Product Differences of Consecutive Factors

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Consider 3-factor products in the following form:

n(n + a)(n + 2a)
(n + a)(n + 2a)(n + 3a)
(n + 2a)(n + 3a)(n + 4a)
(n + 3a)(n + 4a)(n + 5a)
.
.
.

Table 1 reports the results of several 3-factor products. In each case, the third differences are constant.

Table 1.

Factors differ by 1

1 x 2 x 3 = 6			
	18		
2 x 3 x 4 = 24		18	
	36		6
3 x 4 x 5 = 60		24	
	60		6
4 x 5 x 6 = 120		30	
	90		6
5 x 6 x 7 = 210		36	
	126		6
6 x 7 x 8 = 336		42	
	168		
7 x 8 x 9 = 504			

Factors differ by 3

3 x 6 x 9 = 162			
	486		
6 x 9 x 12 = 648		486	
	972		162
9 x 12 x 15 = 1620		648	
	1620		162
12 x 15 x 18 = 3240		810	
	2430		162
15 x 18 x 21 = 5670		972	
	3402		162
18 x 21 x 24 = 9072		1134	
	4536		
21 x 24 x 27 = 13608			

Factors differ by 2

1 x 3 x 5 = 15			
	90		
3 x 5 x 7 = 105		120	
	210		48
5 x 7 x 9 = 315		168	
	378		48
7 x 9 x 11 = 693		216	
	594		48
9 x 11 x 13 = 1287		264	
	858		48
11 x 13 x 15 = 2145		312	
	1170		
13 x 15 x 17 = 3315			

Factors differ by 4

3 x 7 x 11 = 231			
	924		
7 x 11 x 15 = 1155		1056	
	1980		384
11 x 15 x 19 = 3135		1440	
	3420		384
15 x 19 x 23 = 6555		1824	
	5244		384
19 x 23 x 27 = 11799		2208	
	7452		384
23 x 27 x 31 = 19251		2592	
	10044		
27 x 31 x 35 = 29295			

Table 2 reports the factor differences and the third differences between products.

Table 2.

Factor Difference (a)	Third Product Differences
1	$6 = 6(1)^3$
2	$48 = 6(2)^3$
3	$162 = 6(3)^3$
4	$384 = 6(4)^3$

From the results of Table 2, Conjecture I is made. Consider the following 3-factor products:

$n(n + a)(n + 2a)$
$(n + a)(n + 2a)(n + 3a)$
$(n + 2a)(n + 3a)(n + 4a)$
$(n + 3a)(n + 4a)(n + 5a)$
.
.
.

Then, the third differences between consecutive products are $6a^3$.

PROOF — Select four consecutive products of the following form:

$$P_1 = (n + ai)(n + a(i+1))(n + a(i+2))$$
$$P_2 = (n + a(i+1))(n + a(i+2))(n + a(i+3))$$
$$P_3 = (n + a(i+2))(n + a(i+3))(n + a(i+4))$$
$$P_4 = (n + a(i+3))(n + a(i+4))(n + a(i+5))$$

Then,

$$\begin{aligned} D_1 &= P_2 - P_1 \\ &= (n + a(i+1))(n + a(i+2))(n + a(i+3)) - (n + ai)(n + a(i+1))(n + a(i+2)) \\ &= (n + a(i+1))(n + a(i+2))[(n+a(i+3)) - (n + ai)] \\ &= (n + a(i+1))(n + a(i+2))[3a] \end{aligned}$$

$$\begin{aligned} D_2 &= P_3 - P_2 \\ &= (n + a(i+2))(n + a(i+3))(n + a(i+4)) - (n + a(i+1))(n + a(i+2))(n + a(i+3)) \\ &= (n + a(i+2))(n + a(i+3))[3a] \end{aligned}$$

$$\begin{aligned} D_3 &= P_4 - P_3 \\ &= (n + a(i+3))(n + a(i+4))(n + a(i+5)) - (n + a(i+2))(n + a(i+3))(n + a(i+4)) \\ &= (n + a(i+3))(n + a(i+4))[3a] \end{aligned}$$

$$SD_1 = D_2 - D_1$$
$$= (n + a(i+2))(n + a(i+3))[3a] - (n + a(i+1))(n + a(i+2))[3a]$$
$$= (3a)(n + a(i+2))[2a]$$
$$= 6a^2(n + a(i+2))$$

$$SD_2 = D_3 - D_2$$
$$= (n + a(i+3))(n + a(i+4))[3a] - (n + a(i+2))(n + a(i+3))[3a]$$
$$= 6a^2(n+a(i+3))$$

$$TD = SD_2 - SD_1$$
$$= 6a^2(n + a(i+3)) - 6a^2(n + a(i+2))$$
$$= 6a^2(a)$$
$$= 6a^3$$

The third difference, $6a^3$, depends only upon the difference between consecutive factors and not upon the size of the factors themselves. Thus, Conjecture I is established.

Now consider 4-factor products of the form:

$n(n + a)(n + 2a)(n + 3a)$
$(n + a)(n + 2a)(n + 3a)(n + 4a)$
$(n + 2a)(n + 3a)(n + 4a)(n + 5a)$
$(n + 3a)(n + 4a)(n + 5a)(n + 6a)$
$(n + 4a)(n + 5a)(n + 6a)(n + 7a)$
.
.
.

Table 3 reports results of 4-factor products for a = 1, 2, 3, 4.

Table 3.

2 x 3 x 4 x 5 = 120				
	240			
3 x 4 x 5 x 6 = 360		240		
	480		120	
4 x 5 x 6 x 7 = 840		360		24
	840		144	
5 x 6 x 7 x 8 = 1680		504		24
	1344		168	
6 x 7 x 8 x 9 = 3024		672		24
	2016		192	
7 x 8 x 9 x 10 = 5040		864		
	2880			
8 x 9 x 10 x 11 = 7920				

3 x 5 x 7 x 9 = 945				
	2,520			
5 x 7 x 9 x 11 = 3,465		3,024		
	5,544		1,728	
7 x 9 x 11 x 13 = 9,009		4,752		384
	10,296		2,112	
9 x 11 x 13 x 15 = 19,305		6,864		384
	17,160		2,496	
11 x 13 x 15 x 17 = 36,465		9,360		384
	26,520		2,880	
13 x 15 x 17 x 19 = 62,985		12,240		
	38,760			
15 x 17 x 19 x 21 = 101,745				

1 x 4 x 7 x 10 = 280				
	3,360			
4 x 7 x 10 x 13 = 3,640		7,560		
	10.920		6,480	
7 x 10 x 13 x 16 = 14,560		14,040		1,944
	24,960		8,424	
10 x 13 x 16 x 19 = 39,520		22,464		1,944
	47,424		10,368	
13 x 16 x 19 x 22 = 86,944		32,832		1,944
	80,256		12,312	
16 x 19 x 22 x 25 = 167,200		45,144		
	125,400			
19 x 22 x 25 x 28 = 292,600				

2 x 6 x 10 x 14 = 1,680				
	13,440			
6 x 10 x 14 x 18 = 15,120		26,880		
	40,320		21,504	
10 x 14 x 18 x 22 = 55,440		48,384		6,144
	88,704		27,648	
14 x 18 x 22 x 26 = 144,144		76,032		6,144
	164,736		33,792	
18 x 22 x 26 x 30 = 308,880		109,824		6,144
	274,560		39,936	
22 x 26 x 30 x 34 = 583,440		149,760		
	424,320			
26 x 30 x 34 x 38 = 1,007,760				

The fourth differences are constant in each case. Table 4 shows the factor differences and the fourth differences of the products for each case.

Table 4.

Factor Difference (a)	*Fourth Product Difference*
1	$24 = 24(1)^4$
2	$384 = 24(2)^4$
3	$1944 = 24(3)^4$
4	$6144 = 24(4)^4$

An examination of Table 4 yields Conjecture II. Consider the following 4-factor products:

$$n(n + a)(n + 2a)(n + 3a)$$
$$(n + a)(n + 2a)(n + 3a)(n + 4a)$$
$$(n + 2a)(n + 3a)(n + 4a)(n + 5a)$$
$$(n + 3a)(n + 4a)(n + 5a)(n + 6a)$$
$$(n + 4a)(n + 5a)(n + 6a)(n + 7a)$$
$$\vdots$$

Then, the fourth differences between consecutive products are $24a^4$.

PROOF — Select 4-factor products of the following form:

$$P_1 = (n + ai)(n + a(i+1))(n + a(i+2))(n + a(i+3))$$
$$P_2 = (n + a(i+1))(n + a(i+2))(n + a(i+3))(n + a(i+4))$$
$$P_3 = (n + a(i+2))(n + a(i+3))(n + a(i+4))(n + a(i+5))$$
$$P_4 = (n + a(i+3))(n + a(i+4))(n + a(i+5))(n + a(i+6))$$
$$P_5 = (n + a(i+4))(n + a(i+5))(n + a(i+6))(n + a(i+7))$$

Then, $D_1 = P_2 - P_1$
$$= (n+a(i+1))(n + a(i+2))(n + a(i+3))[4a]$$

$$D_2 = P_3 - P_2$$
$$= (n + a(i+2))(n + a(i+3))(n + a(i+4))[4a]$$

$$D_3 = P_4 - P_3$$
$$= (n + a(i+3))(n + a(i+4))(n + a(i+5))[4a]$$

$$D_4 = P_5 - P_4$$
$$= (n + a(i+4))(n + a(i+5))(n + a(i+6))[4a]$$

$$SD_1 = D_2 - D_1$$
$$= (4a)[(n + a(i+2))(n + a(i+3))](3a)$$
$$= 12a^2[(n + a(i+2))(n + a(i+3))]$$

$$SD_2 = D_3 - D_2$$
$$= 12a^2[(n + a(i+3))(n + a(i+4))]$$

$$SD_3 = D_4 - D_3$$
$$= 12a^2[(n + a(i+4))(n + a(i+5))]$$

$$TD_1 = SD_2 - SD_1$$
$$= 12a^2(n + a(i+3))[2a]$$
$$= 24a^3(n + a(i+3))$$

$$TD_2 = SD_3 - SD_2$$
$$= 24a^3(n + a(i+4))$$

$$FD = TD_2 - TD_1$$
$$= 24a^3(a)$$
$$= 24a^4$$

Thus, Conjecture II is established.

Challenges for the Reader

1. We have investigated products resulting from 3-factor and 4-factor combinations in which the factors differed by a fixed amount, and the second factor of any given product became the first factor in the next product. Extend the patterns to include 5-factor products, 6-factor products, . . . Table 5 should result from your computations, where a is the difference between factors.

Table 5.

Number of Factors (j)	*j^{th} Product Differences*
3	$6a^3 = 3!a^3$
4	$24a^4 = 4!a^4$
5	$120a^5 = 5!a^5$
6	$720a^6 = 6!a^6$
.	.
.	.
.	.

2. The results of Table 5 may be generalized by Conjecture III. Consider the following j-factor products:

$n(n + a)(n + 2a)(n + 3a) \ . \ . \ . \ (n + (j-1)a)$
$(n + a)(n + 2a)(n + 3a)(n + 4a) \ . \ . \ . \ (n + ja)$
$(n + 2a)(n + 3a)(n + 4a)(n + 5a) \ . \ . \ . \ (n + (j+1)a)$
. . .

Then, the j^{th} product differences are all $j!(a)^j$. Prove this conjecture.

3. In each of the examples of this article, the second factor of any given product became the first factor of the next product. What happens if the third (fourth, fifth, . . .) factor of a given product becomes the first factor of the next product and then consecutive differences (third, fourth, fifth, . . .) are found? □