Canadian Mathematical Society

1980 Alberta High School Prize Examination Results

Prize	$ Amt.	Student	School
Canadian Mathematical Society Scholarship	400	MOREWOOD, Robert	Medicine Hat High School Medicine Hat
Nickle Foundation Fellowship	400	Not awarded this year.	
First Runner-Up	150	WILTING, Carl	M.E. LaZerte Composite High Edmonton
Second Runner-Up	150	WILLIAMS, Daniel	McCoy High School Medicine Hat

Special Provincial Prizes

Highest Grade 12 student (below first 3)	75	URSENBACH, Charles	Dr. E.P. Scarlett Senior High Calgary
Highest Grade 10/11 student	75	BARAGAR, Arthur	Old Scona Academic High School Edmonton

District Prizes

District No.	$ Amt.	Student	School
1	50	WONG, Garant	Saint Thomas More Fairview
2	50	HAYASHI, Peter	Harry Collinge School Hinton
3	50	NICOL, Bonnie	Fort Saskatchewan High School Fort Saskatchewan
4	50	KOMISHKE, Bradley	William E. Hay Composite High Stettler
5	50	SHELDON, Joanne	Banff Composite High School Banff

District No.	$ Amt.	Student	School
6	50	NEUFELDT, David	Kate Andrews High School Coaldale
7(1)	50	TRUMPENER, John	Strathcona Composite High School Edmonton
7(2)	50	BOWMAN, John	Old Scona Academic High School Edmonton
8(1)	50	BAMBER, Jim	Queen Elizabeth High School Calgary
8(2)	50	MA, Felix	Dr. E.P. Scarlett Senior High Calgary

367 students from 64 schools in Alberta and the Northwest Territories wrote the 1980 examination. The following students took the first 16 places and were nominated for the Canadian Mathematical Olympiad.

Student	School
BAMBER, Jim	Queen Elizabeth High School, Calgary
BARAGAR, Arthur	Old Scona Academic High School, Edmonton
BONDY, Brad	Old Scona Academic High School, Edmonton
BOWMAN, John	Old Scona Academic High School, Edmonton
DVORKIN, Len	Henry Wise Wood Senior High, Calgary
FILIPCHUK, David	Harry Ainlay Composite High, Edmonton
FRAGA, Werner	St. Mary High School, Edmonton
GREENWAYS, Jeannette	St. Mary High School, Edmonton
KENYON, Jim	M.E. LaZerte Composite High School, Edmonton
LEE, Michael	Old Scona Academic High, Edmonton
MA, Ching-Kwok Felix	Dr. E.P. Scarlett Senior High School, Calgary
MOREWOOD, Robert	Medicine Hat High School, Medicine Hat
TRUMPENER, John	Strathcona Composite High School, Edmonton
URSENBACH, Charles	Dr. E.P. Scarlett Senior High School, Calgary
WILLIAMS, Daniel	McCoy High School, Medicine Hat
WILTING, Carl	M.E. LaZerte Composite High School, Edmonton

The following 11 students placed 17-27th. Credit for an additional two or three part one questions or for an additional 1/2 of a part two question would have put any one of these students into the top 16.

Todd Anderson (Dr. E.P. Scarlett Sr. High, Calgary); Jack Chu (Henry Wise Wood Sr. High, Calgary); Scott Craig (Henry Wise Wood Sr. High, Calgary); Mark Jorgenson (Dr. E.P. Scarlett Sr. High, Calgary); John Marko (Louis St. Laurent Sr. High, Edmonton); Michael Markowski (Strathcona Comp. High School, Edmonton); Steven Leikeim (Bishop Grandin High School, Calgary); Aaron Ng (M.E. LaZerte Comp. High School, Edmonton); Adam Parrish (Strathcona Composite High School, Edmonton); Eleanor Stein (Ross Sheppard Composite High School, Edmonton); Raymond Wong (James Fowler Sr. High, Calgary)

The following 11 students placed 28-38th. These students were approximately one full part two question away from being in the top 16.

John Achilles (Harry Ainlay Comp. High School, Edmonton); Allison Chernowski (Old Scona Academic High School, Edmonton); Mike Jackson (Harry Ainlay Comp. High School, Edmonton); David Leung (Harry Ainlay Composite High School, Edmonton); Craig McLurg (Dr. E.P. Scarlett Sr. High School, Calgary); David Neufeldt (Kate Andrews High School, Coaldale); Joyce Oman (William Aberhart High School, Calgary); Ashok Patel (Old Scona Academic High School, Edmonton); Joanne Sheldon (Banff Comp. High School, Banff); Stuart Thompson (Louis St. Laurent Catholic School, Edmonton); Garant Wong (Saint Thomas More School, Fairview)

The following 17 students also placed in the top 15% of all who wrote.

Stephen Bell (Dr. E.P. Scarlett Sr. High, Calgary); David Berg (William Aberhart High School, Calgary); Marc Berner (William Aberhart High School, Calgary); Gordon Boyes (Louis St. Laurent Catholic School, Edmonton); Jean Chan (Harry Ainlay Composite High School, Edmonton); Bobby Chum (Jasper Place Comp. High School, Edmonton); Richard Garside (Raymond High School, Raymond); William Graham (Queen Elizabeth Sr. High School, Calgary); Peter Hayashi (Harry Collinge High School, Hinton); Todd Herron (Dr. E.P. Scarlett Sr. High School, Calgary); Bradley Komishke (William E. Hay Comp. High School, Stettler); Leanne Koziak (Louis St. Laurent Catholic School, Edmonton); Bonnie Nicol (Fort Saskatchewan High School, Fort Saskatchewan); Dwaine Palmer (Raymond High School, Raymond); Arvid Schultz (Noble Central School, Nobleford); Wenona Urquhart (Sir Winston Churchill High School, Calgary); Derek Woolner (Queen Elizabeth Sr. High School, Calgary).

ANSWER SHEET

To be filled in by the Candidate.

PRINT:

Last Name First Name Initial

Candidate's Street Address Town/City POSTAL CODE

Name of Candidate's School

Grade Male or Female

ANSWERS:

1	2	3	4	5	6	7	8	9	10

11	12	13	14	15	16	17	18	19	20

To be completed by the Department of Mathematics, University of Alberta:

Points		Points Correct	Number Wrong
1-20	5	5 × =	1 × =
TOTALS		C = ______	W = ______

SCORE = Correct - Wrong = ______

PART I **Time: 60 Minutes**

Instructions to Candidates

1. *Please do not open this booklet beyond Page 2 until instructed to do so by the supervisor.*

2. *Please turn now to Page 2 (the next page) and fill in the top four lines - Page 2 is your answer sheet.*

3. *This exam is multiple choice. Each question will be followed by 5 possible answers, labelled, A, B, C, D, E. For each question, list your choice of answer in the box on the answer sheet directly above the question number. For example, if you decide the correct answer to question 3 is labelled C, then enter C in the box above 3 on the answer sheet.*

4. *To discourage random guessing, there is a penalty for each incorrect answer; there is no penalty for unanswered questions. Filling in more than one letter in any box counts as a wrong answer for that question.*

5. *At the signal from your supervisor, detach both this page and page 2 (the answer sheet) and begin the examination. Pencil, graph paper, scratch paper, ruler, compass, and eraser are allowed.*

   Calculators are not allowed.

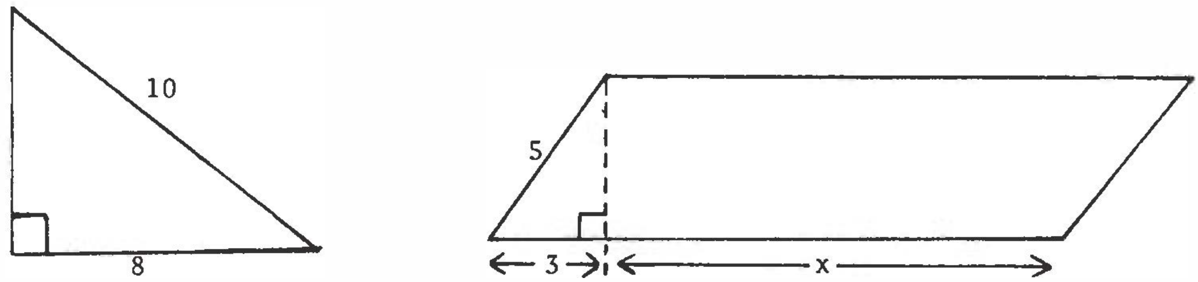
Do all problems. Each problem is worth five points.

1. There are 5 roads between the towns A and B and 4 roads between the towns B and C.

   The number of different ways of driving the roundtrip $A \to B \to C \to B \to A$ without using the same road more than once is:

   (A) 32 (B) 240 (C) 400 (D) 16 (E) none of these.

2.



In the above diagrams, the area of the triangle is two-fifths of the area of the parallelogram. The value of x is therefore:

(A) 30 (B) 12 (C) 15 (D) 24 (E) none of these.

3. A die is thrown repeatedly until a 6 is obtained. Assuming the die to be fair, the probability that this will happen on the third throw is:

(A) 1/6 (B) 1/216 (C) 5/36 (D) 25/216 (E) none of these.

4. For which real values of k does $kx^2 + kx + 1$ have no real root?

(A) $-2 < k < 6$ (B) $-2 < k < 4$

(C) $0 < k < 2$ (D) $0 < k < 4$

(E) none of these

5. For which value(s) of k are the lines $9x + ky = 7$ and $kx + y = 2$ parallel?

(A) $k = 3$ (B) $k = \pm 3$ (C) $k = 1/3$ (D) $k = \pm 1/3$

(E) none of these.

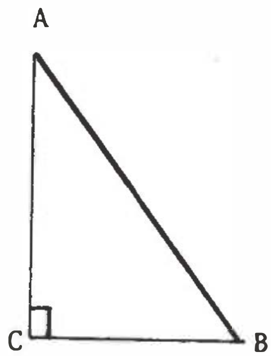
6. Let $x = \log_{a^2} b$. Then $\log_b a^x$ equals

(A) $\sqrt{b}$ (B) b^2 (C) 1/2 (D) 2 (E) none of these.

7. If $\sin 3x = 0$, then $\sin x$ equals

(A) 0 (B) 0 or $\pm 1/2$ (C) 0 or $\pm \sqrt{3}/2$

(D) 0, $\pm 1/2$ or $\pm \sqrt{3}/2$ (E) none of these.

8. 

In the right-angled triangle ABC, the lengths of the sides AB and BC are $(x^2 + y^2)$ and $2xy$ units respectively. The length of the side AC is therefore:

(A) (x^2-y^2) (B) $(x-y)^2$

(C) $(x^2-y^2)^2$ (D) $(x+y)^2$

(E) none of these.

9. In a certain examination, the average mark (out of 100) of the 30 boys in a class was 60. The girls did rather better, their average being 65. If the overall average for the class was 62, the number of girls who took the examination was:

(A) 20 (B) 24 (C) 18 (D) 36 (E) none of these.

10. The following is true for all real numbers x,y:

(A) $(x+y)^2 \geq (x-y)^2$

(B) $|x+y| \geq x + y$

(C) $\sqrt{x^2+y^2} \geq x + y$

(D) $xy \geq x + y$

(E) none of the above is true for all real x,y .

11. $\log_{10}\left[(144)^{144}\right]$ equals

(A) $576 \log_{10}2 + 288 \log_{10}3$

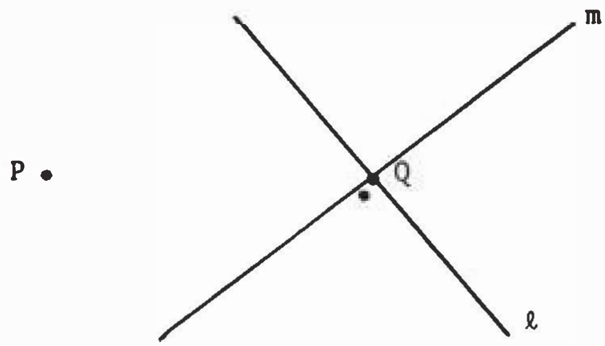
(B) $144 \log_{10}2 + 144 \log_{10}3$

(C) $(\log_{10} 144)^{144}$

(D) $2 \log_{10}144$

(E) none of these.

12.



The line ℓ has the equation $y = -2x - 4$ and the line m has the equation $y = 2x + 4$. If P is the point $(0,0)$ then the equation of the line through P and Q is

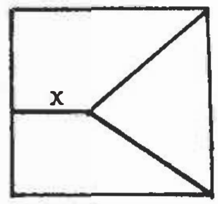
(A) $y = x$ (B) $y = -2x$

(C) $2x + 3y = 1$ (D) $-y - 2x = 1$

(E) none of these.

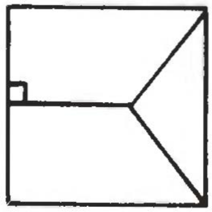
13. The maximum possible value of $3x - 3x^2$ where x is real is
(A) 5/4 (B) 0 (C) -1 (D) 3/4 (D) none of these.

14.



A unit square is divided into three parts of equal area as shown. Then x is

(A) 1/3 (B) $1/\sqrt{2}$ (C) 1/4 (D) $1/\sqrt{3}$ (E) 1/2 .

15. 

A unit square is divided into three parts by three lines of equal length as shown. The length of each line is

(A) 1/2 (B) 5/8 (C) $2 - \sqrt{2}$ (D) $1/\sqrt{2}$ (E) 3/5 .

16. $\sqrt{\dfrac{\sqrt{5} - 2}{\sqrt{5} + 2}}$ equals

(A) $\dfrac{1}{\sqrt{5} - \sqrt{2}}$ (B) $\dfrac{1}{\sqrt{2} + 5}$ (C) $\sqrt{5} - 2$

(D) $\sqrt{5} + 2$ (E) $\dfrac{1}{5 - \sqrt{2}}$.

17. If $f(x) = x^2 - 5$ and $f(4+a) = f(4a)$ then a is

(A) 4/3 or -4/5 (B) 4 or -4/3 (C) 4/3 or 4/5

(D) -4 or 4/3 (E) -4 or -4/5.

18. Two circles and a straight line are drawn in the plane and form exactly N bounded regions. N is <u>not</u>

(A) 3 (B) 4 (C) 5 (D) 6 (E) all these are possible

19. A bowl contains 2 marbles of each of 4 colours. If you randomly remove 3 marbles from the bowl without replacing them, what is the probability that you have removed two of the same colour?

(A) 1/4 (B) 1/7 (C) 3/7 (D) 1/2 (E) none of these

20. If n is a positive integer so that

$$1 + 2 + \ldots + n = (n+1) + (n+2) + \ldots + 118 + 119$$

then n is

(A) 60 (B) 69 (C) 76 (D) 84 (E) 89 .

INSTRUCTIONS TO CANDIDATES

Do all questions. All questions are weighted equally and you may attempt them in any order you wish.

Your paper should be concise and complete and each step of your answer should be clearly justified and presented in a legible and intelligible form. Extra credit will be given for particularly elegant solutions as well as for non-trivial generalizations with proof. If there is any doubt about an interpretation of a problem, make a note of that on your paper and state and solve what you consider to be a valid non-trivial interpretation of the problem.

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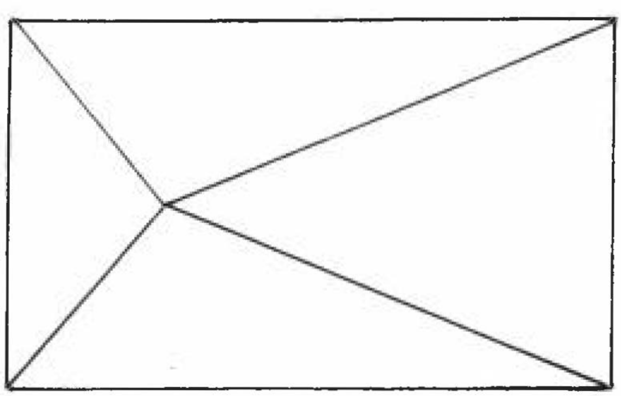
1. Suppose we start with 7 sheets of paper and then some number of them are each cut into 7 smaller pieces. Then some of the smaller pieces are each cut into 7 still smaller pieces and so on. Finally the process is stopped and it turns out that the total number of pieces of paper is some number between 1976 and 1986. Can one determine the exact final number of pieces of paper?

2. A rectangle is divided into 4 triangles as in the diagram.



Three of the triangles have areas of 5, 9, and 11. Find all possible areas for the fourth triangle.

3. A certain sequence of real numbers is defined as follows:

$$x_1 = 2$$

$$x_{n+1} = \frac{2x_n}{3} + \frac{1}{3x_n}; \qquad n = 1,2,3,\ldots.$$

Show that for all values of $n \geq 2$, $1 < x_n < 2$.

4. If all cross-sections of a bounded solid figure are circles, prove that the figure is a sphere.

5. Consider a line segment AB of length $2a$. A point P is chosen at random on this line segment, all points being equally likely.

Show that the probability that the product of the lengths of the segments AP and PB exceeds $a^2/2$ is $1/\sqrt{2}$.

6. Show that one root of the equation

$$x^4 + 5x^2 + 5 = 0$$

is $x = \omega - \omega^4$ where ω is a complex fifth root of unity (that is, $\omega^5 = 1$). Determine the other three roots as polynomials in ω.

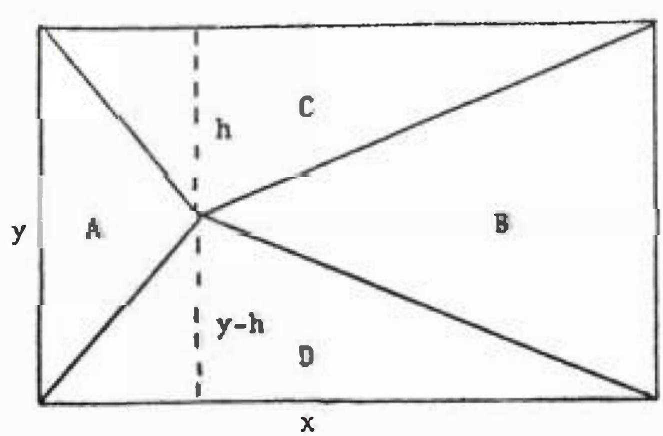
Solutions to Part I

B	B	D	D	B	C	C	E	A	B
1	2	3	4	5	6	7	8	9	10
A	E	D	A	B	C	A	E	C	D
11	12	13	14	15	16	17	18	19	20

Solutions to Part II

1. If x of the pieces are cut into 7, we have $7 - x + 7x = 7 + 6x$ pieces. In fact, at every stage the number of pieces is a number of the form $7 + 6n$ for some integer n. To see this, suppose we have $7 + 6n_1$ pieces and cut n_2 of these into seven. Then we have $7 + 6n_1 - n_2 + 7n_2 =$ $= 7 + 6(n_1+n_2)$ pieces. Now, if we subtract 7 from a number of the form $7 + 6n$, what remains is a multiple of 6. The only number between 1976-7 and 1986-7 which is a multiple of 6 is 1974, so $1974 + 7 = 1981$ is the only possibility for the final number of pieces.

2.



Label the areas of the four triangles A,B,C,D as shown. Suppose the rectangle has length x and width y. If triangle C has height h, then triangle D has height $y - h$. Consequently,

$C + D = \frac{1}{2}hx + \frac{1}{2}x(y-h) = \frac{1}{2}xy$ is half the total area of the rectangle.

But then $A + B$ is also half the area and so $A + B = C + D$. If A and B are 5 and 9, then one of C and D is 11 and the other is <u>3</u>. If A and B are 5 and 11, then one of C and D is 9 and the other is <u>7</u>. Finally, if A and B are 9 and 11, then one of C and D is 5 and the other is <u>15</u>. Thus the possible areas for the fourth triangle are 3, 7 or 15.

3. <u>Solution 1</u> (Without calculus)

$$x_2 = \frac{2(2)}{3} + \frac{1}{3(2)} = \frac{9}{6} \quad \text{so} \quad 1 < x_2 < 2 .$$

Assume, as an induction hypothesis, that $1 < x_k < 2$.

Then $1/x_k < 1$ and so

$$x_{k+1} = \frac{2x_k}{3} + \frac{1}{3}\left(\frac{1}{x_k}\right) < \frac{2(2)}{3} + \frac{1}{3}(1) = \frac{5}{3} < 2 .$$

Also if $x_k > 1$, then $x_k = 1 + a$ for some $a > 0$. Consequently,

$$x_{k+1} = \frac{2(1+a)}{3} + \frac{1}{3(1+a)} = \frac{2}{3} + \frac{2}{3}a + \frac{1}{3(1+a)} .$$

We need to know that $x_{k+1} > 1$.

Now
$$x_{k+1} - 1 = \frac{2}{3} + \frac{2}{3}a + \frac{1}{3(1+a)} - 1$$
$$= \frac{1}{3}\left(2a + \frac{1}{1+a} - 1\right)$$
$$= \frac{1}{3}\left(\frac{2a^2+a}{1+a}\right) \text{ which is } > 0 \text{ since } a > 0 .$$

Thus $x_{k+1} > 1$ as required.

Solution 2 (Calculus)

Let $f(x) = \frac{2x}{3} + \frac{1}{3x}$ $\quad 1 \le x \le 2$.

Then $f(1) = \frac{2}{3} + \frac{1}{3} = 1$ and $f(2) = \frac{2(2)}{3} + \frac{1}{3(2)} = \frac{9}{6}$.

Also $f'(x) = \frac{2}{3} - \frac{1}{3x^2} = \frac{2x^2 - 1}{3x^2} > 0$ for all x , $1 \le x \le 2$.

Thus $f(x)$ is a strictly increasing function on the interval $1 \le x \le 2$. Consequently $1 < x < 2$, implies $f(1) < f(x) < f(2)$. That is $1 < x < 2$ implies $1 < f(x) < \frac{3}{2}$. In particular $1 < x_n < 2$ implies $1 < x_{n+1} < 2$. Since $x_2 = 3/2$ we are again done by induction.

4. Any bounded solid figure has a maximum diameter, d . Consider a line segment, L, which connects two points on the figure and whose length is d . Now consider any plane which includes L. This plane intersects the figure in a circle, C. Moreover because L has maximal length, L must be a diameter of C and the midpoint of L is the center of C . Now using L as the axis of rotation, rotate this plane through space. The intersection of the plane with the figure will always be a circle whose center is the midpoint of L . Hence the figure is exactly the sphere obtained by rotating C around L.

5. Suppose AP has length x so that PB has length $2a - x$. We are interested in the probability that

$$x(2a-x) > a^2/2 .$$

$$\begin{aligned} \text{But} \quad & x(2a-x) > a^2/2 \\ \Leftrightarrow \quad & x^2 - 2ax + a^2/2 < 0 \\ \Leftrightarrow \quad & (x-a)^2 < a^2/2 \end{aligned}$$

$$\Leftrightarrow \qquad -a/\sqrt{2} < x - a < a/\sqrt{2}$$

$$\Leftrightarrow \qquad a - a/\sqrt{2} < x < a + a/\sqrt{2} .$$

Thus for the required inequality to be satisfied x must be in the given interval of length $2a/\sqrt{2}$. Since the total length of the interval is $2a$ the probability of this happening is

$$\frac{\frac{2a}{\sqrt{2}}}{2a} = \frac{1}{\sqrt{2}}$$

6. Since $\omega^5 = 1$, $\omega^6 = \omega$, $\omega^7 = \omega^2$ and so on. In particular, $(\omega^2)^5 = \omega^{10} = 1$, $(\omega^3)^5 = \omega^{15} = 1$, and $(\omega^4)^5 = \omega^{20} = 1$ so that if ω is a fifth root of unity so are ω^2, ω^3 and ω^4 . Also $0 = \omega^5 - 1 = (\omega-1)(\omega^4+\omega^3+\omega^2+\omega+1)$ and since $\omega \neq 1$, this implies $\omega^4 + \omega^3 + \omega^2 + \omega + 1 = 0$. Now by direct calculation

$$(\omega-\omega^4)^2 = \omega^2 - 2\omega^5 + \omega^8 = \omega^2 - 2 + \omega^3$$

and $\quad (\omega-\omega^4)^4 = (\omega^2 - 2 + \omega^3)^2 = \omega^4 - 4\omega^3 - 4\omega^2 + \omega + 6$.

Then

$$\begin{aligned}(\omega-\omega^4)^4 + 5(\omega-\omega^4)^2 + 5 &= \omega^4 - 4\omega^3 - 4\omega^2 + \omega + 6 + 5\omega^2 - 10 + 5\omega^3 + 5 \\ &= \omega^4 + \omega^3 + \omega^2 + \omega + 1 = 0 .\end{aligned}$$

Thus if ω is a fifth root of unity, then $x = \omega - \omega^4$ is a root of the given polynomial. Applying this argument to ω^2, ω^3, and ω^4 which are also fifth roots of unity, we see that $\omega^2 - (\omega^2)^4 = \omega^2 - \omega^3$, $\omega^3 - (\omega^3)^4 = \omega^3 - \omega^2$, and $\omega^4 - (\omega^4)^4 = \omega^4 - \omega$ are the other roots of the polynomial.