Rhombi Ratios on the Extended Multiplication Table and Hundred Square

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Mathematics teachers are interested in situations that can be used for drill and practice using either paper and pencil or a calculator. It is ideal if these situations give rise to pattern discovery in settings involving finite mathematical notions. One of these settings involves the multiplication table, as shown in Figure 1.

FIGURE 1. Multiplication Table

x	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9
2	0	2	4	6	8	10	12	14	16	18
3	0	3	6	9	12	15	18	21	24	27
4	0	4	8	12	16	20	24	28	32	36
5	0	5	10	15	20	25	30	35	40	45
6	0	6	12	18	24	30	36	42	48	54
7	0	7	14	21	28	35	42	49	56	63
8	0	8	16	24	32	40	48	56	64	72
9	0	9	18	27	36	45	54	63	72	81

Figure 2, on the following page, displays the interior of an extended multiplication table. Rhombi of varying sizes have been drawn on the table. For each rhombus:

1. Find the sum of the vertices (V).
2. Find the sum of the interior numbers (I).
3. Find the sum of the entries on the horizontal diagonal (HD) and the sum of the entries on the vertical diagonal (VD).

FIGURE 2.

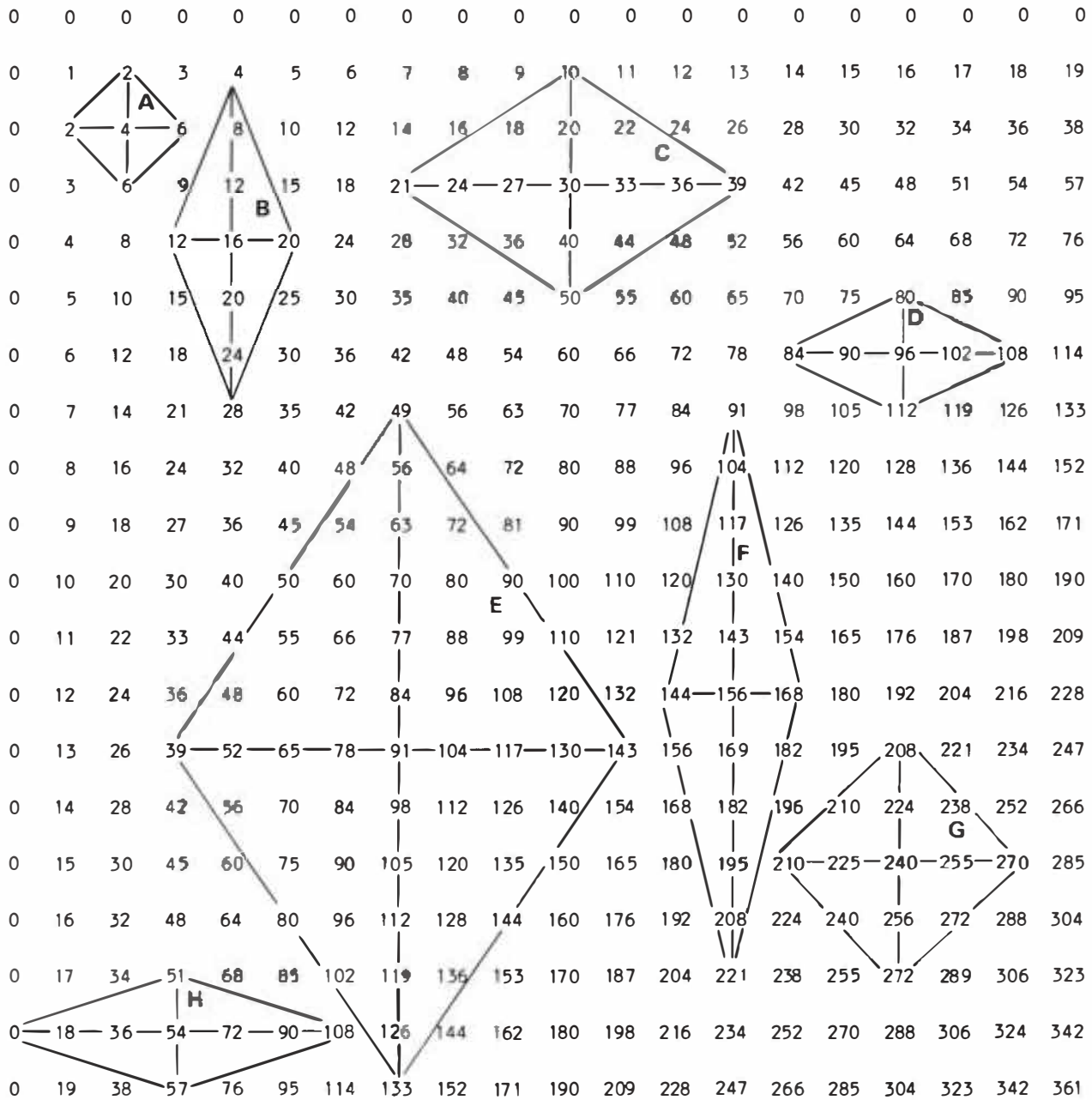


Table I reports the sums, together with the number of entries that were summed. In each case, the sum is listed first with the number of entries in each sum in parentheses.

TABLE I.

Rhombus	V(#)	I(#)	HD(#)	VD(#)
A	16(4)	4(1)	12(3)	12(3)
B	64(4)	80(5)	48(3)	112(7)
C	120(4)	330(11)	210(7)	150(5)
D	384(4)	288(3)	480(5)	288(3)
E	364(4)	4095(45)	819(9)	1183(13)
F	624(4)	1404(9)	468(3)	1716(11)
G	960(4)	1200(5)	1200(5)	1200(5)
H	216(4)	270(5)	378(7)	162(3)

To find a pattern, form ratios by dividing each sum by the parenthesized number which follows. When this is performed, a constant ratio results for each rhombus. Table II reports these constants.

TABLE II.

Rhombus	Constant Ratio
A	4
B	16
C	30
D	96
E	91
F	156
G	240
H	54

In each rhombus, the common ratio is equal to the "centre" number, the point where the two diagonals intersect. Why?

Consider rhombus C. The numbers to the right and to the left of the centre number 30 on the horizontal diagonal are evenly spaced above and below 30. To the right of 30:

$$33 = 30 + 3$$
$$36 = 30 + 6$$
$$39 = 30 + 9.$$

To the left of 30:

$$27 = 30 - 3$$
$$24 = 30 - 6$$
$$21 = 30 - 9.$$

Consequently, 30 is the mean of the seven numbers on the horizontal diagonal. This implies that the sum of the entries on the horizontal diagonal is

$$30 \cdot 7 = 210, \text{ or } 210 \div 7 = 30.$$

On the vertical diagonal below 30:

$$40 = 30 + 10$$
$$50 = 30 + 20.$$

Above 30:

$$20 = 30 - 10$$
$$10 = 30 - 20.$$

Again, 30 is the mean of the five numbers on the vertical diagonal. This implies that the sum of the vertical diagonals is $30 \cdot 5 = 150$ or $150 \div 5 = 30$.

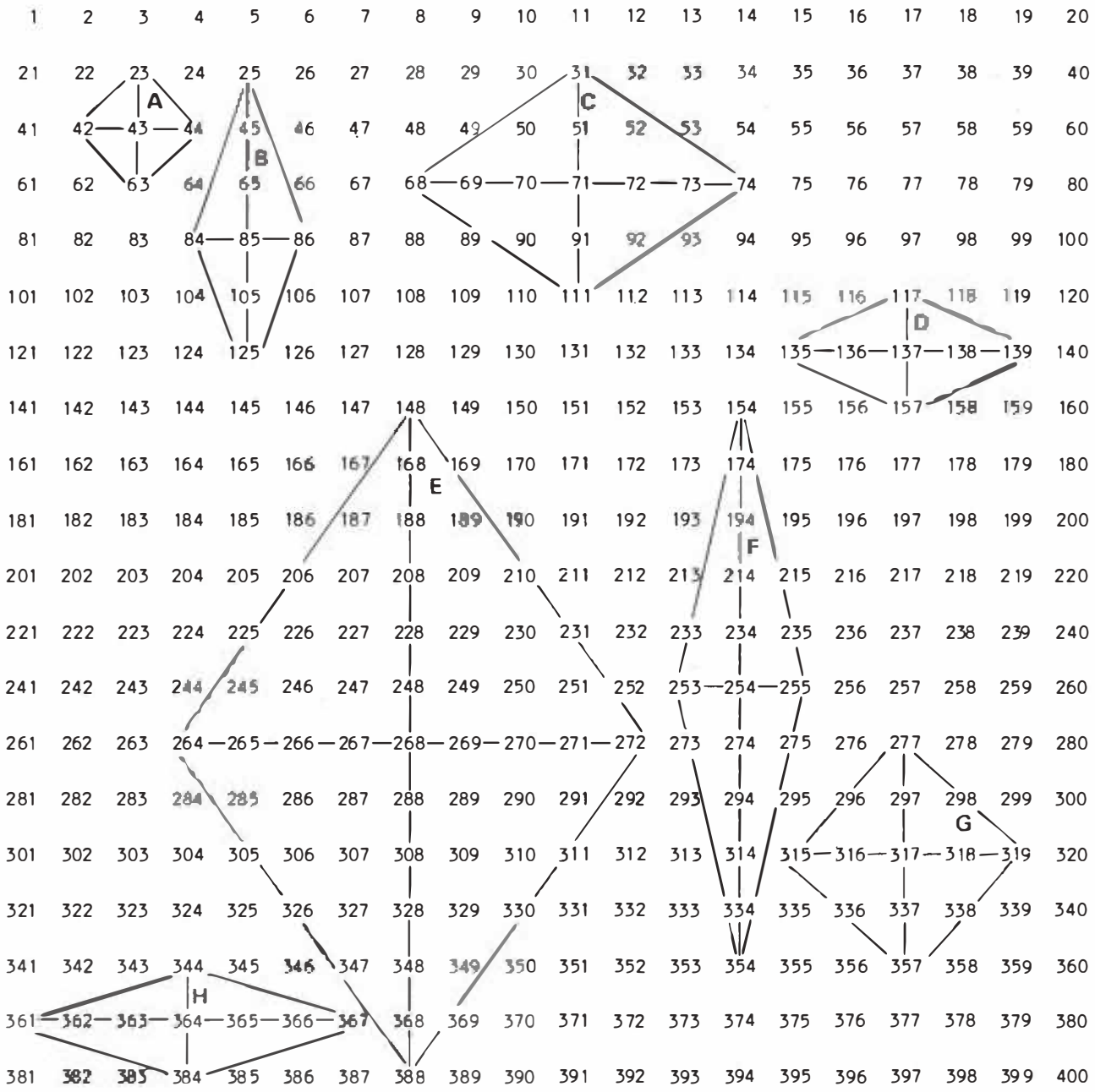
Similar arguments can be generated for the four vertex numbers and the interior numbers of C. These arguments would apply equally well to any of the other rhombi of Figure 2.

The equality of these ratios may be expressed in other ways. For example, in rhombus C, form the ratio V to I ($\frac{120}{330}$). Note this $\frac{4}{11}$ or $\frac{\#V}{\#I}$. We had previously noted that $\frac{120}{4} = \frac{330}{11}$; this observation notes that $\frac{120}{330} = \frac{4}{11}$. This is one of the standard properties of proportions.

Consider the ratio resulting from comparing HD to VD. What pattern do you observe?

Let us investigate patterns on an extended hundred square. We have made a "400 square," and rhombi have been drawn on it. See Figure 3.

FIGURE 3.


1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20
21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40
41 42 43 44 45 46 47 48 49 50 51 52 53 54 55 56 57 58 59 60
61 62 63 64 65 66 67 68 69 70 71 72 73 74 75 76 77 78 79 80
81 82 83 84 85 86 87 88 89 90 91 92 93 94 95 96 97 98 99 100
101 102 103 104 105 106 107 108 109 110 111 112 113 114 115 116 117 118 119 120
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381 382 383 384 385 386 387 388 389 390 391 392 393 394 395 396 397 398 399 400
A
B
C
D
E
F
G
H

On Figure 3, rhombi A through H have been drawn in exactly the same positions as they occupied on Figure 2. Again, find the same sums and ratios as in the previous activity. Tables III and IV report these results.

TABLE III.

Rhombus	V(#)	I(#)	HD(#)	VD(#)
A	172(4)	43(1)	129(3)	129(3)
B	340(4)	425(5)	255(3)	595(7)
C	284(4)	781(11)	497(7)	355(5)
D	548(4)	411(3)	685(5)	411(3)
E	1072(4)	12060(45)	2412(9)	3484(13)
F	1016(4)	2286(9)	762(3)	2794(11)
G	1268(4)	1585(5)	1585(5)	1585(5)
H	1456(4)	1820(5)	2548(7)	1092(3)

TABLE IV.

Rhombus	Constant Ratio
A	43
B	85
C	71
D	137
E	268
F	254
G	317
H	364

Observe that the same patterns hold as in the extended multiplication table, and for the same reasons.

Challenges for the Reader:

1. Draw other rhombi on Figures 2 and 3. Compute the appropriate sums and ratios. Do the same patterns hold?

2. Compute other sums and ratios using the rhombi of Figures 2 and 3. For example, find the sum of the entries which lie on the perimeter of each rhombi. What patterns hold?

3. Draw other geometric shapes on Figures 2 and 3. Find sums and ratios. What patterns hold?

4. Draw extended addition and subtraction tables. If rhombi are drawn and sums and ratios are computed, do the same patterns hold?

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