??? PROBLEM CORNER ???

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Problems suggested here are aimed at students of both the junior and senior high schools of Alberta. Solutions are solicited and a selection will be made for publication in the next issue of *delta-k*. Names of participants will be included. All solutions must be received (preferably in typewritten form) within 30 days of publication of the problem in *delta-k*. *Mail solutions to:*

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Problem 1 was first published in the March 1980 issue and is reproduced here along with its solution.

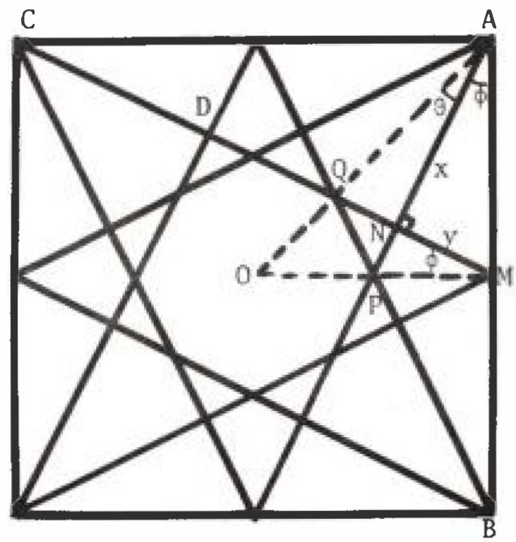
Problem 1:

An 8-point star is formed in a square region of side S units by drawing two lines from each mid-point of a side of the square to opposite corners of the square. Note that these lines also form two identical smaller squares as well as one octagon.

(a) If the star is cut out of the square region, what fraction of the square is wasted?

(b) In terms of S , what is the area of one of the smaller square regions?

(c) In terms of S , what is the area of the octagonal region?



Solution of Problem 1

The following (edited) solutions to parts (a) and (b) were supplied by Ian Weitz, a Mathematics 30 student at Winston Churchill High School in Lethbridge, Alberta.

Note that the two diagonals and the two perpendicular bisectors of the sides are axes of symmetry. Hence only one corner of the diagram need be considered.

Now if $\angle MAN = \angle NMO = \phi$, then $\tan\phi = \frac{1}{2}$. Thus $\angle MNA = 90°$ and if $AB = s$, $MN = x$ and $AN = y$, then $AM = \frac{1}{2}s$,

$$x = AM \cdot \sin\phi = \frac{s}{2} \cdot \frac{1}{\sqrt{5}}$$

and $$y = AM \cdot \cos\phi = \frac{s}{2} \cdot \frac{2}{\sqrt{5}} = \frac{s}{\sqrt{5}} .$$

(a) Area $(\Delta\ MAN) = \frac{1}{2} \cdot x \cdot y = \frac{1}{20}s^2$. Hence the wasted area is $8\left(\frac{s^2}{20}\right) = \frac{2}{5}s^2$.

(b) Now $MC = \sqrt{s^2 + \left(\frac{s}{2}\right)^2} = \frac{\sqrt{5}}{2}s$. So $ND = MC - (x+y) = \frac{s}{\sqrt{5}}$. Hence the area of the square with side ND is $\left(\frac{s}{\sqrt{5}}\right)^2 = \frac{1}{5}s^2$.

(c) By symmetry, each side of the octagon has the same length; however, the angles are not all equal. Now $MP = MA \cdot \tan\phi = \frac{s}{4}$. Thus $OP = \frac{s}{4}$. Also $\theta = \angle NAQ = 45° - \phi$. So $AQ = y \cdot \sec\phi = \frac{s}{\sqrt{5}} \cdot \frac{1}{\cos(45°-\phi)} = \frac{\sqrt{2}}{3}s$, since $\cos(45°-\phi) = \cos 45° \cos\phi + \sin 45° \sin\phi = \frac{3}{\sqrt{10}}$, and $OQ = OA - QA = \frac{s}{\sqrt{2} \cdot 3}$.
Then Area $(\Delta OPQ) = \frac{1}{2} \cdot OP \cdot OQ \sin 45° = \frac{1}{2} \cdot \frac{s}{4} \cdot \frac{s}{3\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = \frac{s^2}{48}$ so that the area of the octagon is $8\left(\frac{s^2}{48}\right) = \frac{s^2}{6}$.

Problem 2:

(submitted by William J. Bruce, University of Alberta)

Clearly $1 = 1^2$ is a perfect square, but 11 and 111 are not. Consider all numbers $11111 \cdots 1 = S$, in which all digits are unity, and prove or disprove that, except for $s = 1$, no such number is a perfect square.*

*EXTENSION (Proof not to be submitted for publication.)

The theorem is true for any number that can be written in the form $100m + 10 + 1$ (m a positive integer). Also, true for $100m + k + 1$ when k is *not* divisible by 4.