Euclid Must Go!

Marshall P Bye

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Euclid must go! Surely anyone who utters such sentiments must be sacrilegious. Yet these are the words of the outstanding mathematician Professor J Dieudonné in his address to the Organization of European Economic Council in France, in 1959.[1] Why did he make the statement? Perhaps I can bring some light to this.

We read so much today about what should and should not be included in the school curriculum that, I am sure, we all wonder just what mathematics will become in another decade. One such indicator is the *Report of the Commission on Mathematics* of the College Entrance Examination Board, published in 1959.[2] The very bold programs set forth in that document (very bold for its day) are being realized in varying degrees around the world today—just 10 years later. A number of topics and concepts listed have yet to be included in the Alberta curriculum, but we are surprisingly close to the programs outlined.

The report of the Cambridge Conference on School Mathematics—*Goals for School Mathematics*,[3] published in 1963—listed a still more startling set of objectives for mathematics from K through 12. This document might well be the preview of the next decade.

Why do I mention these two reports? I do because in both reports strong reference is made to transformations. In the Commission report,[4] some time is spent on the three primary weaknesses of the so-called Euclidean geometry as it has been presented for so many years. I shall be concerned primarily with only one of the weaknesses. Professor Dieudonné had these weaknesses in mind when he made the statement "Euclid must go!"

There has been a pronounced trend away from "traditional" geometry in countries outside of North America. A number of the British programs—to name three, the School Mathematics Project, the Nuffield Project and the Scottish Mathematics group—emphasize transformation.

Belgium, perhaps as a result of Papy's work, leans heavily on transformations from elementary school up. Here in Canada, Professor Dienes of Sherbrooke, Quebec, has made transformations an integral part of his program. Del Grande and Egsgard of Toronto have come out with high school texts integrating transformations into the program.[5] The Secondary School Mathematics Curriculum Improvement Study (SSMCIS),[6] produced by Teachers College, Columbia University—two of the authors are Dr Julius Hlavaty and Professor Ray Cleveland—has utilized transformations in algebra and geometry. The NCTM publication *Geometry in the Secondary School* (1967)[7] devotes nearly half of its space to discussions about transformations of one type or another and hardly mentions traditional Euclidean geometry as it has been taught for years.

Transformations

My objective will be to show quickly and easily how transformations may be used in high school geometry and, at the same time, not get involved with "motion" of a geometric figure or set of points. (At times, I shall call upon your intuition as to the motion of a figure.) I shall not be rigorous in such a brief presentation. I shall also make statements that, in a more formal presentation, would need more firm and rigorous attention.

Definition of Transformation

A transformation is a one-to-one mapping. Since we will be talking about plane geometry, I will say that a transformation is a one-to-one mapping in which the domain and range are the set of points of a plane.

Let us now look at a particular set of transformations—the set known as *isometries*.

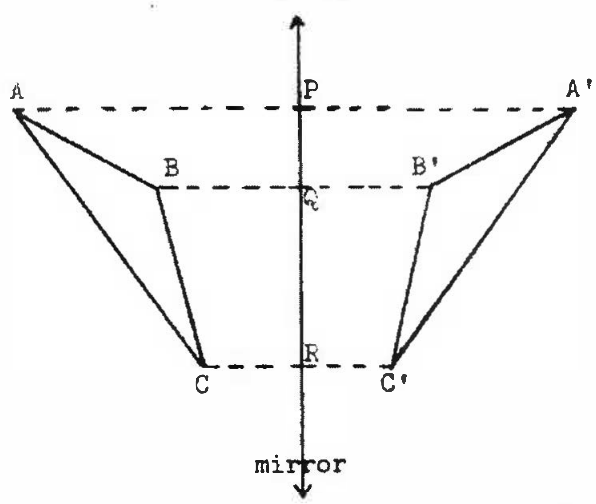
Definition of Isometry

An isometry is a distance-preserving function. Any figure transformed under an isometry is said to be invariant; that is, a figure is its own image under an isometry. Another way to say this, and perhaps crucial to this discussion, is that a figure transformed under an isometry is congruent to its image.

Reflection (in a Line)

Consider a triangle reflected in a mirror.

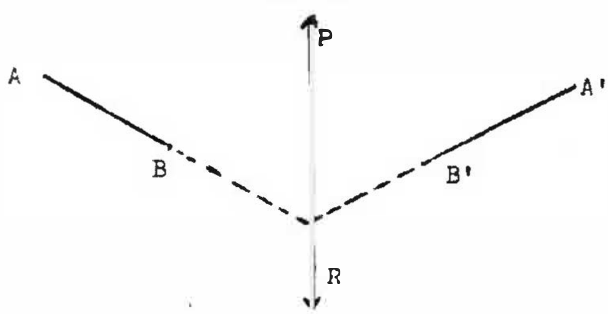
Figure 1



This is the intuitive concept of a reflection. Now let me draw to your attention some of the pertinent details.

1. Every point of the figure ABC is associated with one—and only one—point in its image figure A'B'C'.
2. Points: A→A'; B→B'; C→C'; P→P; Q→Q; R→R.
3. Notice that the points in the mirror line are invariant: each maps on to itself.
4. Segments: $\overline{AB}\rightarrow\overline{A'B'}$; $\overline{AC}\rightarrow\overline{A'C'}$; $\overline{BC}\rightarrow\overline{B'C'}$; $\overline{AP}\rightarrow\overline{PA'}$; $\overline{BQ}\rightarrow\overline{QB'}$; $\overline{CR}\rightarrow\overline{RC'}$
5. Notice that the mirror line is invariant. $\overleftrightarrow{PR}\rightarrow\overleftrightarrow{PR}$
6. Consider the angles formed by $\overrightarrow{AB}$ and $\overrightarrow{A'B'}$ with $\overleftrightarrow{PR}$. The angles are congruent.

Figure 2



7. The perpendicular distance between a point and the mirror is congruent to the perpendicular distance between the image point and the mirror or, stated differently, the axis of reflection is the perpendicular bisector of the segment joining a point and its image.
8. Δ ABC $\cong$ Δ A'B'C'
9. The sense of Δ ABC is opposite to that of Δ A'B'C'. The order of vertices of the object triangle listed clockwise is A-B-C, whereas the order of vertices of the image triangle, clockwise, is A'-C'-B'.

Let us now look at a double reflection—a reflection of a reflection. In the first, the two axes of reflection are parallel (Figure 3), whereas in the second (Figure 4), the two axes are not parallel. Notice that we have one transformation followed by another. This is called *composition of transformations*.

Figure 3

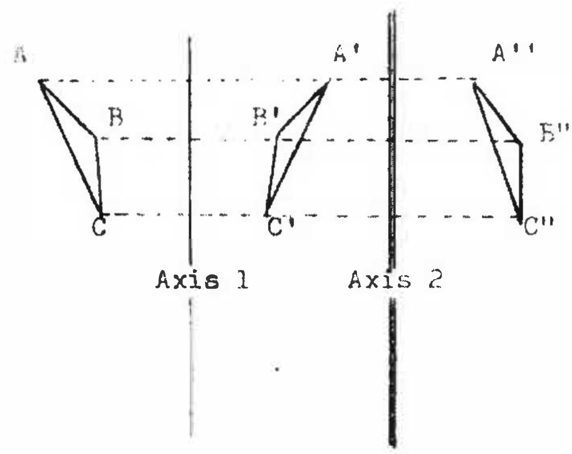
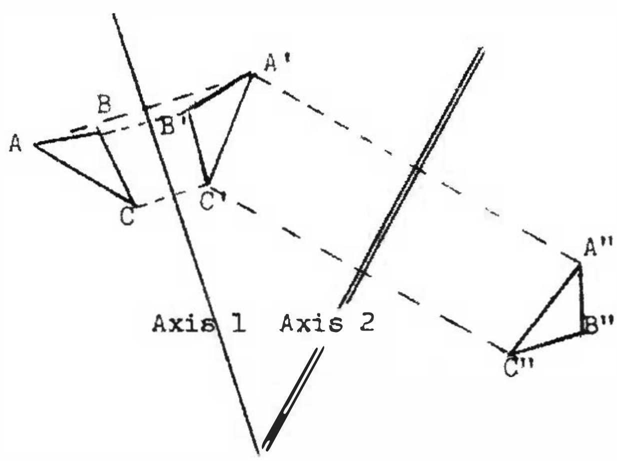


Figure 4

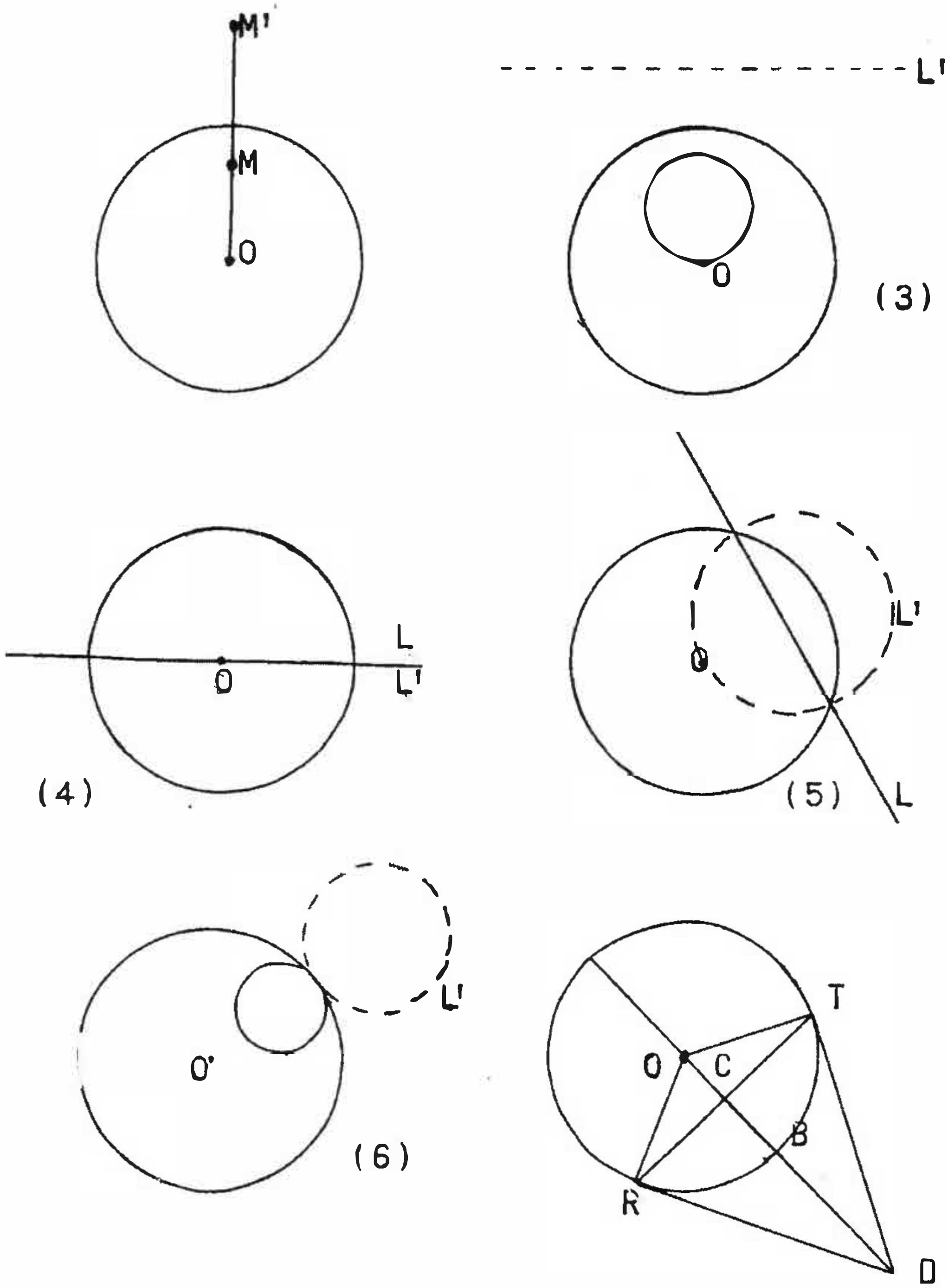


These two illustrations may lead us intuitively to accept the statement that given any two congruent triangles in a plane, there is a series of reflections *such that one triangle is mapped onto the other*. It is an interesting exercise to determine the maximum number of axes of reflection necessary to transform any given triangle into a specific congruent triangle and where those axes of reflection are.

At this time I wish to emphasize that we are not moving triangles or lines or points. When you look in a mirror and see your eyes, you do not, for a moment, have the notion that your eyes have moved behind the mirror. As one author states, in terms of a bowling lane, "We are setting up pins in another alley." As for motion in a plane to explain congruency, there is no motion that would permit you to move Δ ABC to coincide with Δ A'B'C' (Figure 1). The motion would have to come out of the plane.

To make my point clear, let me digress for a moment to a transformation that is not an isometry. Consider the inversive transformation. For this transformation, consider a circle in a plane with centre O and fixed radius r. Any point M is mapped into M' such that $m(\overline{OM}) \cdot m(\overline{OM'}) = r^2$. Refer to Figure 5.

Figure 5



This transformation results in some strange things:

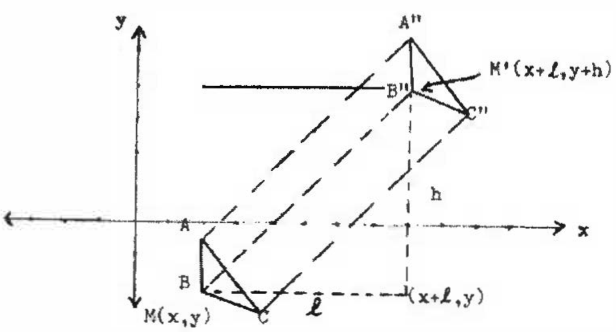
1. Every point in the interior of the circle is mapped into a point in the exterior of the circle and, conversely, every point in the exterior is mapped into the interior.
2. Every point on the circle remains fixed (mapped into itself).
3. Every circle in the interior of the circle that passes through the centre is mapped into a straight line.
4. Every line that contains the centre O of the circle is mapped into itself.
5. Every line that does not contain the centre O of the circle is mapped into a circle.
6. Every circle not containing the centre O of the circle is mapped into another circle.

Clearly, we have not "moved" figures—we have not preserved shape or size.

However, let us return to isometries. While we can use reflections to establish our mappings of one figure into congruent figures, other transformations may be used as well. We shall only spend time with two others in this paper.

Refer to Figure 6. Notice that we can think of A→A", B→B" and C→C". If we place this figure on the coordinate plane, it is easy to think of this transformation as mapping any point M(x,y) into M'(x+l, y+h).

Figure 6



Intuitively, a translation can be thought of as the transformation of the set of points, taken in order, through a certain fixed distance in some direction.

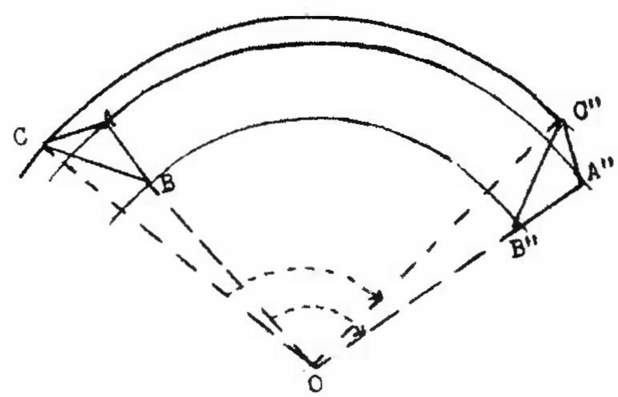
Review some of the properties of this invariant transformation:

1. Corresponding sides are parallel and congruent.
2. Corresponding angles are congruent.
3. The sense of the figure is preserved.

Rotation

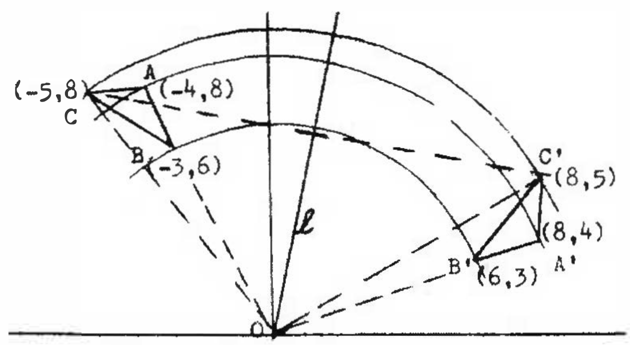
The third and last transformation discussed by me in this paper is illustrated in Figure 7. I have reproduced it here to show the mapping of Δ ABC onto Δ A"B"C". You can visualize the mapping: A→A", B→B" and C→C". As you know, this is a rotation.

Figure 7



In the next figure, we clearly see the mapping on the Cartesian plane.

Figure 8



The point of rotation is the origin. The angle of rotation is the measure of ∠AOA'.

Points to observe in this isometry:

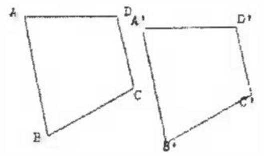
- ∠AOA' ≅ ∠BOB' ≅ ∠COC'
- The perpendicular bisector of the segment determined by two corresponding points contains the point of rotation O (ℓ bisects $\overline{CC'}$).
- The said perpendicular bisector of the segment $\overline{CC'}$ bisects ∠COC'.
- Sense is preserved.
- The point of rotation is the only point in the plane that is invariant.
- The image is congruent to the object.

Another isometry is the glide-reflection. It is a combination of the translation followed by a reflection. Some books use the glide-rotation. These are simply compositions of other isometries.

Properties of an Isometry

At this point we will sum up briefly and state our understandings in the form in which we will be using them.

1. If there is an isometry or isometries which transform one geometric figure into another, the two figures are congruent.
2. Suppose in polygonsABCD and A'B'C'D' the mapping is an isometry and suppose A→A', B→B', C→C', D→D'.
   a. Distance is preserved:
      m(AB) = m(A'B')
      m(BC) = m(B'C')
      m(AC) = m (A'C')
      etc.



   b. Measure of each angle is preserved: m ∠ABC = m ∠A'B'C', etc.
   c. Straightness is preserved—that is, lines map into lines.
   d. Parallel lines map into parallel lines. If AB ∥ CD, then A'B' ∥ C'D'. (Hence, perpendicularity is preserved.)

Now we arrive at the main point of the discussion. I have gone neither into any detail on the method of presentation nor into interesting side trips. I have only laid the foundation for that which I want to present at this time.

Geometric Proofs Using Isometries

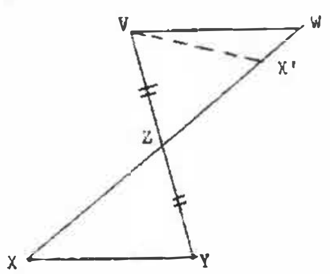
Definition: two figures are said to be *congruent* if there is an isometry (or a composition of isometries) that maps one of the figures onto the other. Let us look at specific instances.

Example 1

Figure XYZVW

Segments $\overline{XW}$ and $\overline{VY}$ intersect such that XY is parallel to $\overline{VW}$ and $\overline{YZ} \cong \overline{VZ}$.

Prove: $\overline{XZ} \cong \overline{WZ}$



Proof: Consider the 180° rotation of Δ XYZ about Z.

Thus Z→Z
Since $\overline{YZ} \cong \overline{VZ}$, Y→V
Let X→X'
Since X lies on $\overleftrightarrow{ZW}$, X' lies on $\overleftrightarrow{ZW}$

Now in a rotation of 180° a line not through Z maps onto a parallel line (property of rotations).

∵ $\overline{XY}$→$\overline{VX'}$
∴ XY ∥ VX' and $\overline{XY} \cong \overline{VX'}$

But ∴ $\overline{XY}$ ∥ $\overline{VW}$
$\overline{VX'}$ ∥ $\overline{VW}$

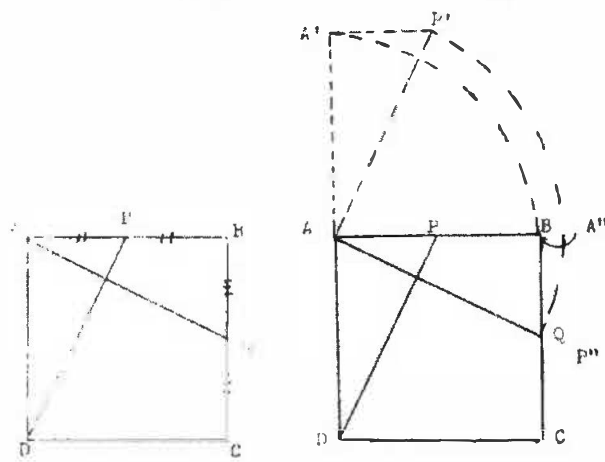
Two parallel segments with one common point must lie in the same line (Euclid—we have not banished him completely. Saccheri does not dethrone Euclid here!).

X' lies in $\overleftrightarrow{VW}$
But X' lies on line $\overleftrightarrow{ZW}$
W is the only point common to the two lines $\overleftrightarrow{ZW}$ and $\overleftrightarrow{VW}$
∴ X' = W
∴ $\overline{XZ}$→$\overline{WZ}$
∴ $\overline{XZ} \cong \overline{WZ}$

Let us look at another example.

Example 2

Consider the square ABCD. P and Q are midpoints of $\overline{AB}$ and $\overline{BC}$ respectively. Prove $\overline{PD} \perp \overline{AQ}$



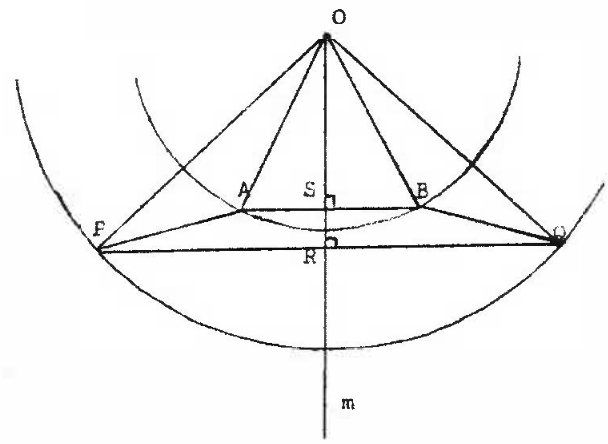
We will not set up the detailed proof, but I will work through the general approach. First, by translation, we transform Δ DAP along the $\overleftrightarrow{DA}$ the distance equal to the measure of $\overline{DA}$. Thus we get the Δ AA'P'. Now we rotate Δ AA'P' about point A, through an angle of rotation of $\pi/2$. We can then show that Δ AA'P' has been mapped onto Δ ABQ. Hence, AD' (which is parallel to $\overline{DP}$) maps onto $\overline{AQ}$ by a rotation of $\pi/2$. Hence $\overline{AQ} \perp$ DP.

Here is a final example to illustrate the use of a reflection.

Example 3

Given: In the figure, $\overline{AP}$ and $\overline{PQ}$ are parallel chords of two circles with a common centre O.

Prove: (a) ∠ AOP ≅ ∠ BOQ
(b) $\overline{AP} \cong \overline{BQ}$



Proof: Consider a reflection of Δ PAO in a line M through O perpendicular to $\overline{PQ}$. From previous work we know line M bisects $\overline{PQ}$. (Can prove, but will accept now.)

$O \rightarrow O$

$S \rightarrow S$

$R \rightarrow R$

$\because \overline{PR} \rightarrow \overline{QR}$ and $\overline{PR} \perp \overline{OR}$ (previously proven)

$\therefore P \rightarrow Q$

Similarly, we can show $A \rightarrow B$.

$\therefore \angle AOP \rightarrow \angle BOQ \quad \therefore \angle AOP \cong \angle BOQ$

$\therefore \overline{PA} \rightarrow \overline{QB}: \quad \therefore \overline{PA} \cong \overline{QB}$.

I have illustrated the use of transformations in proofs from plane geometry. Transformations also may be used to let students discover construction techniques and in turn make the work on constructions far more meaningful and a far more powerful unit in geometry. Time will not permit a discussion of this area.

A Few Concluding Remarks

I have restricted my discussion to plane geometry. There is no need for this restriction; transformations allow an easy transition into 3-dimension or even n-dimension. This is an advantage of transformations.

Transformations can also be used to solve quadratic equations of the form $ax^2 + by^2 + 2yx + 2fy + c = 0$ by simply transforming them to the form $ax^2 + by^2 = c$. This is another advantage of transformations.

Finally, transformations provide ample opportunity to show that Euclidean geometry is one particular element of the set of geometries in which a certain set of properties are invariant. Whenever we change the set of invariant properties we have a new geometry. Within the scope of transformations lies a host of geometries of such a simple nature that students at an early age can develop, if given an opportunity, an intuitive understanding of them.

Professor Dieudonné viewed the broader field of mathematics that is possible through the new freedom provided by a break from Euclid. He does not advocate throwing out all of Euclid, but rather stresses that for young students there is a richness in geometry possible when parts of Euclid are set aside.

Notes

1. *New Thinking in School Mathematics*. Organisation for European Economic Co-operation. 1961, 35.

2. *Report of the Commission on Mathematics*. College Entrance Examination Board. Princeton, NJ. 1959

3. *Goals for School Mathematics*. Report of the Cambridge Conference on School Mathematics. Boston: Houghton Mifflin. 1963.

4. *Report of the Commission on Mathematics*. College Entrance Examination Board. Princeton, NJ. 1959, 109–10.

5. Del Grande, J J, and J C Egsgard. *Mathematics 11*. Toronto, Ont: Gage. 1964.

6. *Unified Modern Mathematics*. Secondary School Mathematics Curriculum Improvement Study. New York: Teachers College, Columbia University. 1968.

7. National Council of Teachers of Mathematics (NCTM). *Geometry in the Secondary School*. Washington, DC: NCTM. 1967.

Marshall P Bye was supervisor of mathematics for Calgary public secondary schools.