Thought Levels in Geometry

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Thought levels in geometry were proposed by two teachers in the Netherlands - Pierre and Dina van Hiele. They were concerned with the difficulty that their students had in learning geometry. After several years of teaching, the van Hieles recognized that levels of thought and language are obstacles to student learning, because the students who are reasoning at a lower level are unprepared to perform at a higher level. The higher level questions are too difficult for the students.

Consider a hurdle race as an analogy. To finish the race one must jump over one hurdle at a time sequentially. (High jump or pole vault competition are also analogies.) It makes no sense to try to jump the last hurdle first and then go back and jump the beginning hurdles. That is just what we do in our geometry programs for students.

Many students are cheated out of adequate sequential geometric experiences in the elementary school and junior high school grades. Then, after one year of algebra, they are thrust unprepared into a high school geometry course and are expected to write formal proofs of theorems.

Here are the thought levels as originally described by the van Hieles.

Level 0 – Recognition

The behavior at this level is primarily visual. Students may recognize pictures of triangles, squares, rectangles as such but they are not actively aware of properties of the figures. It is interesting that even in the junior high school grades many students think there are "good triangles" and "bad triangles." They think that a good triangle must be equilateral, with its base parallel to the floor.

Students at this level think of shapes as a whole. They are not aware of many properties of the figures.

Level 1 – Analysis

The students now begin to analyze properties of the figures. As examples, they realize that the opposite sides and possibly even the diagonals of a rectangle are congruent; that a cube has six square faces; that the diagonals of a rhombus are perpendicular; that the base angles of an isosceles triangle are congruent.

Students at this level are able to work with parts of a figure. They are not aware of interrelationships between figures. For example, at this level squares, rectangles, and parallelograms all seem to be very different.

Level 2 – Ordering

The students begin to logically order figures, to understand interrelationships between figures, and to realize why it is necessary to describe things accurately. For

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example, because they know definitions of figures, students will realize that every square is a rhombus, that every rectangle is a parallelogram.

Students at this level are able to work with concise definitions and to apply elementary logic rules. They are able to classify figures. They are not able to explain (or prove) why certain things are true. For example, at this level students will have difficulty explaining why the opposite sides of a parallelogram are congruent, why a quadrilateral with congruent pairs of opposing sides must be a parallelogram, why the base angles of an isosceles triangle are congruent.

Level 3 – Deduction

The students begin to understand the significance of deduction and the role of postulates, theorems, and proofs.

Students at this level understand the difference between postulates and theorems. They are able to develop a chain of statements to connect a given hypothesis with a conclusion. They can work with given information from a figure and deduce conclusions whether or not the figure is drawn accurately. (They are working logically rather than visually.) They do not understand the foundations of geometry, why there are different geometries, or even how the SAS postulate connects with distance and angle measures in Euclidean geometry.

Level 4 – Rigor

This most advanced level is rarely reached by high school students. At this level students understand the importance of precision in dealing with foundations and interrelationships between structures. For example, students know how the Euclidean parallel postulate relates to the existence of rectangles and why in non-Euclidean geometry rectangles do not exist.

Proper Preparation

For a student to function adequately at one of the advanced levels, he or she must have mastered large chunks of the prior levels. All too often students who are forced to work at a level for which they are not prepared behave like parrots who are imitating the teacher or the textbook without understanding. They depend on memory to get them through. Most high school geometry courses operate at level 3, while students have not fully rounded out level 0. It is no wonder that geometry is a disliked subject. We force many students into failure situations.

The Future Directions of Geometry must involve this awareness of thought levels to better organize learning experiences for students of all ages. In particular, we would do most students a great service if we would devote the first half of the high school geometry course to informal activities and postpone formal proofs to the second semester. Then the students would move through the lower levels so proofs would not be an obstacle for them.

ACTIVITIES

Here are some sample activities for your students. The way the students respond will tell you about their thought level.

A. Make a sheet of quadrilaterals and non-quadrilaterals in various positions. Ask the students to put an S on the squares, R on the rectangles, D on rhombi, P on parallelograms and T on trapezoids. Can they recognize them? Do they realize that a rectangle is a parallelogram?

B. Have several quadrilateral shapes cut out of cardboard. Ask the students to put together the figures that are "alike in some way." Ask the students why they chose that grouping. Repeat this several times to look for different ways of sorting.

C. Ask students to pretend they are telling a blind friend what certain figures are. How would they describe a square, rhombus, rectangle, parallelogram, etc. Then ask them to give the shortest descriptions that they can think of.

D. Ask students to list some properties of a parallelogram. Discuss the properties with them. What else can they observe? Then pick one of the properties and ask the student to explain why it is true. Formulate the converse of the statement and ask the students if that is also true, and why.

E. Show the students a picture of a kite and ask what type of a figure it is (some will say parallelogram). Ask about properties of the figure (many will say that the opposite angles are congruent). Can they explain why their claims are true?

You can try these and similar activities with students at all ages. You will be surprised at some of the responses. In fact, many students who have studied a year of high school geometry still seem to be thinking at level 1.

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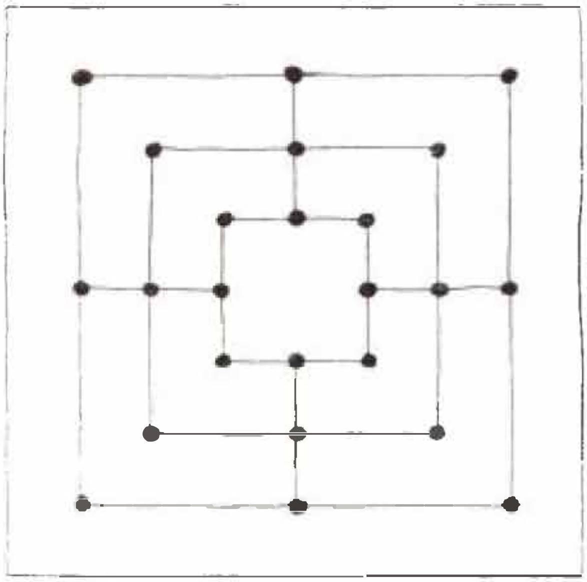

Plan Ahead

(A Strategy game for two players)

by Janet Hewitt, Spokane Lutheran School

Equipment: above gameboard (20cm square), 6 markers (3 of each color).

Rules: The object is to get your three pieces lined up along any one of the lines before your opponent can do so. In your first three turns, each of you places a piece on the board anywhere you desire. Thereafter, you can only move to an adjacent dot, and only along a line. You are not allowed to jump over a piece, nor can you move between dots not connected by a line.

I had my third and fourth graders make this game for their parents' Christmas gift, and they all seemed to have a good time playing the game before wrapping it up. It was a good exercise in measuring and using a ruler. We made the gameboard on tagboard, using colored pens to make it colorful. Then they printed the rules on the back and we covered both sides with contac paper. The children collected beer bottle caps and painted them two colors for markers (the twist off type are best and ones with a rebus inside are additional fun).

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