Teaching Problem Solving Using Core Topics in Mathematics

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An old proverb has been paraphrased: "Teach people a fact and they will know that fact; teach people to think and they can learn all the facts they'll ever need and be able to use them."*

Today, teachers are challenged to teach students how to be critical and creative thinkers. Mathematics programs across the nation are being revamped to provide greater dimensions to achieve this goal. One such dimension is problem solving. However, concerns are being voiced about making problem solving a mandatory part of the math curriculum: "How can problem solving be worked into a program that is already filled, and which already takes up all the time available?" "How can we get students to use the recommended problem-solving method?"

The authors contend that these two issues can be addressed by incorporating the four-step problem-solving model into the teaching of core topics in mathematics. Four distinct reasons for using this approach while developing the core concepts are outlined explicitly:

1. The four-step model can be developed without requiring additional time or cutting of topics.
2. It reassures students that the process is a bona fide one that should be used regularly.
3. It reassures students that the process is an essential part of mathematics.
4. It helps to ensure that the model becomes a natural part of the students' operational mode.

Just how the four-step model can be used while developing the core of the mathematics program will be illustrated in the balance of this paper. While the method can be used in many ways and many places within the mainstream of the program, examples drawn from the following three aspects of the core program are provided:

1. reviewing previous topics,
2. developing new topics, and
3. solving traditional word problems and applying solutions.

The examples have been written to illustrate some of the clues that teachers might provide to encourage a class to start with what they already know and proceed to build new concepts and skills. The sample questions are generally open-ended and are models of what the students might ask themselves when working alone. Since it is desirable that the students choose the

*John D. Long, Administrator of Special Training for General Motors of Canada.

direction to take in solving a problem, it is recommended that, after introducing the model, teachers resist the temptation to provide too much structure and direction. Students should and can be creative in choosing how, and in which direction, to proceed. This will require teachers to be creative in structuring the questions and directions they give students.

Reviewing Previous Topics

OBJECTIVE:
Calculate the surface area of cylinders.

PROBLEM:
Calculate the surface area of a cylinder with a radius of 14 cm and a height of 23 cm.

Understanding the Problem

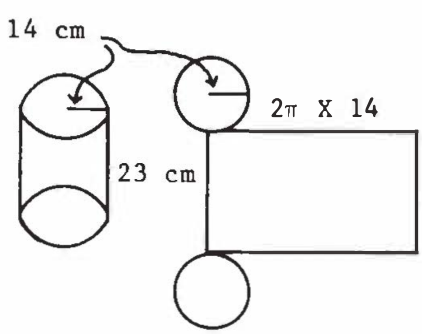
Can I simulate the situation to help me understand the problem?
Will making a drawing help me to understand the problem?

Developing a Plan

Can I simulate the problem?
Can I draw a diagram?
Can I partition the problem and solve each part?

Carrying Out the Plan

Where should I start in carrying out the plan?
 Use a tomato juice tin.
 Make a cylinder and two circles from paper.
 Draw a diagram.



How can I break the problem into simpler parts?
 For which three regions should I find the area?

How can I find the area of the two circles and the rectangle?
 The area of the two circles and the rectangle is
 $2\pi(14)^2 + 2\pi(14)(23)$
 or $3255\ cm^2$.

Looking Back

What is the answer to the question?
 The surface area is $3255\ cm^2$.

Can I generalize the solution?
In words:
 The surface area of the cylinder is the sum of the areas of the two ends and the lateral surface area.
By formula:
 $S.A. = 2\pi r^2 + 2\ rh$

Can I simplify?
 $S.A. = 2\pi r(r + h)$

Can I make and solve a similar or related problem?
 Find the surface area of
 - rectangular solids
 - pyramid.

Developing New Topics

OBJECTIVE:
Derive and apply: $S_n = n(a_1 + a_n)/2$

PROBLEM:
Find the sum of the arithmetic series $1 + 4 + 7 + \ldots$ to 100 terms.

Understanding the Problem

Do I know the meaning of the key words sum, arithmetic series, and terms?
Is there sufficient information to solve the problem?
What information do I know already that would help me with this problem?
Is there a pattern to the problem?
Is this problem like any that I have seen before?

Developing a Plan

Can I solve a simpler problem by using smaller numbers?
Can I apply the discovered patterns?

Carrying Out the Plan

Can I solve a simpler problem?

(a) $1 + 4 + 7 = ?$
What is the average of the first and last term?

(b) $1 + 4 + 7 + 10 = ?$
What is the average of the first and last term? What is the average of the second and second last term? How many averages are there?

(c) $1 + 4 + 7 + 10 + 13 = ?$
What is the average of the first and last term? What is the average of the second and second last term? How many averages are there?

(d) Repeat part (c) as necessary for 6, 7 . . . terms until students see the pattern of the first and last terms divided by 2 gives the average and there are n of these averages.

What would I get by applying the pattern?
The sum of five terms is found by taking the product of the average of the first and last terms and 5.

Can I apply this to the original question?
The sum of 100 terms is found by taking the product of the average of the first and last terms and 100.

Do I have enough information to find this product?
I do not know the last term.

How can I find that term?
I can apply the formula I used when I studied arithmetic sequences yesterday, $a_n = a_1 + (n - 1)d$. The 100th term is $1 + (99)3$ or 298.

Can I find the sum of 100 terms now?
Yes, it is the product of the average of the first term (1) and the last term (298) and 100, which is 14 950.

Looking Back

What is the answer to the question?
The sum of 100 terms of the series is 14 950.

Can I generalize the answer?
$S_n = n(a_1 + a_n)/2$

What other methods are there of doing the problem?

(a) Find the sum of the series and the reverse of the series; that is,

$$\begin{array}{r}1 + 4 + 7 + \ldots + 295 + 298\\ \underline{298 + 295 + 293 + \ldots + 4 + 7}\\ 299 + 299 + \ldots \qquad + 299\end{array}$$

(b) Use symbols for the terms; that is, $a_1 + d\ldots$, $a_1 + (n - 1)d$, and find the sum in the manner in (a) above.

Solving Traditional Word Problems and Applying Solutions

OBJECTIVE:
To solve word problems.

PROBLEM:
Sara is 4a older than her brother Hanif. The sum of their ages is 7/10 that the their mother's age. In 12a time, the sum of their ages will equal their mother's age. Find Sara's, Hanif's, and their mother's present age.

Understanding the Problem

Have I read and reread the problem carefully?
What am I being asked?

Is there a pattern to the problem that will help me to understand it?
Do I understand the relationships in the problem; that is, who is older, how the mother's age compares to the children's, and the comparison of ages in 12a?

Developing a Plan

Can I see the relationships in the problem by using a chart?
Can I formulate equations to show the relationships?
Can I make an intelligent guess, refine, and recheck?

Carrying Out the Plan

Where do I start in carrying out the plan?

A relationship between the ages of the children and the age of their mother is given both at the present time and in 12a. Summarize the information in chart form.

	Hanif	Sara	Mother
Present Ages	x	$x+4$	$\frac{10(x+x+4)}{7}$
Ages in 12a	$x+12$	$x+16$	$\frac{10(2x+4)}{7}+12$

How do I use the chart to consolidate the information?

Write an equation using the ages in 12a and solve the equation for x, which will give us Hanif's present age.

$$(x+12)+(x+16) = 10/7(2x+4)+12$$

$$2x+28 = \frac{20}{7}x+\frac{40}{7}+12$$

$$14x+196 = 20x+40+84$$

$$72 = 6x$$

$$x = 12$$

Looking Back

Have I answered the question asked in the problem?

Since x represents Hanif's present age, Hanif is 12, Sara is 16, and their mother is 40.

Is the answer reasonable?

It is reasonable for the children to be 12 and 16 when the mother is 40.

Have I checked the answer?

At present, the sum of Hanif's and Sara's ages is 28, which is 7/10 of their mother's age. In 12a, Hanif will be 24, Sara will be 28, and their mother will be 52, which is the sum of Sara's and Hanif's ages.

Can I solve the problem in another way?

I could make a reasonable guess, check, and refine the answer.

Can I make and solve a similar or related problem?

For any similar age problem, I could make a chart showing the relationships of the ages, and then write and solve the equations.

Can I solve similar problems such as distance, money, or mixture problems?

By setting up a chart to summarize the problem and then writing equations to show the relationships in the chart, any problem of this type can be solved.

In Conclusion

Topics in which the four-step model is used should be selected carefully. Each example used should have a distinct purpose, such as to illustrate or to practise a particular strategy, to illustrate how the model can be used for a particular purpose, or simply to illustrate its usefulness in solving a problem.

It is not recommended that this method be the only one used in the classroom, but rather that it be added to the variety of developmental teaching strategies already employed to meet the needs of all students.

With appropriate use of the four-step problem-solving method, teachers can not only cover the topics prescribed in the program, but can also achieve the objective of getting students to be critical and creative thinkers so they, too, can "learn all the facts they'll ever need and be able to use them."

WHOLE NUMBER CHALLENGE

Addition, Subtraction, Multiplication, Division, and Order of Operations

PREPARATION

Prepare fraction cards: two each of 0/12 - 12/12, 0/6 - 6/6, 0/4 - 4/4, 0/3 - 3/3, 0/2 - 2/2 (64 cards). One die or spinner.

HOW TO PLAY
(2 to 6 players)

Each player is dealt five cards. One player rolls the die. All players may use two or more of their cards and any of the operations of addition, subtraction, multiplication, or division to write a number sentence that equals the number rolled on the die.

EXAMPLE: cards 7/12, 3/4, 1/6, 1/2, 1/12
die rolled 1

$$\frac{1}{6} \div [\frac{3}{4} - (\frac{7}{12} + \frac{1}{12})] \times \frac{1}{2}$$

SCORE:
1 point for every card used
1 point for each different operation used
3 bonus points for using all five cards
3 bonus points for using all four operations

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