Geometric Patterns for Combinatorial Results

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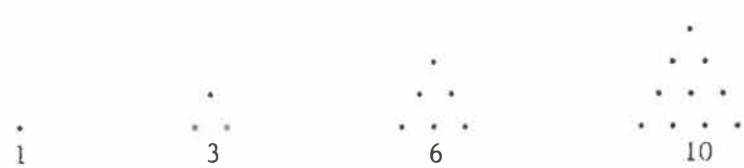
Teachers and their students frequently wish to use activities that provide for drill and practice and also encourage the review or discovery of number patterns. A rich source for such activities is found in sets of figurate numbers; that is, numbers denoting points that can be arranged in geometric shapes.

A familiar set of figurate numbers is the set of triangular numbers. The first four triangular numbers are shown below; they are so named because of the triangular shapes they form.



Verify, by drawing, the appropriate pictures that the first 10 triangular numbers are 1, 3, 6, 10, 15, 21, 28, 36, 45, and 55.

A three-dimensional extension of the notion of triangular numbers results in the tetrahedral numbers 1, 4, 10, 20, 35, 56, 84, 120, 165, 220, ··· These tetrahedral numbers can be depicted as pyramidal "stacks" of triangular numbers, similar to those formed by stacks of cannonballs in an American Civil War exhibit. Verify that the differences between consecutive numbers (4 - 1 = 3; 10 - 4 = 6; 20 - 10 = 10; 35 - 20 = 15; ···) are consecutive triangular numbers. That this should be true is apparent if one recalls that tetrahedral numbers are formed three-dimensionally by stacking two-dimensional layers of triangular numbers. We will now generate these two sets of numbers in a different way.

Recall that $\binom{n}{r} = \frac{n!}{r!(n-r)!}$

Activity 1

$$\binom{2}{2} = \frac{2!}{2!\ 0!} = 1$$

$$\binom{3}{2} = \frac{3!}{2!\ 1!} = 3$$

$$\binom{4}{2} = \frac{4!}{2!\ 2!} = \frac{4 \cdot 3}{2 \cdot 1} = 6$$

$$\binom{5}{2} = \frac{5!}{2!\ 3!} = \frac{5 \cdot 4}{2 \cdot 1} = 10$$

Verify that in general $\binom{n}{2}$ generates the (n −1) st triangular number.

Activity 2

$$\binom{3}{3} = 1$$

$$\binom{4}{3} = 4$$

$$\binom{5}{3} = \frac{5 \cdot 4}{2 \cdot 1} = 10$$

$$\binom{6}{3} = \frac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1} = 20$$

$$\binom{7}{3} = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 35$$

Verify that in general $\binom{n}{3}$ generates the (n − 2) nd tetrahedral number.

Activity 3

$$\binom{4}{4} = 1$$

$$\binom{5}{4} = 5$$

$$\binom{6}{4} = \frac{6 \cdot 5}{2 \cdot 1} = 15$$

$$\binom{7}{4} = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 35$$

$$\binom{8}{4} = \frac{8 \cdot 7 \cdot 6 \cdot 5}{4 \cdot 3 \cdot 2 \cdot 1} = 70$$

Continue this pattern. Note that the differences of consecutive numbers in this sequence form consecutive tetrahedral numbers (5 - 1 = 4; 15 - 5 = 10; 35 - 15 = 20; 70 - 35 = 35; ...). To give a geometric explanation would require the notion of a four-dimensional "stack" of three-dimensional tetrahedral numbers. How might it be pictured?

Activity 4

Continue this pattern for $\binom{r}{r}$, $\binom{r+1}{r}$, $\binom{r+2}{r}$, $\binom{r+3}{r}$, ... for

r = 5, 6, 7, ... How can the results be described and/or

pictured?

Activity 5

Depict "rectangular" numbers as shown by the following n X (n + 1) rectangles.

1 X 2 = 2	2 X 3 = 6	3 X 4 = 12	4 X 5 = 20
(1st rectangular number)	(2nd rectangular number)	(3rd rectangular number)	(4th rectangular number)

Note that:
2 = 2
6 = 2 + 4
12 = 2 + 4 + 6
20 = 2 + 4 + 6 + 8
⋮ ⋮

Continue this pattern. Is it always the case that the n th rectangular number is the sum of the first n positive even integers?

Activity 6

Note that:
2 = 2 X 1
6 = 2 X 3
12 = 2 X 6
20 = 2 X 10
⋮ ⋮

Continue this pattern. The n th rectangular number appears to be equal to twice the n th triangular number. To verify the pattern, we can partition the triangular numbers as follows:

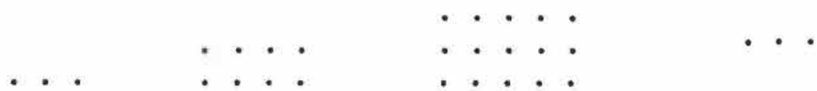
Activity 7

The following figure depicts the interior of a multiplication table. Note that the circled set represents the rectangular numbers of Activities 5 and 6.

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
0	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38
0	3	6	9	12	15	18	21	24	27	30	33	36	39	42	45	48	51	54	57
0	4	8	12	16	20	24	28	32	36	40	44	48	52	56	60	64	68	72	76
0	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90	95
0	6	12	18	24	30	36	42	48	54	60	66	72	78	84	90	96	102	108	114
0	7	14	21	28	35	42	49	56	63	70	77	84	91	98	105	112	119	126	133
0	8	16	24	32	40	48	56	64	72	80	88	96	104	112	120	128	136	144	152
0	9	18	27	36	45	54	63	72	81	90	99	108	117	126	135	144	153	162	171
0	10	20	30	40	50	60	70	80	90	100	110	120	130	140	150	160	170	180	190
0	11	22	33	44	55	66	77	88	99	110	121	132	143	154	165	176	187	198	209
0	12	24	36	48	60	72	84	96	108	120	132	144	156	168	180	192	204	216	228
0	13	26	39	52	65	78	91	104	117	130	143	156	169	182	195	208	221	234	247
0	14	28	42	56	70	84	98	112	126	140	154	168	182	196	210	224	238	252	266
0	15	30	45	60	75	90	105	120	135	150	165	180	195	210	225	240	255	270	285
0	16	32	48	64	80	96	112	128	144	160	176	192	208	224	240	256	272	288	304
0	17	34	51	68	85	102	119	136	153	170	187	204	221	238	255	272	289	306	323
0	18	36	54	72	90	108	126	144	162	180	198	216	234	252	270	288	306	324	342
0	19	38	57	76	95	114	133	152	171	190	209	228	247	266	285	304	323	342	361

Activity 8

Replicate the steps of Activities 5, 6, and 7 using rectangular numbers in an n X (n + 2) shape:



1 X 3   2 X 4   3 X 5   . . .

What patterns can be found?

This article has only scratched the surface of pattern-finding involving figurate numbers. Can you and your students find more?