socratic teaching*—a variant of the Socratic debating method, which involves inquiry and debate between individuals with opposing points of view. In Socratic teaching, the focus is on posing questions to stimulate student thinking.

We also discuss their level of engagement during the question-posing experience. They invariably tell me that it was high. I ask them to imagine that I had used a show-and-tell approach instead and to compare their level of engagement with it to their level of engagement with the question-posing approach. Without fail, they inform me that they would not be as engaged with a show-and-tell approach to teaching. The point is made.

I conclude the methods course session by opening the door to another bout of painful thinking by presenting the following problem for them to think about for the next session.

> There are two pies on the table. Pie #1 is 30 cm across. Pie #2 is 15 cm across. Joe eats 1/2 of pie #1. Hank eats ¼ of pie #2. How much pie is eaten in all?

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