Summing Sets of Odd Integers: Patterns in Powers

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Teachers are constantly searching for activities that will aid in developing skills in computation and the use of calculators. It is motivational if number patterns can also be discovered.

Throughout this article we will use the set of positive odd integers (1, 3, 5, 7, ...). We will present several activities which are based upon sums of different sets of these integers.

Activity I

A rather well-known number pattern results from summing the first n odd integers for different values of n as shown in Table 1.

Table 1.

Indicated Sum	No. of Odd Integers in the Sum	Sum
1	1	$1 = 1^2$
1+3	2	$4 = 2^2$
1+3+5	3	$9 = 3^2$
1+3+5+7	4	$16 = 4^2$
1+3+5+7+9	5	$25 = 5^2$
.	.	.
.	.	.
.	.	.

In general the sum of the first n odd integers $[1+3+5+...+(2n-1)]$ yields n^2. The proof of this is found in chapters on mathematics induction in many mathematics texts.

Activity II

To generate cubes, again add consecutive odd integers. These sums are organized quite differently from those of Activity I. See Table 2.

Table 2.

Indicated Sum	No. of Odd Integers in the Sum	Sum
1	1	$1=1^3$
3+5	2	$8=2^3$
7+9+11	3	$27=3^3$
13+15+17+19	4	$64=4^3$
21+23+25+27+29	5	$125=5^3$
.	.	.
.	.	.
.	.	.

In general the sum of n appropriately chosen consecutive odd integers is n^3. In contrast to Activity I, these summed sets do not all begin at 1. In Activity II, each summed set begins immediately after the previous summed set concludes. No odd integer is used in more than one summed set. Because the summed sets partition the set of odd integers, call this the "partition sum method."

Let us verify the correctness of this summing procedure. We wish to sum n consecutive odd integers, so we need to determine the first odd integer to be summed. To find 1^3, 2^3, 3^3, ..., $(k-1)^3$ requires the summing of 1 odd integer, 2 odd integers, 3 odd integers, ..., (k-1) odd integers. Consequently, to find 1^3, 2^3, 3^3, ... , $(k-1)^3$ requires $1+2+3+...+(k-1)$ odd integers; that is, the first $\frac{(k-1)(k)}{2}$ odd integers have already been used in sums.

The last odd integer used in summing to find $(k-1)^3$ was $2(\frac{(k-1) \cdot k}{2}) -1$ or (k^2-k-1). The first odd integer to be summed to find k^3 is therefore $(k^2-k-1) + 2$ or (k^2-k+1). Since k odd integers are to be summed to generate k^3, the odd integers are:

$$\underbrace{k^2-k+1}_{1st}, \quad \underbrace{k^2-k+3}_{2nd}, \quad \underbrace{k^2-k+5}_{3rd}, \ldots, \underbrace{k^2-k+(2k-1)}_{4th}$$

This sum is thus:

$$\underbrace{(k^2-k+1) + (k^2-k+3) + (k^2-k+5) +\ldots+ (k^2-k+(2k-1))}_{k \text{ terms}}$$

$$= k(k^2-k) + [1+3+5+\ldots+(2k-1)]$$
$$= k^3-k^2 + [1+3+5+\ldots+(2k-1)]$$
$$= k^3-k^2 + k^2 \quad \text{(Activity I)}$$
$$= k^3.$$

Activity III

To generate the fourth powers, sum the sets as shown in Table 3.

Table 3.

Indicated Sum	No. of Odd Integers in the Sum	Sum
1	1	$1 = 1^4$
1+(3+5+7)	4	$16 = 2^4$
1+(3+5+7) + (9+11+13+15+17)	9	$81 = 3^4$
1+(3+5+7) + (9+11+13+15+17)+ (19+21+23+25+27+29+31)	16	$256 = 4^4$
1+(3+5+7) + (9+11+13+15+17)+ (19+21+23+25+27+29+31)+ (33+35+37+39+41+43+45+47+49)	25	$625 = 5^4$
.	.	.
.	.	.
.	.	.

Activity III resembles Activity I in that all summed sets begin with 1. It differs from Activity I in the number of odd integers summed. In Activity I, n^2 was found by summing the first n odd integers; in this activity, n^4 is found by summing the first n^2 odd integers.

Algebraically, this pattern is easy to verify. The sum of the first n^2 odd integers is $1+3+5+\ldots+(2n^2-1)$. By the results of Activity I, this sum is $(n^2)^2 = n^4$.

In general, this method enables us to generate all the even powers. To find n^{2k}, sum the first n^k odd integers, producing $1+3+5\ldots+(2n^k-1) = (n^k)^2 = n^{2k}$.

For example, $3^8 = 1+3+5+\ldots+(2 \cdot 3^4-1) = 1+3+5+\ldots+ 161 = 6561$.

Activity IV

To generate the fifth powers, sum the disjoint sets of odd integers as shown in Table 4.

Table 4.

Indicated Sum	No. of Odds Used	Odd Integers Skipped	No. of Odds Skipped	Sum
1	1	None	0	$1 = 1^5$
5+7+9+11	4	3	1	$32 = 2^5$
19+21+23+25+...+35	9	13,15,17	3	$243 = 3^5$
49+51+53+...+79	16	37,39,...,47	6	$1024 = 4^5$
101+103+105+...+149	25	81,83,...,99	10	$3125 = 5^5$
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.

Activity IV resembles Activity II in that disjoint sums are used. It differs from Activity II in the following ways:

1. The number of odd integers summed to achieve n^5 is n^2 (to achieve n^3, add n odd integers).

2. Odd integers are skipped between summed sets. The number of odd integers skipped is 1, 3, 6, 10, ... ; this is the set of triangular numbers.

A more general pattern connecting Activities I and II and Activities III and IV may be noticed. In generating either n^2 or n^3, n consecutive odd integers are summed. The summed sets which generate n^2 all begin with 1, while the summed sets which generate n^3 are disjoint.

In generating either n^4 or n^5, n^2 consecutive odd integers are summed. The summed sets generating n^4 all begin with 1, while the summed sets which generate n^5 are disjoint.

Challenges for the Reader

1. We know that the pattern for generating n^6 is to sum the first n^3 odd integers where the summed sets all start with 1 (Activity III). From the relationship just noted among Activities I through IV, it might be conjectured that values of n^7 could be found by summing disjoint sets of n^3 odd integers.
   a) Is this conjecture true?
   b) If the conjecture is true, how many odd integers must be skipped between the summed sets?

2. Generate other odd powers, and conjecture a general pattern for these cases.

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