Reconciling Differing Anti-Derivatives

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Three students, Cindy, Ginger, and Roy, took a calculus test. One of the questions was to evaluate the indefinite integral $\int(\sin x \cdot \cos x)dx$.

Cindy first made the substitution $u = \sin x$, yielding $du = \cos x\, dx$ and $dx = \frac{du}{\cos x}$. Then,

$$\int(\sin x \cdot \cos x)dx = \int(u \cos x)\frac{du}{\cos x} = \int u\, du = \frac{u^2}{2} + C = \frac{\sin^2 x}{2} + C .$$

Ginger first made another substitution, namely $u = \cos x$. Then, $du = -\sin x\, dx$ and $dx = \frac{-du}{\sin x}$.

$$\int(\sin x \cdot \cos x)dx = \int(\sin x\, du)\left(\frac{-du}{\sin x}\right) = -\int u(du) = \frac{-u^2}{2} + C = \frac{-\cos^2 x}{2} + C .$$

In his work, Roy recalled that $\sin 2x = 2(\sin x)(\cos x)$, and thus $\sin x \cdot \cos x = \frac{\sin 2x}{2}$.

$$\int(\sin x \cdot \cos x)dx = \int\left(\frac{\sin 2x}{2}\right)dx = \frac{1}{2}\int(\sin 2x)dx .$$

Next, Roy substituted $u = 2x$, yielding $du = 2dx$ and $dx = \frac{du}{2}$. Now, by substitution, the integral is written

$$= \frac{1}{2}\int(\sin u)\frac{du}{2} = \frac{1}{4}\int \sin u\, du = -\frac{1}{4}\cos u + C = -\frac{1}{4}\cos 2x + C .$$

Are these three answers, which differ in appearance, all equivalent? Recall that the constant added to the results of the integration is intended to reflect the fact that an infinite number of anti-derivatives (all differing by constants) may arise from the same integration problem.

To see the meaning of this concept in the integral problem $\int(\sin x \cdot \cos x)dx$, rewrite Ginger's and Roy's results in forms resembling Cindy's.

GINGER: $$\frac{-\cos^2 x}{2} = \frac{-(1-\sin^2 x)}{2}$$
$$= \frac{\sin^2 x - 1}{2}$$
$$= \frac{\sin^2 x}{2} - \frac{1}{2}$$
$$= \text{Cindy's answer} - \frac{1}{2}.$$

ROY: $$-\frac{1}{4}\cos 2x = -\frac{1}{4}(1 - 2\sin^2 x)$$
$$= \frac{1}{2}\sin^2 x - \frac{1}{4}$$
$$= \frac{\sin^2 x}{2} - \frac{1}{4}$$
$$= \text{Cindy's answer} - \frac{1}{4}.$$

It is now apparent that these three answers do, in fact, differ by a constant. This shows that a calculus teacher must be alert in grading tests!

Challenges for the Reader:

1. Integrate $\int(\tan x \cdot \sec^2 x)dx$ in several ways and show that the answers differ by a constant.

2. Find other examples of this type of problem.

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