

Product Differences of Consecutive Factors

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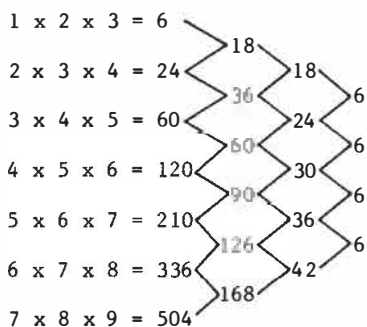
Consider 3-factor products in the following form:

$$\begin{aligned} &n(n + a)(n + 2a) \\ &(n + a)(n + 2a)(n + 3a) \\ &(n + 2a)(n + 3a)(n + 4a) \\ &(n + 3a)(n + 4a)(n + 5a) \\ &\vdots \\ &\vdots \\ &\vdots \end{aligned}$$

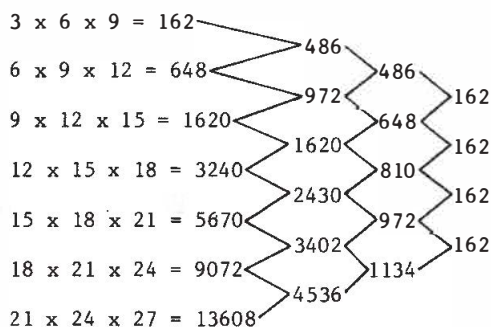
Table 1 reports the results of several 3-factor products. In each case, the third differences are constant.

Table 1.

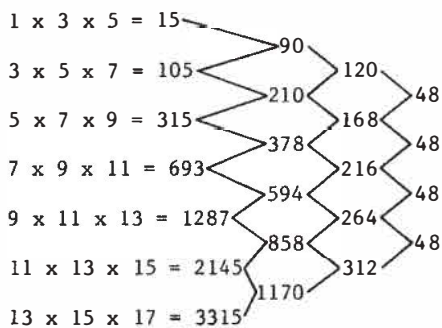
Factors differ by 1



Factors differ by 3



Factors differ by 2



Factors differ by 4

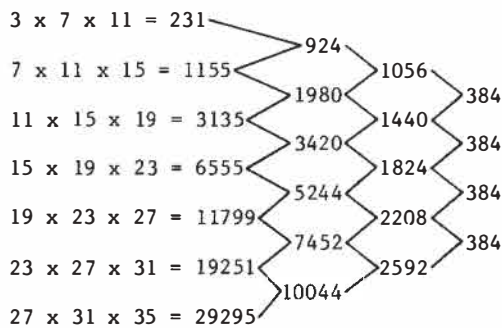


Table 2 reports the factor differences and the third differences between products.

Table 2.

<i>Factor Difference (a)</i>	<i>Third Product Differences</i>
1	$6 = 6(1)^3$
2	$48 = 6(2)^3$
3	$162 = 6(3)^3$
4	$384 = 6(4)^3$

From the results of Table 2, Conjecture I is made. Consider the following 3-factor products:

$$\begin{aligned}
 &n(n + a)(n + 2a) \\
 &(n + a)(n + 2a)(n + 3a) \\
 &(n + 2a)(n + 3a)(n + 4a) \\
 &(n + 3a)(n + 4a)(n + 5a) \\
 &\vdots \\
 &\vdots \\
 &\vdots
 \end{aligned}$$

Then, the third differences between consecutive products are $6a^3$.

PROOF — Select four consecutive products of the following form:

$$\begin{aligned}
 P_1 &= (n + ai)(n + a(i+1))(n + a(i+2)) \\
 P_2 &= (n + a(i+1))(n + a(i+2))(n + a(i+3)) \\
 P_3 &= (n + a(i+2))(n + a(i+3))(n + a(i+4)) \\
 P_4 &= (n + a(i+3))(n + a(i+4))(n + a(i+5))
 \end{aligned}$$

Then,

$$\begin{aligned}
 D_1 &= P_2 - P_1 \\
 &= (n + a(i+1))(n + a(i+2))(n + a(i+3)) - (n + ai)(n + a(i+1))(n + a(i+2)) \\
 &= (n + a(i+1))(n + a(i+2))[(n+a(i+3)) - (n + ai)] \\
 &= (n + a(i+1))(n + a(i+2))[3a]
 \end{aligned}$$

$$\begin{aligned}
 D_2 &= P_3 - P_2 \\
 &= (n + a(i+2))(n + a(i+3))(n + a(i+4)) - (n + a(i+1))(n + a(i+2))(n + a(i+3)) \\
 &= (n + a(i+2))(n + a(i+3))[3a]
 \end{aligned}$$

$$\begin{aligned}
 D_3 &= P_4 - P_3 \\
 &= (n + a(i+3))(n + a(i+4))(n + a(i+5)) - (n + a(i+2))(n + a(i+3))(n + a(i+4)) \\
 &= (n + a(i+3))(n + a(i+4))[3a]
 \end{aligned}$$

$$\begin{aligned}
SD_1 &= D_2 - D_1 \\
&= (n + a(i+2))(n + a(i+3))[3a] - (n + a(i+1))(n + a(i+2))[3a] \\
&= (3a)(n + a(i+2))[2a] \\
&= 6a^2(n + a(i+2))
\end{aligned}$$

$$\begin{aligned}
SD_2 &= D_3 - D_2 \\
&= (n + a(i+3))(n + a(i+4))[3a] - (n + a(i+2))(n + a(i+3))[3a] \\
&= 6a^2(n + a(i+3))
\end{aligned}$$

$$\begin{aligned}
TD &= SD_2 - SD_1 \\
&= 6a^2(n + a(i+3)) - 6a^2(n + a(i+2)) \\
&= 6a^2(a) \\
&= 6a^3
\end{aligned}$$

The third difference, $6a^3$, depends only upon the difference between consecutive factors and not upon the size of the factors themselves. Thus, Conjecture I is established.

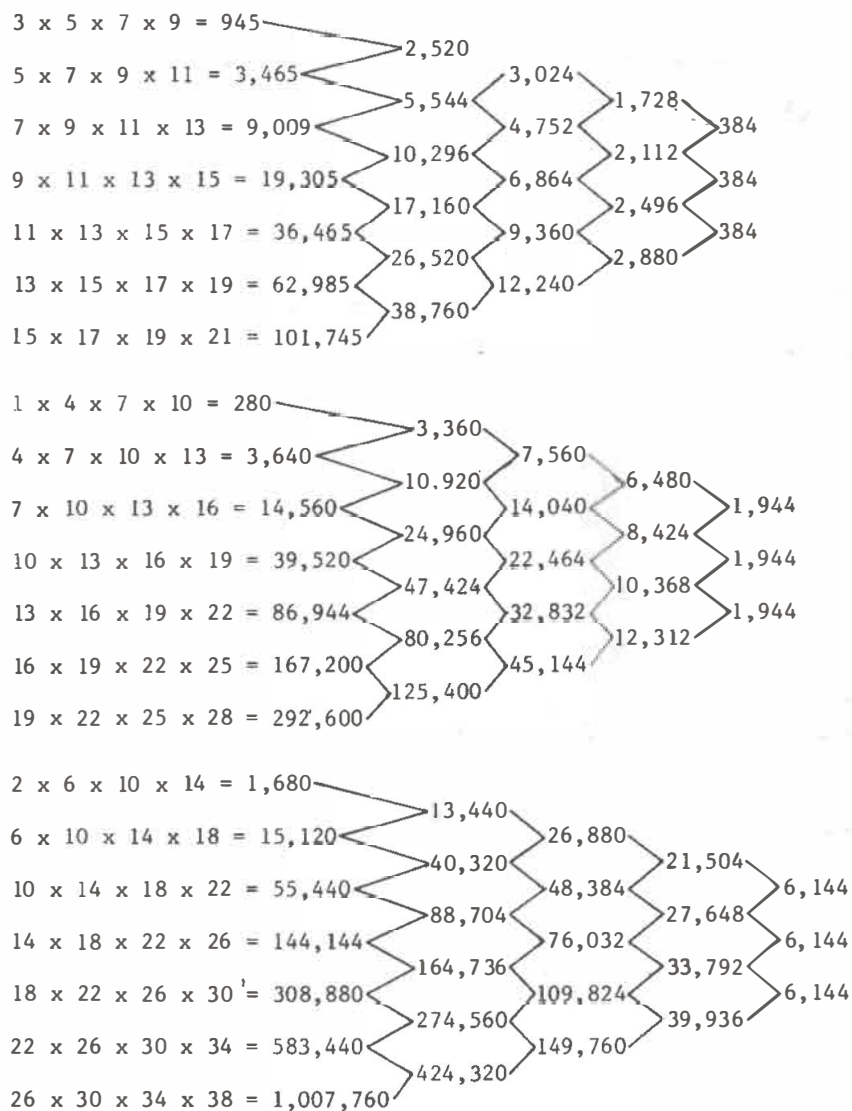
Now consider 4-factor products of the form:

$$\begin{aligned}
&n(n + a)(n + 2a)(n + 3a) \\
&(n + a)(n + 2a)(n + 3a)(n + 4a) \\
&(n + 2a)(n + 3a)(n + 4a)(n + 5a) \\
&(n + 3a)(n + 4a)(n + 5a)(n + 6a) \\
&(n + 4a)(n + 5a)(n + 6a)(n + 7a) \\
&\quad \vdots \\
&\quad \vdots \\
&\quad \vdots
\end{aligned}$$

Table 3 reports results of 4-factor products for $a = 1, 2, 3, 4$.

Table 3.

2 x 3 x 4 x 5 = 120					
3 x 4 x 5 x 6 = 360	240				
4 x 5 x 6 x 7 = 840	480	240			
5 x 6 x 7 x 8 = 1680	840	360	120	24	
6 x 7 x 8 x 9 = 3024	1344	504	144	24	
7 x 8 x 9 x 10 = 5040	2016	672	168	24	
8 x 9 x 10 x 11 = 7920	2880	864	192	24	



The fourth differences are constant in each case. Table 4 shows the factor differences and the fourth differences of the products for each case.

Table 4.

Factor Difference (a)	Fourth Product Difference
1	$24 = 24(1)^4$
2	$384 = 24(2)^4$
3	$1944 = 24(3)^4$
4	$6144 = 24(4)^4$

An examination of Table 4 yields Conjecture II. Consider the following 4-factor products:

$$\begin{aligned}
 & n(n+a)(n+2a)(n+3a) \\
 & (n+a)(n+2a)(n+3a)(n+4a) \\
 & (n+2a)(n+3a)(n+4a)(n+5a) \\
 & (n+3a)(n+4a)(n+5a)(n+6a) \\
 & (n+4a)(n+5a)(n+6a)(n+7a) \\
 & \quad \cdot \\
 & \quad \cdot \\
 & \quad \cdot
 \end{aligned}$$

Then, the fourth differences between consecutive products are $24a^4$.

PROOF — Select 4-factor products of the following form:

$$\begin{aligned}
 P_1 &= (n+ai)(n+a(i+1))(n+a(i+2))(n+a(i+3)) \\
 P_2 &= (n+a(i+1))(n+a(i+2))(n+a(i+3))(n+a(i+4)) \\
 P_3 &= (n+a(i+2))(n+a(i+3))(n+a(i+4))(n+a(i+5)) \\
 P_4 &= (n+a(i+3))(n+a(i+4))(n+a(i+5))(n+a(i+6)) \\
 P_5 &= (n+a(i+4))(n+a(i+5))(n+a(i+6))(n+a(i+7))
 \end{aligned}$$

$$\begin{aligned}
 \text{Then, } D_1 &= P_2 - P_1 \\
 &= (n+a(i+1))(n+a(i+2))(n+a(i+3))[4a]
 \end{aligned}$$

$$\begin{aligned}
 D_2 &= P_3 - P_2 \\
 &= (n+a(i+2))(n+a(i+3))(n+a(i+4))[4a]
 \end{aligned}$$

$$\begin{aligned}
 D_3 &= P_4 - P_3 \\
 &= (n+a(i+3))(n+a(i+4))(n+a(i+5))[4a]
 \end{aligned}$$

$$\begin{aligned}
 D_4 &= P_5 - P_4 \\
 &= (n+a(i+4))(n+a(i+5))(n+a(i+6))[4a]
 \end{aligned}$$

$$\begin{aligned}
 SD_1 &= D_2 - D_1 \\
 &= (4a)[(n+a(i+2))(n+a(i+3))](3a) \\
 &= 12a^2[(n+a(i+2))(n+a(i+3))]
 \end{aligned}$$

$$\begin{aligned}
 SD_2 &= D_3 - D_2 \\
 &= 12a^2[(n+a(i+3))(n+a(i+4))]
 \end{aligned}$$

$$\begin{aligned}
 SD_3 &= D_4 - D_3 \\
 &= 12a^2[(n+a(i+4))(n+a(i+5))]
 \end{aligned}$$

$$\begin{aligned}
 TD_1 &= SD_2 - SD_1 \\
 &= 12a^2(n+a(i+3))[2a] \\
 &= 24a^3(n+a(i+3))
 \end{aligned}$$

$$\begin{aligned} TD_2 &= SD_3 - SD_2 \\ &= 24a^3(n + a(i+4)) \end{aligned}$$

$$\begin{aligned} FD &= TD_2 - TD_1 \\ &= 24a^3(a) \\ &= 24a^4 \end{aligned}$$

Thus, Conjecture II is established.

Challenges for the Reader

1. We have investigated products resulting from 3-factor and 4-factor combinations in which the factors differed by a fixed amount, and the second factor of any given product became the first factor in the next product. Extend the patterns to include 5-factor products, 6-factor products, . . . Table 5 should result from your computations, where a is the difference between factors.

Table 5.

Number of Factors (j)	j^{th} Product Differences
3	$6a^3 = 3!a^3$
4	$24a^4 = 4!a^4$
5	$120a^5 = 5!a^5$
6	$720a^6 = 6!a^6$
.	.
.	.
.	.

2. The results of Table 5 may be generalized by Conjecture III. Consider the following j -factor products:

$$\begin{array}{ccccccc} n(n+a)(n+2a)(n+3a) & \dots & (n+(j-1)a) & & & & \\ (n+a)(n+2a)(n+3a)(n+4a) & \dots & (n+ja) & & & & \\ (n+2a)(n+3a)(n+4a)(n+5a) & \dots & (n+(j+1)a) & & & & \\ \vdots & & \vdots & & & & \\ \vdots & & \vdots & & & & \\ \vdots & & \vdots & & & & \end{array}$$

Then, the j^{th} product differences are all $j!(a)^j$. Prove this conjecture.

3. In each of the examples of this article, the second factor of any given product became the first factor of the next product. What happens if the third (fourth, fifth, . . .) factor of a given product becomes the first factor of the next product and then consecutive differences (third, fourth, fifth, . . .) are found? □