Enlightened Problem Solving: A Parable

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The Lord said, "Let there be light," and so there was, and it was good. The mathematician said, "Let x be the unknown quantity," and so it was. Many problems were solved, and it was good. The NCTM said, "Let problem solving be the focus," and indeed it was, and it was good. In fact, there was a great flurry of activity and enthusiasm and commitment, and this was very good.

It came to pass that the ideas of a great scholar named Polya were carried far and wide, eventually to be accepted and enshrined as "The Four Commandments," which all disciples preached and sometimes practiced. Children, both young and old, were encouraged and guided to practice the Commandments, for it was believed that, in this way, genuine problem solvers would come about.

One day, a young lad, a most successful problem solver to be sure, realized that it was the Fourth Commandment, often called "looking back," that enabled him to continue to grow and to harvest the fruits of his past activity, for it was the Fourth Commandment that indeed commanded him to harvest, although he had not realized this before, since he thought it was simply a commandment to review, a commandment that his teachers routinely practised in many other settings almost to an excess.

He continued to solve problems, for the environment was rich in problems. He devoted more and more of his time to the Fourth Commandment, looking back at his work with anticipation of seeing something he had overlooked, and each time was rewarded handsomely, so perceptive had he become. He had, indeed, become a master of looking back, of reflecting upon his activities, sometimes to the exclusion of other activities.

"Have you ever wondered what happens to a problem on its way to a solution?"

His teacher noticed that he was no longer the first one to ask for new problems. He had taken to only solving the "problem of the week." He did this masterfully and with keen insight, sometimes programming general solutions on the microcomputer when this was appropriate, but his teacher was worried about his lack of productivity. Noticing that his teacher was concerned, and knowing his teacher was truly a believer in the Four Commandments, he sat down beside her one day and asked, "Have you ever wondered what happens to a problem on its way to a solution?" Somewhat taken aback, but not totally surprised, she replied, "Well not really. The FC (Four Commandments) don't really lead us to ask such a question, do they?"

"No, I guess they don't," he said, "and I suppose that's one reason why I haven't felt that it was appropriate to ask."

Sensing that he was still somewhat hesitant to ask, she said, "Now that the question's been raised, I find it intriguing. What led you to it?"

"I'm not really sure, but in following the Fourth Commandment, it came to me one day that all these problems that you have provided for us are really not that different at all."

"You mean, for example, that the chickens and pigs problem is similar to the motorcycles and cars problem?" she asked.

"That's part of it, I guess, but only a small part of it. It's much bigger than two problems being alike. It's all problems being alike. Like there's more to it than the problems. In fact, the problems themselves aren't very important. Even the mathematics in the problems isn't very important!" His voice quivered.

This was the first time she had ever had a student who had dared to probe beyond the FC. "I'm fascinated. Tell me more."

"I'm having trouble understanding it myself, but I'll try. Actually, it is more than just the question of what happens to a problem on its way to a solution - much more, in fact. Even so, I think the best place to start is with the question that got me started. That question is this: What is an answer? For some time now, I have been asking myself this question each time I came up with an answer to a problem. Now I ask myself this question before I come up with an answer, and believe me, it really helps me to understand both the problem and what I'm doing to solve it."

"That sounds fascinating, but I'm not quite sure I understand," she said. "Could you give me an example?"

"I'll try. Remember last week when you asked me to help Marvin with his multiplication facts? I think I learned more than he did. He couldn't seem to remember anything and didn't want to. I said it was important. He said it wasn't. I tried to explain why it was, but to be honest, I really didn't even convince myself that it was important. That bothered me. I remember one thing very well. When I asked him what is 2x8, he said 8+8. I was just about to tell him he was wrong again, but I didn't. I just looked at him and he looked at me. He wasn't wrong! 2x8 = 8+8. In fact, there were a whole lot of correct answers: 4x4, 20-4, 4 squared, 10+6. At that moment, it dawned on me that I could keep on giving correct answers forever, and all of them would be different, yet correct. Then why do we insist that 16 is *the* correct answer? I asked Marvin, but he didn't know. Marvin asked me. As I stumbled my way to an explanation, I eventually stated that 16 is *the* answer simply because it is the simplest way to write the number 2x8. Marvin wouldn't let me off the hook with that. He wanted to know what was so simple about 16. It didn't seem so simple to him. Perhaps it was simple only to simple minded persons! I told him then that I really didn't know. He said that's why he hated math. Then it dawned on me: 16 is simple because it is the answer according to tens and ones. We always group by tens and ones. If we always write a number according to tens and ones, it becomes simple. We get used to it. We expect it. We have gotten so used to expecting the answer to be in tens and ones that we have forgotten that we expected it in that way."

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"I think you're really onto something here," she said. "What you're saying is that basic facts like 2x8 = ? are really problems, but that we have gotten so used to experiencing them as facts that we overlook what constitutes an answer to these problems. I think you may be right. But I think I may have interrupted your story."

"Not really, but there is more. When I told Marvin that 16 was *the* answer because it was the name for 2x8 in tens and ones, he shrugged his shoulders as if to say, 'So what?' I started to get excited because I thought I saw a way to get him into this. Even though you have said that the math balance is not a very good device, I got it off the shelf and put two weights on the eighth peg on the left side, and told Marvin that that was two groups of eight or 2x8. He had no trouble with that. I then put four weights on the fourth peg on the right-hand side, and said that was 4x4. The balance was level, so we knew that 4x4 was a correct answer. I asked Marvin to put on some other correct answers. He put two weights on the eighth peg of the right-hand side, and it worked. I wouldn't have done that, but 2x8 certainly is a correct answer to 2x8. 'But what is *the* answer, Marvin?' Without hesitating, he put a weight on the tenth peg and a weight on the sixth peg, and nearly glared at me. We must have done 40 or 50 basic facts on the balance that morning, and I don't think it entered Marvin's mind that he should be bored."

"That explains Marvin's behavior. This week he hasn't been nearly as disruptive. But what has this got to do with problem solving?"

"That's exactly what I asked myself when I did the Fourth Commandment after working with Marvin. I asked myself why Marvin pursued so many examples on the math balance. He didn't get bored, but I certainly did. I tried to look back at what we were doing. First, we had some information about a number (such as 2x8), but the information was not in the form that we wanted it. What we wanted was the same information to be in the form of tens and ones. So next, we put the initial information (2x8) on the math balance, and then arranged it in tens and ones on the other side. And finally, we read off the number as 16. And Marvin knew that was the answer because it was in the form we wanted."

"This is interesting. You're telling me that problem solving is simply transforming information. That we begin with some information as given in the problem. Then, knowing that the answer is simply asking for a particular form for the initial information, we transform the information (using the math balance in this case) until we get it into the desired form. Have I got it?"

"I think you've got it exactly. What was really bothering me earlier was that I didn't realize that for basic facts, there was a desirable form for the answer. It now seems perfectly clear to me that if I didn't know the desired form for the information, finding an answer would make no sense whatsoever. And I think that was Marvin's case."

"I think you're onto something important here, and I hate to dampen your enthusiasm, but perhaps other problems do not fit your scheme," she said.

"You may be right. All I can say is that I've tried it with all the problems you gave as problems of the week, and they all fit the scheme. The pigs and chickens problem is a good example. Remember: 18 animals; 50 legs. How many chickens; how many pigs? I'll never forget when you asked us to imagine that we were commanders-in-chief of the barnyard and all animals would obey us. 'All animals on your back.' We see 50 legs up. 'Each animal put down two legs.' Now we see 2x18, or 36 legs, go down, leaving 50-36, or 14 legs, remaining up. We also see that only pigs have legs up and that each pig has two legs up. So, with 14 legs up and two per pig, there must be 14/2 or seven pigs. There must then be 18-7, or 11 chickens. We had the initial information: 18 animals (pigs and chickens) and 50 legs. We knew the final form for the information: so many chickens, so many pigs. So we began transforming the initial information, putting it into various forms, each hopefully getting us closer to the desired form. When we saw the 14 legs up with two per pig, we knew we had the information nearly the way we wanted it. At this point, the problem was all but solved (we nearly had the information in final form). Seeing the seven pigs and the 18-7=11 chickens was then seeing the answer, the final form for the information."

"Yes, that does fit nicely. But don't we also have problems in which we don't know the form for the answer? In fact, sometimes we don't really know what an answer looks like, and the problem really is finding out what an answer might be. For example, when a mathematician is trying to solve an unsolved problem, he or she may not know the exact form for the answer, making it very difficult to transform the initial information until it is in final form." She paused almost as if she were trying to answer her own question. "But yet, in such a case, the problem is to find an appropriate form for an answer and to defend the appropriateness of any form we may create."

The lad did not choose to argue with his teacher. Somehow he felt that what she had just said indeed did fit what he had done when he thought he had come up with what, for him, were new forms for problem solving. His scheme was an answer to the open-ended problem, "What happens to a problem on its way to an answer?" Whether or not his scheme was an appropriate form for the answer, only time would tell. In the meantime, Marvin sure seemed to be a changed person, and so was his teacher.

Dr. Sawada, professor of education in the Department of Elementary Education at the University of Alberta, has been a frequent contributor to MCATA annual meetings. He has given presentations at NCTM annual meetings and has published in The Arithmetic Teacher. *Currently, Dr. Sawada is on sabbatical leave from the University of Alberta.*