Mathematical Problem-Solving Classics

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While the National Council of Teachers of Mathematics has once again increased the visibility of problem-solving activity in the junior high school mathematics classroom, solving mathematical problems has been going on for years. Over those years, certain problems appear over and over again, and they have become "classics." This article is limited to discussing only 10 such problems.

Approach the article as follows: (1) try to solve each problem; (2) compare your solution with the one provided; (3) decide how the problem best fits your class (if at all); and (4) use the problem. The problems are not presented in any particular order. Good luck, and have fun!

1. **A farmer's chickens and sheep have 24 heads and 76 feet. How many chickens and how many sheep does the farmer have?**

SOLUTION:

I'm sure every reader has seen at least one problem that is a takeoff on the one above. The standard solution is solution (a) given below. The not-so-standard solution is given as (b) below.

(a) Let X = the number of chickens; Y = the number of sheep.
$X + Y = 24$, the number of heads, so $X = 24 - Y$;
$2X + 4Y = 76$, the number of feet.
Substituting for X, one gets

$$\begin{aligned} 2(24 - Y) + 4Y &= 76 \\ 48 - 2Y + 4Y &= 76 \\ 48 + 2Y &= 76 \\ 2Y &= 28 \\ Y &= 14, \end{aligned}$$

so there are 14 sheep and $24 - 14 = 10$ chickens.

(b) Every animal has either 2 or 4 feet. Since there are 24 heads, there are at least $2 \times 24 = 48$ feet. There are, in fact, 76 feet, so one has $76 - 48 = 28$, or 14 pairs of extra feet. This means 14 animals have 4 feet, so there are 14 sheep and $24 - 14 = 10$ chickens.

While both solutions are interesting, the first is more algebraic. The second one, however, shows more use of logic and also demonstrates how a problem solution may be descriptive.

2. **This problem is a magic trick. Follow these four steps:**

(a) Pick any three-digit number in which no digit is repeated.

(b) Write down all the two-digit numbers that can be created using the three digits of the number selected in step (a).

(c) Add up all the two-digit numbers created in step (b).

(d) Divide the sum found in step (c) by the sum of the digits of the original three-digit number selected in step (a).

The answer is always 22.

Examples:

(a)	482	987	123
(b)	48 42 82 84 24 28	98 97 87 89 78 79	12 13 23 21 32 31
(c)	308	528	132
(d)	$4 + 8 + 2 = 14$ $308 \div 14 = 22$	$9 + 8 + 7 = 24$ $528 \div 24 = 22$	$1 + 2 + 3 = 6$ $132 \div 6 = 22$

The problem, of course, is to show that it works for all numbers selected in step (a).

SOLUTION:

You will probably want to have your class try lots of examples first to see the magic; then, let them loose to try to find a solution to the magic.

One can represent all three-digit numbers by the expression $100a + 10b + c$, in the magic trick $a \neq b \neq c$ and $a \neq c$.

By step (b), one creates

$$\begin{array}{lll} 10a + b & 10a + c & 10b + a \\ 10b + c & 10c + a & 10c + b \end{array}$$

In step (c), one finds the sum of the previously given 6 numbers:
$(10a + b) + (10a + c) + (10b + a) + (10c + a) + (10c + b) =$
$22a + 22b + 22c = 22(a + b + c)$.

In step (d), one divides the sum by $a + b + c$, so
$22(a + b + c) \div (a + b + c) = 22$.

Not only is this problem excellent for its "magic" qualities, but it also gives the students experience in working with a general representation of numbers.

Along the same lines as problem #2 is another "magic" trick that is even better known, namely 1089:

3. **Follow these three steps:**

(a) Pick a three-digit number (no palindromes).

(b) Reverse the digits and subtract the lesser number from the greater number.

(c) Reverse the digits of the difference computed in step (b) and add this new number to the number computed in step (b).

The answer is always 1089.

Examples:

(a) 375	594	632
(b) 573 573 - 375 = 198	495 594 - 495 = 099	236 632 - 236 = 396
(c) 198 + 891 = 1089	099 + 990 = 1089	396 + 693 = 1089

The problem, as with the last magic trick, is to show that it works with all the numbers selected in step (a).

SOLUTION:

As teachers, we rarely see the "proof" for 1089 and, even more rarely, have a class try to prove it. It is a great problem, even if we simply have the class try some examples and dazzle them with the magic of it all. The proof, at least the one presented below, is not simple. The proof, on the other hand, does demonstrate some excellent problem-solving strategies.

In step (a), pick any three-digit number that is not a palindrome, for example, pick $100x + 10y + z$, where $x \neq z$. For the sake of saving space, let $x > z$. If $x < z$, we still get the same result.

In step (b), we reverse the digits and subtract the lesser from the greater: $(100x + 10y + z) - (100z + 10y + x)$, and we get $100(x - z) + (z - x)$.

In step (c), we are supposed to reverse the digits of the difference. How can we reverse the digits of a number that is not in standard form? What are the digits of $100(x - z) + (z - x)$? We will have to try to find out. We will look for a pattern.

You can try many examples to see the the pattern; I, of course, cannot do so in this article. I have systematically selected six examples.

x y x	100(x-z) + (z-x)
2 0 1	099
3 0 1	198
3 0 2	099
4 0 1	297
4 0 2	198
4 0 3	099

Based on many examples (using many more examples than are presented above), I now have a guess as to the digits of 100(x - z) + (z - x).

My guess is, Units: 10 + z - x
Tens: 9
Hundreds: x - z - 1. Let's test the guess.

$$100(x-z-1) + 10(9) + (10+z-x) = 100x-100z-100+90+10+z-x = 100(x - z) + (z - x).$$

Since $100(x-z-1) + 10(9) + (10+z-x) = 100(x - z) + (z - x)$, it works, and we can now reverse the digits and add.

$$[100(x-z-1)+10(9)+(10+z-x)] + [100(10+z-x)+10(9)+x-z-1] =$$
$$(100x-100z-100+90+10+z-x) + (1000+100z-100x+90+x-z-1) =$$
$$(100x-x-100x+x) + (-100z+100z+z-z) + (-100+90+10+1000+90-1) =$$
$$0 + 0 + 1089 = 1089.$$

4. **Which is greater, the tenth root of 10 or the cube root of 2?**

SOLUTION:

Here is a problem that is simple to state, but appears to be mathematically difficult.

$$\sqrt[10]{10} = 10^{\frac{1}{10}} \qquad \sqrt[3]{2} = 2^{\frac{1}{3}}$$

$$\text{Assume } 10^{\frac{1}{10}} > 2^{\frac{1}{3}}$$

$$(10^{\frac{1}{10}})^{30} > (2^{\frac{1}{3}})^{30}$$

$$10^{\frac{30}{10}} > 2^{\frac{30}{3}}$$

$$10^3 > 2^{10}$$

$$1000 > 1024 \ \# \text{ so } \sqrt[3]{2} > \sqrt[10]{10}$$

Just by knowing the definition of "root" and the basic laws of exponents (both topics well within the junior high school mathematics program), one is able to solve a problem that appeared to be mathematically difficult.

Here is another problem that is along the same lines as #4.

5. **Find two whole numbers a and b such that a X b = 1 000 000; however, neither a nor b can have a zero in its standard representation.**

SOLUTION:

You will probably get a lot of moans from the students who couldn't solve the problem when they see the solution.

$1\ 000\ 000 = 10^6 = (2 \times 5)^6 = 2^6 \times 5^6$
$2^6 = 64$ and $5^6 = 15625$ and $64 \times 15625 = 1\ 000\ 000$, so
$a = 64$ and $b = 15625$ (or vice versa).

There are problems that are difficult simply because of their statement. For example:

6. **A computer engineer is twice as old as his wife was when he was as old as his wife is now. He is 24. How old is his wife?**

SOLUTION:

This problem has to be read over and over again to get the relationships straight. Another difficulty with this problem is that the variables stand for things that are different, as opposed to different names of the same thing. Take, for example, the farmer problem (see problem #1). The variables represented the number of animals; X was the number of chickens and Y was the number of sheep. In the current problem, let X be his wife's age and Y be the number of years that have elapsed. X and Y stand for two different things.

X: his wife's age (his former age)
Y: amount of time that has elapsed
$24 = 2(X - Y)$
$X = 24 - Y$

Substituting 24 - Y for X in the first equation, one gets
$24 = 2[(24 - Y) - Y]$
$24 = 48 - 4Y$
$4Y = 24$
$Y = 6$. His wife is $24 - Y = 24 - 6 = 18$ years old.

Problems #7 and #8 are real shockers when you first see them, but the mathematics is actually from the elementary school level. The common difficulty with problem #7 is that students panic when they first read it; it seems impossible.

7. **The host at a party turned to a guest and said, "I have three daughters and I will tell you how old they are. The product of their ages is 72. The sum of their ages is my house number. How old is each?"**

The guest rushed to the door, looked at the house number, and informed the host that he needed more information. The guest said, "Oh, the oldest likes strawberry pudding." The guest then announced the ages of the three girls.

What are the ages of the three girls?

SOLUTION:

The three ages multiply together to yield a product of 72, so list all the triples that yield a product of 72 (be systematic):

1	1	72	1	6	12	2	4	9
1	2	36	1	8	9	2	6	6
1	3	24	2	2	18	3	3	8
1	4	18	2	3	12	3	4	6

The sum of the ages is the house number, so

1 + 1 + 72 = 74	1 + 6 + 12 = 19	2 + 4 + 9 = 15
1 + 2 + 36 = 39	1 + 8 + 9 = 18	2 + 6 + 6 = 14
1 + 3 + 24 = 28	2 + 2 + 18 = 22	3 + 3 + 8 = 14
1 + 4 + 18 = 23	2 + 3 + 12 = 17	3 + 4 + 6 = 13

Once the guest saw the house number, he still couldn't answer the question. Why? Because there are two combinations that sum to 14, namely 2, 6, 6 and 3, 3, 8. Therefore, we know the house number must be 14 because, if it were any other number, the guest could tell the ages of the three girls. But which of the two combinations is the correct one? Ah, the **oldest** likes strawberry pudding. In the combination 2, 6, and 6, there is no oldest, so the girls are 3, 3, and 8 years old.

8. **At Gauss High School, there are 1000 students and 1000 lockers (numbered 1 through 1000). At the beginning of our story, all the lockers are closed, and then the first student goes by and opens every locker. The second student goes along and closes every second locker. The third student "changes the state" (if a locker is closed, he opens it; if a locker is open, he closes it) of every third locker. The fourth student changes the state of every fourth locker, and so on. Finally, the 1000th student changes that state of the 1000th locker.**

When the last student has changed the state of the last locker, which lockers are open?

SOLUTION:

This is an excellent problem for showing junior high school mathematics students the importance of solving a simpler problem. I can't solve it for 1000 lockers, but how about 16 lockers? Why 16? It is all I could get on one line of my word processor.

STUDENT #	LOCKERS 1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	o	o	o	o	o	o	o	o	o	o	o	o	o	o	o	o
2		c		c		c		c		c		c		c		c
3			c			o			c			o			c	
4				o				o				c				o
5					c					o					o	
6						c						o				
7							c							o		
8								c								c
9									o							
10										c						
11											c					
12												c				
13													c			
14														c		
15															c	
16																o
	1			4					9							16

So, for the case of N = 16 lockers, 1, 4, 9, and 16 are left open. Do you see a pattern? Yes, the lockers numbered with a perfect square are the ones left open.

Lockers 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225, 256, 289, 324, 361, 400, 441, 484, 529, 576, 625, 676, 729, 784, 841, 900, and 961 are left open.

It is up to you whether you continue the story and show that any number with an odd number of divisors is left open because the first, third, fifth . . . times the lockers are touched they are opened, and the second, fourth, sixth . . . times the lockers are touched they are closed. You could go on still further to show that only numbers with an odd number of divisors are perfect squares.

9. **Can you split 100 into four parts in such a way that when you add 4 to one part, you get the same answer as when you subtact 4 from another part or multiply 4 by another part or divide the last part by 4?**

SOLUTION:

Let W, X, Y, and Z be the four numbers, so $W + X + Y + Z = 100$.

But $W + 4 = X - 4 = 4Y = Z/4$, so

$$X = W + 8$$
$$Y = (W + 4)/4$$
$$Z = 4W + 16.$$

Substituting the values for X, Y, and Z in terms of W into the equation $W + X + Y + Z = 100$, one gets the following:

$$W + (W + 8) + [(W + 4)/4] + (4W + 16) = 100$$
$$4W + (4W + 32) + (W + 4) + (16W + 64) = 400$$
$$25W + 100 = 400$$
$$25W = 300$$
$$W = 12$$

$$X = W + 8 = 12 + 8 = 20$$
$$Y = (W + 4)/4 = (12 + 4)/4 = 4$$
$$Z = 4W + 16 = 4(12) + 16 = 64$$
$$12 + 20 + 4 + 64 = 100$$
$$12 + 4 = 20 - 4 = 4 \times 4 = 64 \div 4.$$

Because the next problem is the tenth and final problem, I must be running out ot time, so I'll end with a time problem.

10. **A mathematics teacher whose clock had stopped wound it, but did not bother to set it correctly. Then she walked from her home to the home of a friend for an evening of solving mathematical word problems. Afterward, she walked back to her own home and set her clock exactly. How could she do this without knowing the time her trip took?**

SOLUTION:

When she gets to her friend's home, she finds out the correct time. When she leaves her friend's home, she notes the correct time.

Let T_1 = the time on her clock when she wound it
T_2 = the time on her clock when she returned home
T_3 = the correct time when she arrived at her friend's home
T_4 = the correct time when she left her friend's home
T_5 = the correct time when she finally sets her clock

$T_5 = T_4 + 0.5[(T_2 - T_1) - (T_4 - T_3)]$, but the teacher knows each of T_1, T_2, T_3, and T_4. Since all the quantities on the right side of the equation are known, the teacher can compute T_5.

Clearly, these problems do not represent the standard collection of story problems. They were picked for three reasons:

1. They are "classics," that is, they keep coming up in discussions of problem solving as great examples to show certain aspects of problem solving.

2. The solutions demonstrate a variety of problem-solving strategies; for example, analysis, logic, use of mnemonics, solving a simpler problem, searching for patterns.

3. The mathematics is well within the grasp of junior high school students.

It is equally clear that I could have picked a different set of 10 problems. In fact, I'm sure each person reading this article can pick a set of 10 problems fulfilling the three criteria. The difficult part is to limit yourself to 10.

Have you heard the one about the hunter who wanted to take his one-piece rifle on a plane, but the law requires that rifles be stored in the baggage compartment? Unfortunately, the law also states that any item in the baggage compartment cannot have a length, width, or height exceeding 100 cm, and his rifle is 170 cm long. What can he do?

Assume a person can carry four days' supply of food and water for a trip across a desert that takes six days to cross. Obviously, a person cannot make the trip alone because the food and water will be gone in four days and he or she will die. How many people would have to start out in order for one person to get across the desert and for the others to get back to the starting point?

But those are different problems for another day.