Dissection-Transformation Activities: Rectangulations in the Plane

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This article is dedicated to the Canadian Mathematics Education Study Group/ Geometry Subgroup.

"Geometry is the study of space and spatial relationships."

Geometry is a subject rich in history and broad in scope. It was developed from practical activities and from the problems of daily life. The properties of geometric concepts as well as the concepts themselves have been abstracted from the world around us. It was necessary for people to draw many straight lines before they developed the axiom that a straight line can be drawn through any two arbitrary distinct points. They had to move various plane regions about and apply them to one another on many occasions before they could come to generalize their experiences to the notion of superposition of plane regions and employ this notion for the proof of theorems. This was done in the famous theorems about the congruence of traingular regions.

The following are some hand-mind investigations which might offer students straightforward physical experiences vital to formal conceptions of some goemetric ideas.

Objectives

1. To develop a general pattern in dissecting a triangular region into a rectangular region of equal area.

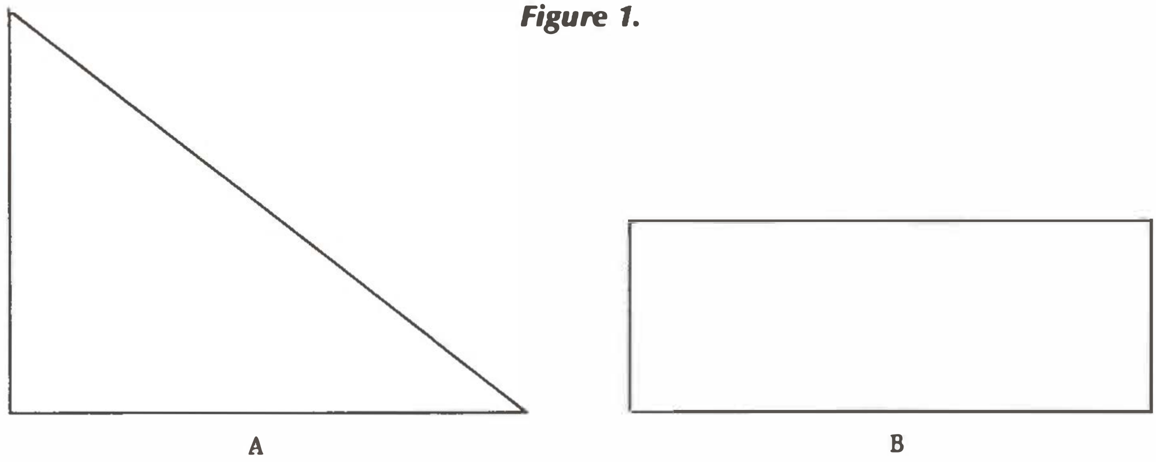
2. To develop the ability to distinguish between the perimeter and area of a given triangular region and to correct the common misconception that area is measured by the perimeter.

3. To expose interrelationships that exist among various polygnal regions.

Investigation I

A is a right triangular region and B is a rectangular region.

Figure 1.



There are certain relations between A and B related to their perimeters, areas, and corresponding sides. To investigate and identify them:

1. Trace and cut out a copy for each region. Dissect region A into pieces (two, three, or more; use the fewest pieces possible) and try to cover region B completely.

   Use = , < , or > to complete the following statement:

   Area of A _____ Area of B

2. Measure the sides of each region and calculate their perimeters. Use = , < , or > to complete the following statement:

   Perimeter of A ____ Perimeter of B.

3. Complete the following statement:

   A right triangular region and a rectangular region of ____ areas can be of ____ perimeters.

Instructional Suggestions:

1. Clues and leading questions may be given based on the following facts:

   a) There is a linear congruency between the bases (this is crucial to recognize).
   b) The altitude of A equals twice the altitude of B, and hence the midpoint concept is vital.
   c) In particular, the midpoints of the altitude and the hypotenuse are to be identified, and cutting through the line joining them is the key to a successful dissection.
   d) A half-turn motion is involved.

2. The dissection of A into B below can be looked at as:

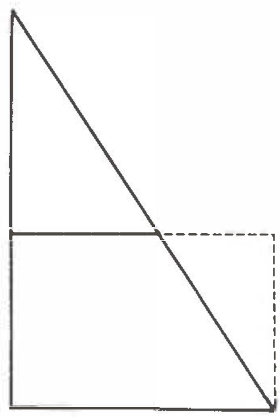


Figure 2.

A dissection of one-half of an isosceles triangular region and a dissection of an isosceles triangular region into an area-equivalent rectangular region would follow where the third side is taken as a base.

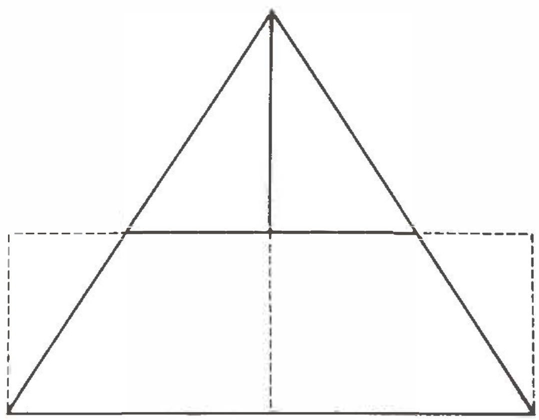


Figure 3.

3. The dissection of A into B can be used further to dissect any triangular region into an area-equivalent rectangular region, provided that in the obtuse case, the longest side is to be taken as a base.

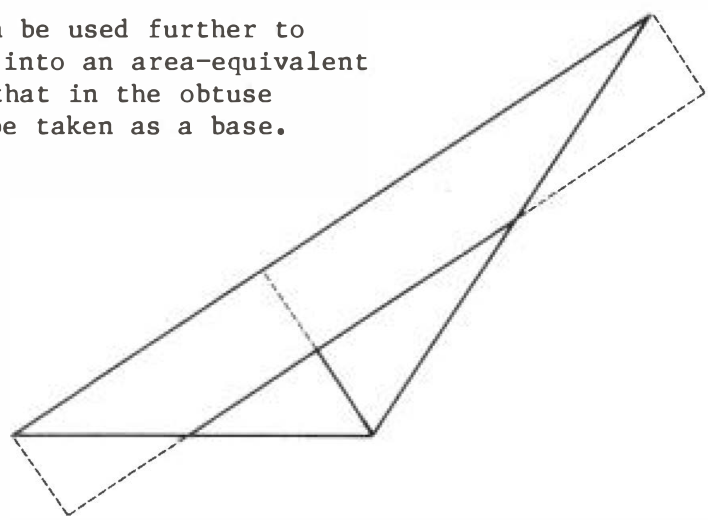


Figure 4.

Investigation II

Below are three copies of a right triangular region with the lengths of the sides as indicated.

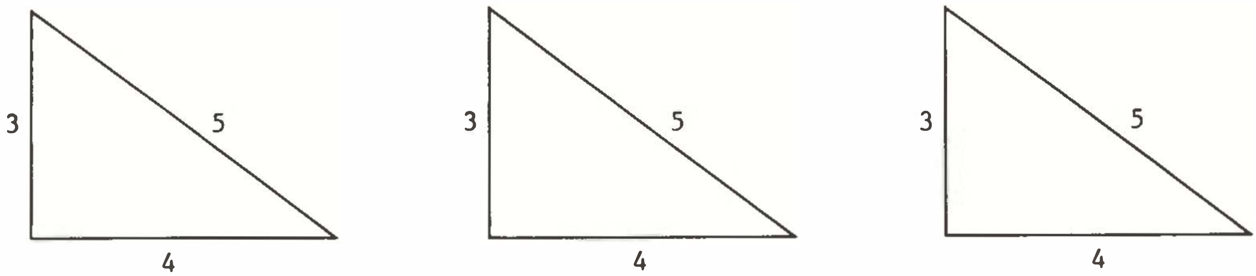


Figure 5.

1. Dissect each region into a rectangular region such that a different side at a time is taken as a base for both the proposed rectangular region and the triangular region.

2. Complete the following table:

	First Copy	Second Copy	Third Copy
Base of □ Altitude of □	4	3	5
Base of ◺ Altitude of ◺			
Perimeter of □ Perimeter of ◺			
Area of □ Area of ◺			

3. Using the data in the table, choose a word from those in the brackets to make a valid statement out of the following statement:

   While the areas for the two regions are (equal/not equal), the perimeters inscribed by them are (varied/not varied).

4. Justify your statement by numerical evidences from the table.

5. Accordingly, make a valid conclusion out of the following:

The perimeter of a region (can/cannot) be a measure for the inscribed area.

Instructional Suggestions:

1. For the case where the hypotenuse is taken as a base for the proposed rectangular region, it is quicker to calculate the altitude of the resultant rectangular region by:

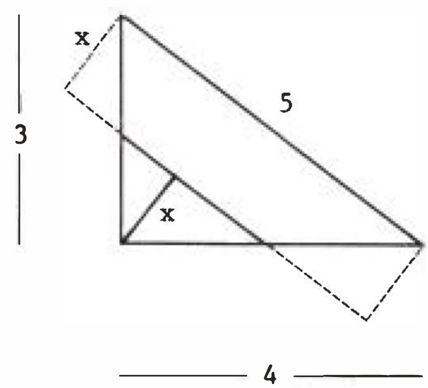


Figure 6.

Area of $\qquad$ = Area of $\qquad$ (by dissection)

$$5(x) = \frac{1}{2}(4)(3)$$

$$x = 1.2$$

rather than by the Pythagorean Theorem:

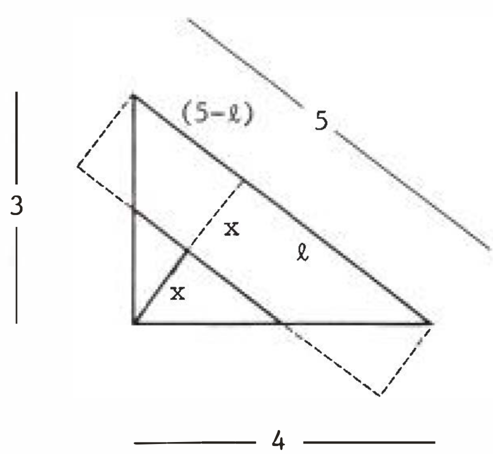


Figure 7.

$$(2x)^2 + \ell^2 = 4^2$$

$$(2x)^2 + (5 - \ell)^2 = 3^2 \qquad \text{and so on.}$$

2. The table above indicates that while the areas for the two regions in all cases are equal (6 area units), the perimeters inscribed by them vary (taking the values 10, 11, 12, or 12.4 length units). *Therefore, the perimeter cannot be a measure for the inscribed area.*

3. Clues and leading questions related to such key concepts as midpoints and the type of motion are to be offered throughout the instructional period so that continuous work will be maintained. In the third copy, for example, and depending on the level of the class, the corresponding altitude of the triangular region (2.4 length units) may be given.

Investigation III

A, B, C, and D are triangular regions. Trace, cut out copies, and then use for the following:

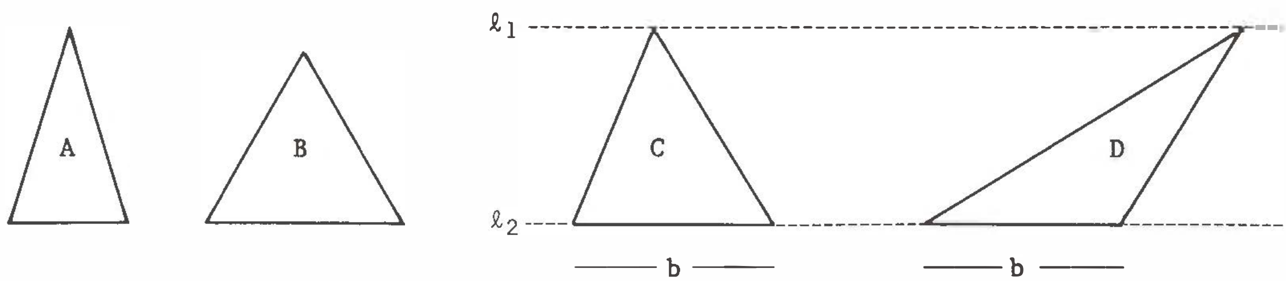


Figure 8.

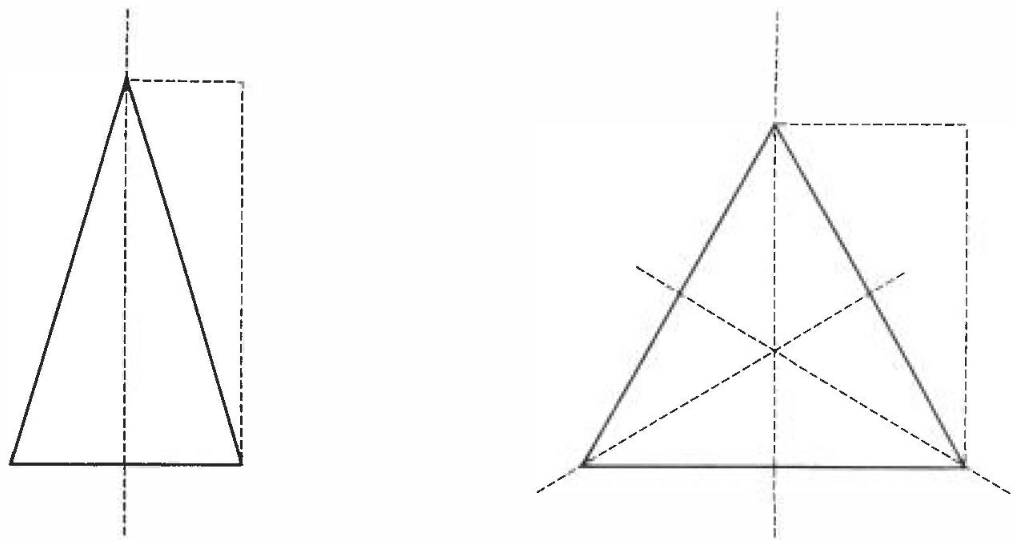
1. Dissect each of them into a rectangular region of equal area. (Use the fewest pieces possible.)

2. Dissect each of A and B into an area-equivalent rectangular region using only one cut. How many lines of symmetry does A have? Does B have? You may use these lines.

3. Dissect C and D into area-equivalent rectangular regions with base = b length units. Note that $\ell_1 \parallel \ell_2$. Are the resulting rectangular regions congruent? What can you tell, then, about the areas of the triangular regions? How many lines of symmetry does C have? Does D have?

Instructional Suggestions:

1. The concept of symmetry may be introduced throughout the dissection procedures of A and B where the lines of symmetry would give simple cuts for A and B in terms of the number of pieces used.



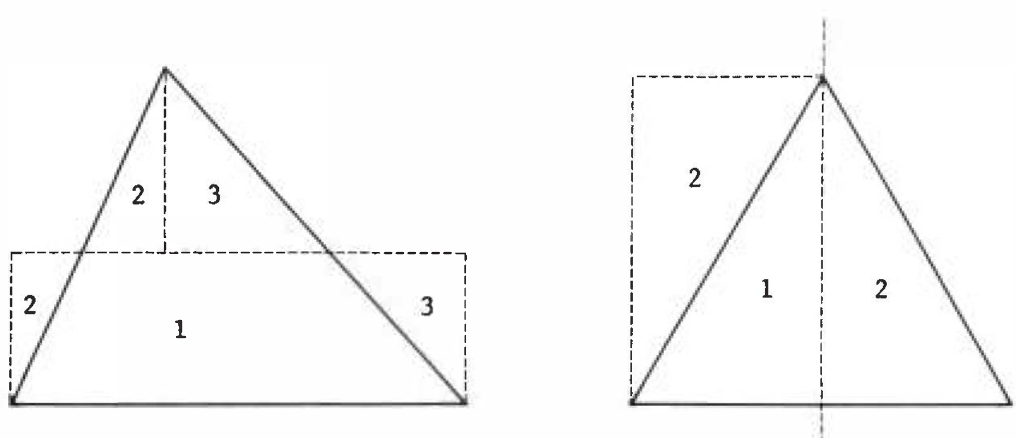
One line of symmetry indicates one option for the cutting.

The lines of symmetry indicate three options for the cutting.

Figure 9.

2. Clues and leading questions on some involved concepts such as symmetry, midpoints, and the motions of slide, turn, and flip are to be offered.

3. A comparison may be made on the role of the line of symmetry, when it exists, similar to the following:



In a triangular region, whenever a line of symmetry exists, the number of pieces will be less, and the dissection will be simpler.

Figure 10. □