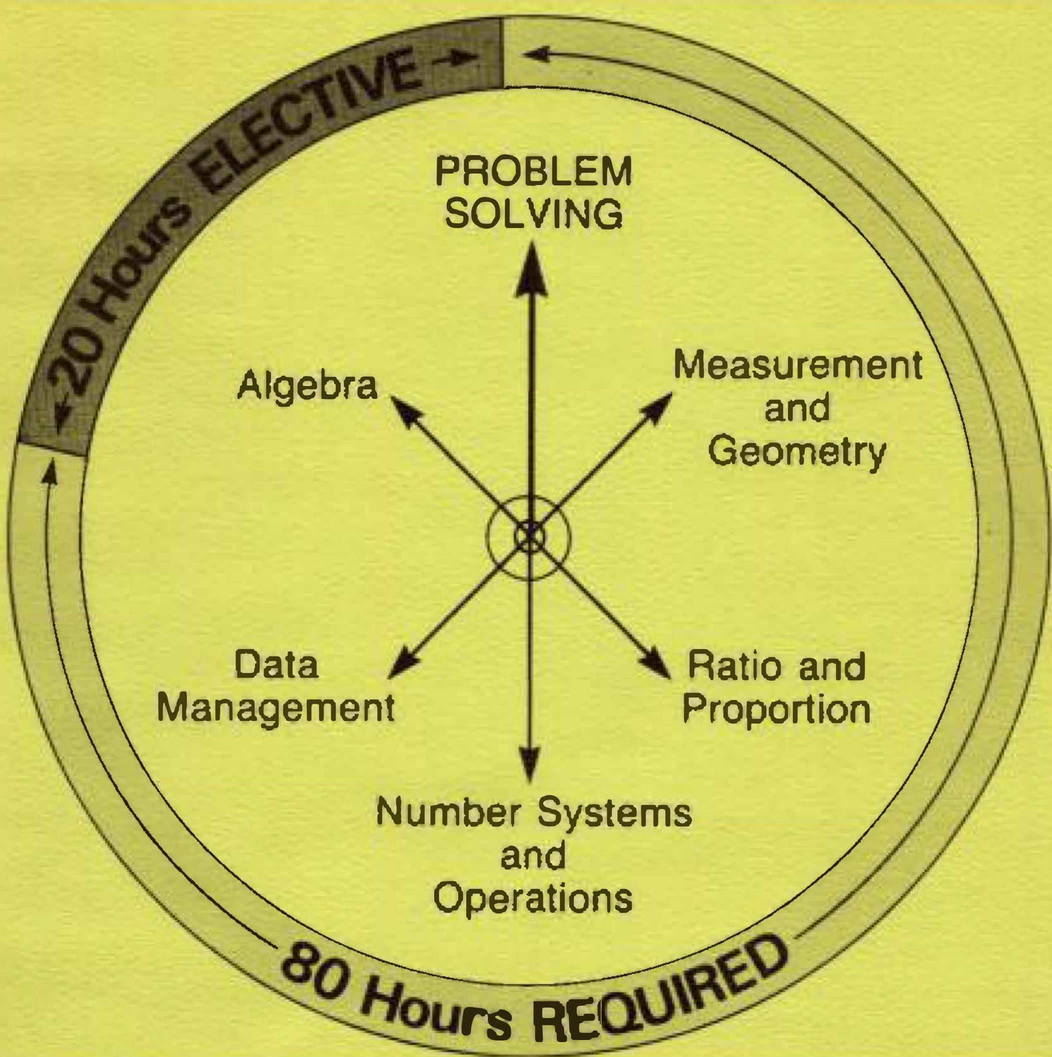
ISSN 0319-8367

delta-k

JOURNAL OF THE
MATHEMATICS COUNCIL
OF THE ALBERTA
TEACHERS' ASSOCIATION

Volume XXVI, Number 2 June 1987

SIX STRANDS
The Junior High School Curriculum



Estimation Thinking Manipulatives

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EDPRESS

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EDITORIAL

Direction

The junior and senior high school mathematics curricula are being reviewed, and it is assumed that the changes being considered will reflect the recommendations of the Review of Secondary Education report. Of particular importance to mathematics teachers is the statement that students should become **critical** thinkers. Problem-solving skills and strategies are a component of thinking.

Subsequent to the development of the article "Problem-Solving Relationships in Algebra," I had the opportunity to present the content of the article at two workshops. Preliminary remarks included my view that in the future: (1) problems chosen should introduce the student to thinking procedures and the logical-deductive nature of mathematics; (2) other problems should reflect or show an application of mathematics theory in a nontext problem situation; (3) another category of problems should show the need for intuitive or other thinking skills; and (4) problems should be selected from other disciplines to show that the mathematical problem-solving strategies also apply in other subjects.

During one of the workshops, a teacher asked: "John, do you have any research evidence that the use of different symbols will transfer into the algebraic approach?" The logical-deductive mode of thinking was evident. The question received an honest answer: "No!" I had not, and have not since then, read an article on the potential of this problem in introducing algebraic concepts. An equally academic response could have been: "Do you have any research evidence to show that transfer of thought processes from an intuitive to a deductive approach does not happen?" An issue is defined. Is mathematics purely logical-deductive, or can mathematics be done in other thinking modes? As students become more proficient at organizing thought, they should become more proficient in the logical-deductive mode.

At the University of Lethbridge, I encounter many highly successful education students with grade point averages greater than three (on a four-point scale) who detest mathematics, who view mathematics as a system of rules, and who do not enjoy mathematics. The irony of the situation is that these education students have been taught mathematics in a logical-deductive manner, and those who teach mathematics will probably emulate the teachers who taught them mathematics.

Will the new Alberta secondary mathematics curriculum provide resource materials that encourage a problem-solving approach? Will the problems be coordinated to reflect content? Will the classroom environment be organized to encourage students to think? Both issues are the prerogative of the mathematics teacher.

Comments

The MCATA executive authorized this larger issue of delta-K so that some of the presentations delivered at the NCTM Canadian Conference held in Edmonton, October 16-18, 1986, could be shared with a larger audience.

John Heuver's letter (below) is the first that the editor has received. **Art Jorgensen** provides a tribute to Joan Worth, the second recipient of the Mathematics Educator of the Year Award. **Gary Hill** offers opinions on teaching problem solving. **James Sherrill** analyzes some problems and shows how they can be solved without using mathematical-deductive, algebraic reasoning. Student thinking, often the result of instruction, is analyzed by **Eric Wood. Bonnie Litwiller and David Duncan** present a problem-solving situation using an addition table, and explore patterns. Is there an algebraic base for the patterns? **Irvin Burbank** uses inductive thinking to solve probability problems.

Two articles on evaluation of problem solving are presented by **Marie Hauk** and **Gary Flewelling**. Flewelling's article allows teachers to rate themselves on the degree to which they make problem solving a part of their classroom environment.

Marshall Bye and Bob Midyette encourage the use of problems that support the core curriculum. **John Percevault** and **Mary-Jo Maas** demonstrate how problems used at the elementary school level can also be applied at the junior and senior high school levels. Two problems are presented by **Kevin Sherratt** and **Karen Gibling**.

John B. Percevault

LETTER TO THE EDITOR

I noticed that the theme of the next issue of delta-K is to be "Problem Solving in the Junior High School." I would like to point out to you that there appeared in the September 1985 issue of the Mathematics Magazine, published by the Mathematical Association of America, an article entitled "An Example of an Error-Correcting Code" written by Mark Roben Stein, a Grade 8 student at McKernan Elementary-Junior High School in Edmonton. Mark mentions that he attended an enrichment program run by professor Andy Liu from the University of Alberta. In my opinion, it would be a serious oversight if delta-K did not mention this fact. The article gives evidence of the fact that, when some of our students are properly guided, they can achieve a very high degree of skill in mathematics. Coding theory is not a conventional secondary school topic and, therefore, requires the instructor to possess both mathematical knowledge and great pedagogical skills.

I also noticed that in the July 1986 issue, two errors appear in "The Road to Four Villages" problem by L.G. Hoye. A radical sign is missing. It should read "$\sqrt{3}$ + 1 mile(s)," and a reference should be cited: page 23 in the second edition of Coxeter's book.

With kind regards,
John G. Heuver, Grande Prairie

EDITOR'S COMMENT: The error noted by Mr. Heuver was corrected in the October 1986 issue of delta-K.

A Tribute to Joan Worth

Arthur Jorgensen

Dr. Arthur Jorgensen is presently working for the Ministry of Education, Kingston, Jamaica.

Dr. Joan Worth is the second recipient of the Mathematics Educator of the Year Award, which is awarded annually by the Mathematics Council of The Alberta Teachers' Association (MCATA).

Joan's interest and involvement in mathematics education spans a number of years in this province. In fact, she has been affectionately labeled the "Mother of Mathematics Education in Alberta" because of her significant contribution.

Joan is currently a professor in the Department of Elementary Education at the University of Alberta, where she specializes in undergraduate and graduate mathematics education. Before assuming responsibilities at the university, Joan had a wide variety of experiences and taught all subjects at several elementary schools in Alberta. Joan has taken an active interest in the MCATA since its inception, serving in a number of key executive positions. Although, from time to time, she has not been an active member of the board, she willingly comes back when her talents are required. Her most recent significant contribution to MCATA was in serving as conference director for the National Council of Teachers of Mathematics (NCTM) Name of Site meeting held in Edmonton, October 16-18, 1986. This conference will go down in history as being one of the best ever held in the province.

Throughout the years, Joan has also taken an active role in the affairs of the NCTM, where once again she has served admirably in a variety of roles. Her special interest recently has been the NCTM's Agenda for Action, particularly the recommendations dealing with problem solving and the use of computers and calculators in elementary mathematics instruction.

Because of her extensive knowledge of elementary mathematics education, Joan's expertise as a conference speaker is widely sought both nationally and internationally.

Joan is also an author. She has written numerous articles for The Arithmetic Teacher and various MCATA publications. Currently, Joan is working with the Addison Wesley Publishing Company as coauthor of its new elementary mathematics textbook series.

The following poem, I believe, accurately reflects Joan's greatness. She has lived its message.

True Greatness

Women are as great as the dreams they dream
As great as the love they bear;
As great as the values they redeem,
As the happiness they share.
Women are as great as the thoughts they think
As the worth they have attained;
As the fountain at which their spirits drink,
As the insights they have gained.
Women are as great as the truth they speak
As great as the help they give;
As great as the destiny they seek,
As great as the lives they live.

Let's Do Some Problem Solving in Junior High

Gary Hill

Gary Hill, past president of the MCATA, is a mathematics consultant for the Lethbridge Regional Office of Alberta Education. Gary was vice-principal of Gilbert Paterson Community School in Lethbridge and a junior high school teacher before being seconded by Alberta Education.

Problem Solving—An Integral Part of a Junior High Mathematics Program

In the 1980 publication An Agenda for Action, the National Council of Teachers of Mathematics' first recommendation states: "Problem solving must be the focus of school mathematics in the 1980s." Accompanying this statement are six recommended actions that we can follow to help bring this to fruition. The purpose of this paper is to suggest to junior high mathematics teachers how they might make problem solving the focal point of their programs.

Prior to 1980, mathematics teachers had generally incorporated varying degrees of problem solving into their programs, occasionally as an additional exercise if there was time. In the past seven years, teachers have become more aware of its importance and more effort has been made to interweave problem solving throughout the strands of the mathematics curriculum. This awareness level has been raised largely through the leadership of the many mathematics educators' associations and authors of the most recent mathematics resources, in which problem solving is highlighted in virtually every chapter.

In Alberta, not only do the curriculum guides for every level of mathematics emphasize problem solving, but also separate problem-solving monographs (similar in appearance to curriculum guides) have been devised for the elementary and junior high levels. A monograph is also being developed for the senior high programs. The purpose of these monographs is to encourage and assist teachers to make problem solving an integral part of their mathematics curricula.

What constitutes a problem?

For a problem to exist, two criteria must be fulfilled:

1. The person must accept that something needs to be overcome. Such involvement may be either intrinsically or extrinsically stimulated.
2. There is no immediate method of solution. Thus, what is a problem for one person may not be a problem for another. Also, problems need not be mathematical in nature. Teachers often assign students a series of similar "problems." Once the students have solved the first problem, the others are merely exercises to reinforce the algorithm or concept (an important type of activity to have students do, but not necessarily what the teacher had intended that the students accomplish). This leads to the next question.

What constitutes problem solving?

Problem solving is a process that will serve people throughout their lives. Although problems are a fact of life, not all problems are mathematical in nature. It is essential that mathematics teachers develop in students those problem-solving skills that can be applied to any situation.

Problem solving should not be viewed as a separate entity in a mathematics program; rather, it must be interwoven throughout the strands of the junior high school math curriculum. It must be used to develop and reinforce the skills and concepts of numeration, operation, geometry, graphing, algebra, measurement, and ratio and proportion.

Why should problem solving be taught in the junior high school?

Students often ask why they have to learn a particular concept or skill. When problem solving is incorporated throughout the mathematics curriculum, students can see a closer relationship between the sometimes abstract world of the classroom and the real world.

Teachers who carefully select problems to present to their students create an atmosphere of positivism, success, and enthusiasm in the classroom. As was mentioned before, problem solving can also reinforce and/or clarify previously taught concepts and skills.

Most importantly, students must be taught a model or framework to help them organize their attempts at solving any problem. This can readily be done during the junior high mathematics program. A successful model is illustrated later in this paper.

How can a teacher foster the development of problem-solving skills in the junior high school?

As with any classroom situation, the enthusiasm and positivism of the teacher are the most influential factors in promoting an atmosphere for the development of problem-solving skills. In addition, the teacher must be open-minded and accepting of a variety of methods and solutions presented by students, ask carefully worded questions, and carefully select the problems presented to students. It is desirable to reduce the total number of problems and to pose a daily problem, as well as keep a balance of success and challenge.

What constitutes a good problem for junior high school students?

When selecting problems, the teacher should be aware of several criteria. No problem will satisfy all criteria. However, a good problem will satisfy several of them:

1. The reading level is below the students' frustration level.
2. A challenge is presented.
3. A variety of methods and/or solutions is possible.
4. Required skills, concepts, and vocabulary have been previously taught.
5. A realistic situation is evident.
6. Reinforcement of skills and concepts is accomplished.
7. Challenging problems have been preceded with simple problems.
8. Students will be able to achieve successful solutions.
9. The problem is fun and/or interesting to students of that age.

What are some hints for developing a positive attitude toward problem solving in the junior high school?

Some teachers may be uncomfortable with problem solving simply because they have not had enough successful experience in this area. This feeling may be overcome simply with more experience and by following as many of these suggestions as possible:

1. Select problems carefully and assign them in a sequence of easiest to most challenging, shortest to longest.

2. Assign fewer problems, but on a regular basis; for example, assign a problem daily.

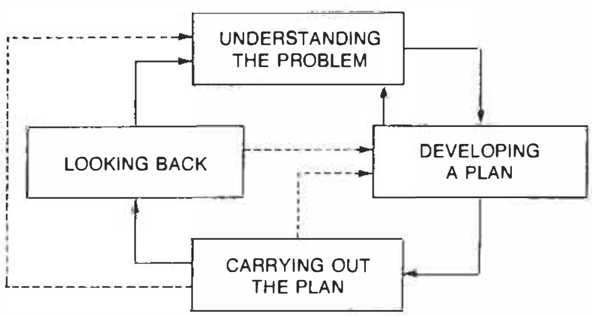
3. Select problems that are within the students' knowledge and ability levels.

4. Find problems that are of interest to the age group and to both sexes.

5. Find problems that have an extension component.

6. Feel free to modify a problem to suit the age, ability, or interests of the students.

7. Teach students a framework for problem solving. The following model, taken from the Alberta Education monograph Problem Solving Challenge for Mathematics, 1985, p. 4, has proven to be successful.



   Other models are equally acceptable. Whichever model you choose, it is well worth the time and effort to have students record the framework in their notebooks, and to create a wall or bulletin board display for easy reference.

8. Communicate your expectations to the students. Inform them that partial marks for each problem will be awarded for each of the stages suggested by the illustration in item 7.

9. Select a variety of problems that will allow students to try as many of the strategies as possible. Encourage students to keep a set of mathematics notes, including a list of the strategies discovered under the four headings in the diagram in item 7. At the same time, add to the classroom chart. Eventually, you may wish to give the students a list of strategies or have them copy it from the blackboard. A listing of skills and strategies may be found in Alberta Education's Problem Solving Challenge for Mathematics (1985).

10. Whenever possible, involve the students in the problem--by giving it a name, by acting it out, by talking about it, by working through it as a group.

11. Allow students to create problems and exchange them for solving. Generally, a stimulus in the form of a story, a picture, or a previous problem is required. Encourage students to include diagrams, pictures, or extraneous information.

12. Allow students to work together in pairs or small groups. Brainstorming and exchanging ideas can lead students to try other strategies when they reach a dead end.

13. Encourage the students to have and use a calculator. The focus of problem solving should be on the process, not on the computation.

14. If you are comfortable with computers, incorporate available software to enhance problem solving.

What are some sources of problem-solving exercises for junior high school programs?

Textbooks (even those not prescribed by Alberta Education) are always good sources. Some publishers have supplementary booklets or packets of problems that can be very useful. A school or a school system might find it useful to have some of the publications that can be obtained from the following sources:

The National Council of Teachers of Mathematics
1906 Association Drive
Reston, Virginia 22091

1. A $35 (US) annual membership fee entitles you to receive The Mathematics Teacher or The Arithmetic Teacher. Both are journals of NCTM and contain problems and suggested activities each month.

2. Hirsch, C.R., ed. Activities for Implementing Curricular Themes, 1986. Price: $12.15.

Mathematics Council
The Alberta Teachers' Association
11010 - 142 Street
Edmonton, Alberta T5N 2R1

Math Monograph No. 7: Problem Solving in the Mathematics Classroom, April 1982.

Alberta Education
Curriculum Branch
Devonian Building
11160 Jasper Avenue
Edmonton, Alberta T5K 0L2

Problem Solving Challenge for Mathematics, 1985.

CONCENTRATE

Equivalent Fractions and Memory

PREPARATION

Prepare fraction cards: 0/12 - 12/12, 0/6 - 6/6, 0/4 - 4/4, 0/3 - 3/3, and 0/2 - 2/2 (32 cards). Place each set in an envelope.

HOW TO PLAY
(2 to 4 players)

Turn all cards face down on the table. Have each player choose one card. The person with the highest denominator goes first. If there is a tie, the person with both the highest denominator and highest numerator goes first. The first person picks up two cards. If they are equivalent, the player keeps them and chooses two more. The player continues until an equivalent pair is not chosen. These two cards are then replaced on the table, and the player to the right has a turn.

Once all equivalent pairs have been found, the person with the greatest number of pairs is the winner.

Contributed by Karen Gibling, Elboya Elementary-Junior High School, Calgary.

Mathematical Problem-Solving Classics

James M. Sherrill

Dr. Sherrill is a professor of education and director of educational graduate studies at the University of British Columbia. The content of this article was presented at the NCTM Canadian Conference in Edmonton, October 16-18, 1986.

While the National Council of Teachers of Mathematics has once again increased the visibility of problem-solving activity in the junior high school mathematics classroom, solving mathematical problems has been going on for years. Over those years, certain problems appear over and over again, and they have become "classics." This article is limited to discussing only 10 such problems.

Approach the article as follows: (1) try to solve each problem; (2) compare your solution with the one provided; (3) decide how the problem best fits your class (if at all); and (4) use the problem. The problems are not presented in any particular order. Good luck, and have fun!

1. **A farmer's chickens and sheep have 24 heads and 76 feet. How many chickens and how many sheep does the farmer have?**

SOLUTION:

I'm sure every reader has seen at least one problem that is a takeoff on the one above. The standard solution is solution (a) given below. The not-so-standard solution is given as (b) below.

(a) Let X = the number of chickens; Y = the number of sheep.
$X + Y = 24$, the number of heads, so $X = 24 - Y$;
$2X + 4Y = 76$, the number of feet.
Substituting for X, one gets
$$\begin{aligned} 2(24 - Y) + 4Y &= 76 \\ 48 - 2Y + 4Y &= 76 \\ 48 + 2Y &= 76 \\ 2Y &= 28 \\ Y &= 14, \end{aligned}$$
so there are 14 sheep and $24 - 14 = 10$ chickens.

(b) Every animal has either 2 or 4 feet. Since there are 24 heads, there are at least $2 \times 24 = 48$ feet. There are, in fact, 76 feet, so one has $76 - 48 = 28$, or 14 pairs of extra feet. This means 14 animals have 4 feet, so there are 14 sheep and $24 - 14 = 10$ chickens.

While both solutions are interesting, the first is more algebraic. The second one, however, shows more use of logic and also demonstrates how a problem solution may be descriptive.

2. **This problem is a magic trick. Follow these four steps:**

(a) Pick any three-digit number in which no digit is repeated.

(b) Write down all the two-digit numbers that can be created using the three digits of the number selected in step (a).

(c) Add up all the two-digit numbers created in step (b).

(d) Divide the sum found in step (c) by the sum of the digits of the original three-digit number selected in step (a).

The answer is always 22.

Examples:

(a)	482	987	123
(b)	48 42 82 84 24 28	98 97 87 89 78 79	12 13 23 21 32 31
(c)	308	528	132
(d)	4 + 8 + 2 = 14 308 ÷ 14 = 22	9 + 8 + 7 = 24 528 ÷ 24 = 22	1 + 2 + 3 = 6 132 ÷ 6 = 22

The problem, of course, is to show that it works for all numbers selected in step (a).

SOLUTION:

You will probably want to have your class try lots of examples first to see the magic; then, let them loose to try to find a solution to the magic.

One can represent all three-digit numbers by the expression $100a + 10b + c$, in the magic trick $a \neq b \neq c$ and $a \neq c$.

By step (b), one creates

$$\begin{array}{ccc} 10a + b & 10a + c & 10b + a \\ 10b + c & 10c + a & 10c + b \end{array}$$

In step (c), one finds the sum of the previously given 6 numbers:
$(10a + b) + (10a + c) + (10b + a) + (10c + a) + (10c + b) =$
$22a + 22b + 22c = 22(a + b + c)$.

In step (d), one divides the sum by $a + b + c$, so
$22(a + b + c) \div (a + b + c) = 22$.

Not only is this problem excellent for its "magic" qualities, but it also gives the students experience in working with a general representation of numbers.

Along the same lines as problem #2 is another "magic" trick that is even better known, namely 1089:

3. **Follow these three steps:**

(a) Pick a three-digit number (no palindromes).

(b) Reverse the digits **and** subtract the lesser **number from** the greater number.

(c) Reverse the digits of the difference computed in step (b) and add this new number to the number computed in step (b).

The answer is always 1089.

Examples:

(a) 375	594	632
(b) 573 573 - 375 = 198	495 594 - 495 = 099	236 632 - 236 = 396
(c) 198 + 891 = 1089	099 + 990 = 1089	396 + 693 = 1089

The problem, as with the last magic trick, is to show that it works with all the numbers selected in step (a).

SOLUTION:

As teachers, we rarely see the "proof" for 1089 and, even more rarely, have a class try to prove it. It is a great problem, even if we simply have the class try some examples and dazzle them with the magic of it all. The proof, at least the one presented below, is not simple. The proof, on the other hand, does demonstrate some excellent problem-solving strategies.

In step (a), pick any three-digit number that is not a palindrome, for example, pick $100x + 10y + z$, where $x \neq z$. For the sake of saving space, let $x > z$. If $x < z$, we still get the same result.

In step (b), we reverse the digits and subtract the lesser from the greater: $(100x + 10y + z) - (100z + 10y + x)$, and we get $100(x - z) + (z - x)$.

In step (c), we are supposed to reverse the digits of the difference. How can we reverse the digits of a number that is not in standard form? What are the digits of $100(x - z) + (z - x)$? We will have to try to find out. We will look for a pattern.

You can try many examples to see the the pattern; I, of course, cannot do so in this article. I have systematically selected six examples.

x y x	100(x-z) + (z-x)
2 0 1	099
3 0 1	198
3 0 2	099
4 0 1	297
4 0 2	198
4 0 3	099

Based on many examples (using many more examples than are presented above), I now have a guess as to the digits of 100(x - z) + (z - x).

My guess is, Units: 10 + z - x
Tens: 9
Hundreds: x - z - 1. Let's test the guess.

$$100(x-z-1) + 10(9) + (10+z-x) = 100x-100z-100+90+10+z-x$$
$$= 100\,(x - z) + (z - x).$$

Since $100(x-z-1) + 10(9) + (10+z-x) = 100\,(x - z) + (z - x)$, it works, and we can now reverse the digits and add.

$$[100(x-z-1)+10(9)+(10+z-x)] + [100(10+z-x)+10(9)+x-z-1] =$$
$$(100x-100z-100+90+10+z-x) + (1000+100z-100x+90+x-z-1) =$$
$$(100x-x-100x+x) + (-100z+100z+z-z) + (-100+90+10+1000+90-1) =$$
$$0 + 0 + 1089 = 1089.$$

4. **Which is greater, the tenth root of 10 or the cube root of 2?**

SOLUTION:

Here is a problem that is simple to state, but appears to be mathematically difficult.

$$\sqrt[10]{10} = 10^{\frac{1}{10}} \qquad \sqrt[3]{2} = 2^{\frac{1}{3}}$$

$$\text{Assume } 10^{\frac{1}{10}} > 2^{\frac{1}{3}}$$
$$(10^{\frac{1}{10}})^{30} > (2^{\frac{1}{3}})^{30}$$
$$10^{\frac{30}{10}} > 2^{\frac{30}{3}}$$
$$10^{3} > 2^{10}$$
$$1000 > 1024 \;\# \text{ so } \sqrt[3]{2} > \sqrt[10]{10}$$

Just by knowing the definition of "root" and the basic laws of exponents (both topics well within the junior high school mathematics program), one is able to solve a problem that appeared to be mathematically difficult.

Here is another problem that is along the same lines as #4.

5. **Find two whole numbers a and b such that a X b = 1 000 000; however, neither a nor b can have a zero in its standard representation.**

SOLUTION:

You will probably get a lot of moans from the students who couldn't solve the problem when they see the solution.

$1\ 000\ 000 = 10^6 = (2 \times 5)^6 = 2^6 \times 5^6$
$2^6 = 64$ and $5^6 = 15625$ and $64 \times 15625 = 1\ 000\ 000$, so
a = 64 and b = 15625 (or vice versa).

There are problems that are difficult simply because of their statement. For example:

6. **A computer engineer is twice as old as his wife was when he was as old as his wife is now. He is 24. How old is his wife?**

SOLUTION:

This problem has to be read over and over again to get the relationships straight. Another difficulty with this problem is that the variables stand for things that are different, as opposed to different names of the same thing. Take, for example, the farmer problem (see problem #1). The variables represented the number of animals; X was the number of chickens and Y was the number of sheep. In the current problem, let X be his wife's age and Y be the number of years that have elapsed. X and Y stand for two different things.

X: his wife's age (his former age)
Y: amount of time that has elapsed
$24 = 2(X - Y)$
$X = 24 - Y$

Substituting 24 - Y for X in the first equation, one gets
$24 = 2[(24 - Y) - Y]$
$24 = 48 - 4Y$
$4Y = 24$
$Y = 6$. His wife is $24 - Y = 24 - 6 = 18$ years old.

Problems #7 and #8 are real shockers when you first see them, but the mathematics is actually from the elementary school level. The common difficulty with problem #7 is that students panic when they first read it; it seems impossible.

7. **The host at a party turned to a guest and said, "I have three daughters and I will tell you how old they are. The product of their ages is 72. The sum of their ages is my house number. How old is each?"**

The guest rushed to the door, looked at the house number, and informed the host that he needed more information. The guest said, "Oh, the oldest likes strawberry pudding." The guest then announced the ages of the three girls.

What are the ages of the three girls?

SOLUTION:

The three ages multiply together to yield a product of 72, so list all the triples that yield a product of 72 (be systematic):

1	1	72	1	6	12	2	4	9
1	2	36	1	8	9	2	6	6
1	3	24	2	2	18	3	3	8
1	4	18	2	3	12	3	4	6

The sum of the ages is the house number, so

1 + 1 + 72 = 74	1 + 6 + 12 = 19	2 + 4 + 9 = 15
1 + 2 + 36 = 39	1 + 8 + 9 = 18	2 + 6 + 6 = 14
1 + 3 + 24 = 28	2 + 2 + 18 = 22	3 + 3 + 8 = 14
1 + 4 + 18 = 23	2 + 3 + 12 = 17	3 + 4 + 6 = 13

Once the guest saw the house number, he still couldn't answer the question. Why? Because there are two combinations that sum to 14, namely 2, 6, 6 and 3, 3, 8. Therefore, we know the house number must be 14 because, if it were any other number, the guest could tell the ages of the three girls. But which of the two combinations is the correct one? Ah, the **oldest** likes strawberry pudding. In the combination 2, 6, and 6, there is no oldest, so the girls are 3, 3, and 8 years old.

8. **At Gauss High School, there are 1000 students and 1000 lockers (numbered 1 through 1000). At the beginning of our story, all the lockers are closed, and then the first student goes by and opens every locker. The second student goes along and closes every second locker. The third student "changes the state" (if a locker is closed, he opens it; if a locker is open, he closes it) of every third locker. The fourth student changes the state of every fourth locker, and so on. Finally, the 1000th student changes that state of the 1000th locker.**

When the last student has changed the state of the last locker, which lockers are open?

SOLUTION:

This is an excellent problem for showing junior high school mathematics students the importance of solving a simpler problem. I can't solve it for 1000 lockers, but how about 16 lockers? Why 16? It is all I could get on one line of my word processor.

STUDENT #	LOCKERS 1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	o	o	o	o	o	o	o	o	o	o	o	o	o	o	o	o
2		c		c		c		c		c		c		c		c
3			c			o			c			o			c	
4				o				o				c				o
5					c					o					o	
6						c						o				
7							c							o		
8								c								c
9									o							
10										c						
11											c					
12												c				
13													c			
14														c		
15															c	
16																o
	1			4					9							16

So, for the case of N = 16 lockers, 1, 4, 9, and 16 are left open. Do you see a pattern? Yes, the lockers numbered with a perfect square are the ones left open.

Lockers 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225, 256, 289, 324, 361, 400, 441, 484, 529, 576, 625, 676, 729, 784, 841, 900, and 961 are left open.

It is up to you whether you continue the story and show that any number with an odd number of divisors is left open because the first, third, fifth . . . times the lockers are touched they are opened, and the second, fourth, sixth . . . times the lockers are touched they are closed. You could go on still further to show that only numbers with an odd number of divisors are perfect squares.

9. **Can you split 100 into four parts in such a way that when you add 4 to one part, you get the same answer as when you subtact 4 from another part or multiply 4 by another part or divide the last part by 4?**

SOLUTION:

Let W, X, Y, and Z be the four numbers, so $W + X + Y + Z = 100$.

But $W + 4 = X - 4 = 4Y = Z/4$, so

$$X = W + 8$$
$$Y = (W + 4)/4$$
$$Z = 4W + 16.$$

Substituting the values for X, Y, and Z in terms of W into the equation $W + X + Y + Z = 100$, one gets the following:

$$W + (W + 8) + [(W + 4)/4] + (4W + 16) = 100$$
$$4W + (4W + 32) + (W + 4) + (16W + 64) = 400$$
$$25W + 100 = 400$$
$$25W = 300$$
$$W = 12$$

$$X = W + 8 = 12 + 8 = 20$$
$$Y = (W + 4)/4 = (12 + 4)/4 = 4$$
$$Z = 4W + 16 = 4(12) + 16 = 64$$
$$12 + 20 + 4 + 64 = 100$$
$$12 + 4 = 20 - 4 = 4 \times 4 = 64 \div 4.$$

Because the next problem is the tenth and final problem, I must be running out ot time, so I'll end with a time problem.

10. **A mathematics teacher whose clock had stopped wound it, but did not bother to set it correctly. Then she walked from her home to the home of a friend for an evening of solving mathematical word problems. Afterward, she walked back to her own home and set her clock exactly. How could she do this without knowing the time her trip took?**

SOLUTION:

When she gets to her friend's home, she finds out the correct time. When she leaves her friend's home, she notes the correct time.

Let T_1 = the time on her clock when she wound it
T_2 = the time on her clock when she returned home
T_3 = the correct time when she arrived at her friend's home
T_4 = the correct time when she left her friend's home
T_5 = the correct time when she finally sets her clock

$T_5 = T_4 + 0.5[(T_2 - T_1) - (T_4 - T_3)]$, but the teacher knows each of T_1, T_2, T_3, and T_4. Since all the quantities on the right side of the equation are known, the teacher can compute T_5.

Clearly, these problems do not represent the standard collection of story problems. They were picked for three reasons:

1. They are "classics," that is, they keep coming up in discussions of problem solving as great examples to show certain aspects of problem solving.

2. The solutions demonstrate a variety of problem-solving strategies; for example, analysis, logic, use of mnemonics, solving a simpler problem, searching for patterns.

3. The mathematics is well within the grasp of junior high school students.

It is equally clear that I could have picked a different set of 10 problems. In fact, I'm sure each person reading this article can pick a set of 10 problems fulfilling the three criteria. The difficult part is to limit yourself to 10.

Have you heard the one about the hunter who wanted to take his one-piece rifle on a plane, but the law requires that rifles be stored in the baggage compartment? Unfortunately, the law also states that any item in the baggage compartment cannot have a length, width, or height exceeding 100 cm, and his rifle is 170 cm long. What can he do?

Assume a person can carry four days' supply of food and water for a trip across a desert that takes six days to cross. Obviously, a person cannot make the trip alone because the food and water will be gone in four days and he or she will die. How many people would have to start out in order for one person to get across the desert and for the others to get back to the starting point?

But those are different problems for another day.

Pitfalls to Avoid in Teaching Mathematics

Eric F. Wood

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One of the things that makes the teaching of mathematics a difficult job is that some methods that appear to work well initially can cause problems for the student several years later. Teachers may have no suspicion of the problem, however, since they do not normally have the opportunity to see the results of their teaching techniques. This article discusses some common teaching strategies and the problems that can result when they are employed. The intent is not to criticize but to sensitize and, in so doing, to emphasize the importance of considering the future implications of the instructional methods we use.

One day, in a Grade 9 class, I asked a girl for the result when -6 was divided by 2. She responded immediately that it was -12. Without reacting to this answer, I asked for an explanation of her logic. The reply was that since numbers get smaller in proportion to the divisor when they are divided, she just thought of the number that was twice as small as -6, and this was -12!

It was clear that this student had some knowledge of integers and that she was using a logical thinking process to calculate the answer. However, she was applying an idea that works only when dividing a positive number by another positive whole number. Perhaps in an earlier grade, the teacher found it helpful to tell the class that division makes the answer smaller, not realizing the problems that could occur both when using integers and when dividing by fractions or decimals smaller than one. These kinds of rules should be avoided unless they are **always** true and unlikely to be misinterpreted by the student.

Another good case in point is the use of the phrase "two negatives make a positive." It is easy to tell if students have been taught this rule because they will tell you that -2 - 5 = +7 and vociferously defend the answer as being correct! Unfortunately, it seems to be very hard to eradicate such ideas when they have been acquired at an early age. A better statement would be: When two integers of the same sign are multiplied together, the result is positive. It is a little longer to say, but it is correct and quite hard for students to misapply.

Students in elementary school often learn the method of transposition from their teachers (or parents) and come into high school saying things like "move it over and change the sign." The very capable students may be able to apply this technique correctly, but many students will solve an equation and end the solution with the following sequence of steps:

$$9x = 9$$

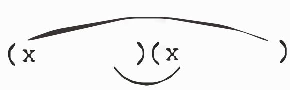
$$x = \frac{9}{-9} = -1$$

When asked why they divided by -9 rather than 9, they always respond that whenever you move a number over, you have to change the sign. Technically, the rule is correct as stated because, on the left side, the 9 is multiplied, and hence, on the right side, the operation should become division. This subtlety is hard for most students to figure out, however, and so they often mix up the signs as in the example.

The problem can be avoided by not teaching transposition at all, but simply having the students add or subtract the same quantity on both sides, or multiply and divide both sides by the same number. Most students will develop short cuts in this procedure, but it will be when they are ready, and when they are comfortable with doing some of the steps mentally.

Lest the reader think that high school teachers are immune to such problems, consider the following explanation of how to factor the trinomial $x^2 + 5x + 6$. The teacher puts two brackets on the board with an x in each and says: "You need two numbers that multiply to give 6 and add to give 5. What are they?" Students will readily give 2 and 3 as the values and, after a few more examples, proceed with the rest of the problems with little difficulty. Does this mean that there is nothing wrong with this method? Nothing so long as we restrict ourselves to factoring trinomials of the form $ax^2 + bx + c$ where $a = 1$. However, the teacher in the next grade will have a hard time convincing the students that to factor $6x^2 - 13x - 5$, they are not looking for two numbers that multiply to give -5 and add to give -13!

This difficulty can be overcome by using a more careful choice of words. After writing the two brackets on the board like this

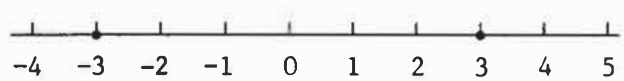


complete with the curved lines to indicate where the middle term is to come from, the teacher asks: "What two numbers multiply to give 6 that will also give a middle term when multiplied out of 5x?" A small difference perhaps, but significant all the same. Students will now ask why they can't just multiply to get the last number and add to get the one in the middle. This gives the teacher a chance to emphasize why this method is a poor one.

Teachers typically teach students that inequations and equations are very similar because you can do anything to one side as long as you do the same thing to the other. The only difference is that multiplication or division by a negative value requires the inequality sign to be reversed. This method is fine as long as the students see only linear inequations, but what about solving an inequation like $x^2 < 9$? A significant number of my student teachers, all of whom have a minimum of two courses at the university level, solved this by finding the square root of both sides and giving the solution as $x < \pm 3$. This is not correct because $-4 < -3$, but -4 does not satisfy the original inequation.

Actually, there are two problems of pedagogy here. In the first place, students should be taught to solve equations such as $x^2 = 9$ by rearranging into a quadratic form such as $x^2 - 9 = 0$. This helps to ensure that they get two roots, rather than the single value that often results when the plus or minus sign is forgotten after taking a square root. In addition, if students are taught to factor, they will be less likely to use the method of finding the square root of both sides, which does not work with quadratic inequations.

An inequation such as $x^2 < 9$ can be factored to $(x - 3)(x + 3) < 0$ and then the boundary (or zero) values graphed on a number line:



We can then verify each of the three regions that these boundary values divide the number line into by picking values such as $x = -4$, $x = 0$, and $x = 4$ and substituting them into $x^2 < 9$ to get a true or false statement. These considerations give the correct solution that $-3 < x < 3$.

This method has the advantage of being applicable to more complicated quadratic inequations such as $x^2 + 3x - 4 > 0$. Furthermore, the idea of graphing the boundary and checking test points is consistent with the method used to graph linear (and quadratic) inequations in two variables in the x-y plane. Thus, the student learns one method that helps integrate various parts of mathematics.

Needless to say, there are many more examples that could be discussed. Indeed, many teaching techniques have inherent and unavoidable difficulties associated with them. The message, however, is clear: When planning lessons, try to be aware of future implications of the method you plan to use. Perhaps most important of all, talk to your colleagues! Find out where the ideas that you are teaching today will lead in the next grade. You will grow professionally and the ultimate winners will be your students.

MAGIC SQUARES
Addition of Fractions

PREPARATION

Photocopy 3 X 3 grids, or have students draw their own.

HOW TO PLAY
(2 or 4 players)

The numbers 1/6, 1/3, 1/2, 2/3, 5/6, 1, 1 1/6, 1 1/3, and 1 1/2 will be placed in the grid. Each number may be used only once.

The first player (or pair) places one of the fractions in one of the squares. The next player (or pair) places a different number in a different square.

Every time a player (or pair) completes a row or column or diagonal so that the sum of the three numbers equals 2 1/2, they receive one point. It is possible to write all nine numbers so that the sum of each row, column, and diagonal is 2 1/2. This is called a magic square.

* * * * * * *

Have students make up their own magic squares, and then write the fractions to be used on the top of a new page. Let the class play the new game.

Contributed by Karen Gibling, Elboya Elementary-Junior High School, Calgary.

Hexagonal Combinations on Familiar Operation Tables

Bonnie H. Litwiller and David R. Duncan

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Number patterns often occur in geometric settings where they are not expected. In this article, we will present polygonal combination activities on the extended addition, subtraction, and multiplication tables. These activities result in interesting number patterns.

Activity 1

Figure 1 is an extended addition table with hexagonal combinations drawn upon it. A hexagonal combination includes a centre hexagon with six surrounding hexagons. For example, D is a centre hexagon and it is surrounded by hexagons A, B, E, G, F, and C.

For each hexagonal combination:

1. Find the sum of the interior numbers of the centre hexagon; call this sum C.
2. Find the sum of the interior numbers of the six hexagons that surround the centre hexagon; call this sum H.
3. Divide by 6 to find the average of the interior numbers of the surrounding hexagons; call this quotient A.
4. Compare C and A.

Table 1 reports the results of our computations.

TABLE 1.

Centre Hexagon	C	Surrounding Hexagons	H	A
D	17	A B E G F C	102	17
K	43	H I L N M J	258	43
R	37	O P S U T Q	222	37
Y	53	V W Z B' A' X	318	53

Observe that in each case, C = A; that is, the sum of the interior numbers of the centre hexagon is equal to the average of the interior number sums of the six surrounding hexagons. Draw other hexagonal combinations and check to see that the conjecture holds.

Activity 2

Figure 2 is an extended subtraction table with hexagonal combinations drawn upon it. Perform steps 1 through 4 of Activity 1. Does the same pattern hold?

Table 2 summarizes the results of our calculations.

TABLE 2.

Centre Hexagon	C	Surrounding Hexagons	H	A
D	-3	A B E G F C	-18	-3
K	-21	H I L N M J	-126	-21
R	9	O P S U T Q	54	9
Y	-1	V W Z B' A' X	-6	-1

Observe that the same pattern holds as for that of Activity 1. Draw other hexagonal combinations and check to see that the pattern continues to hold.

Activity 3

Does the pattern of Activities 1 and 2 also hold on the extended multiplication table of Figure 3?

The results of our computations are shown in Table 3.

TABLE 3.

Centre Hexagon	C	Surrounding Hexagons	H	A
D	36	A B E G F C	216	36
K	176	H I L N M J	1056	176
R	190	O P S U T Q	1140	190
Y	108	V W Z B' A' X	648	108

Again the conjecture holds. Draw other hexagonal combinations and check that the pattern continues to hold.

What happens if hexagonal combinations of "larger size" are shown upon the extended operation tables.

Activity 4

On the extended addition table of Figure 4 with the hexagonal combinations, perform these steps:

1. Find the sum of the interior numbers of the centre hexagon; call this sum C.

2. Find the sum of all the interior numbers of the six surrounding hexagons; call this sum H.

3. Divide by 6, the number of surrounding hexagons; this will give the average, A.

4. Compare C and A.

The results of steps 1 through 4 are displayed in Table 4.

TABLE 4.

Centre Hexagon	C	Surrounding Hexagons	H	A
E	187	A B F J I D	1122	187
F	231	B C G K J E	1386	231
I	209	D E J M L H	1254	209
J	253	E F K N M I	1518	253

CONJECTURE: The sum of the interior numbers of the centre hexagon is equal to the average of the interior number sums of the six surrounding hexagons.

Activity 5

On the extended subtraction table of Figure 5, check to see if the same pattern holds.

Table 5 is given so that you will be able to check your results.

TABLE 5.

Centre Hexagon	C	Surrounding Hexagons	H	A
E	-11	A B F J I D	-66	-11
F	-55	B C G K J E	-330	-55
I	55	D E J M L H	330	55
J	11	E F K N M I	66	11

The pattern holds; that is, the sum of the interior numbers of the centre hexagon is equal to the average of the interior number sums of the six surrounding hexagons.

Activity 6

On the extended multiplication table of Figure 6, check to see if the pattern holds.

Compare your results with those of Table 6. The pattern again holds.

TABLE 6.

Centre Hexagon	C	Surrounding Hexagons	H	A
E	792	A B F J I D	4752	792
F	1144	B C G K J E	6864	1144
I	924	D E J M L H	5544	924
J	1452	E F K N M I	8712	1452

CHALLENGE: Draw other polygonal combinations upon the extended operation tables and find number patterns.

FIGURE 1.

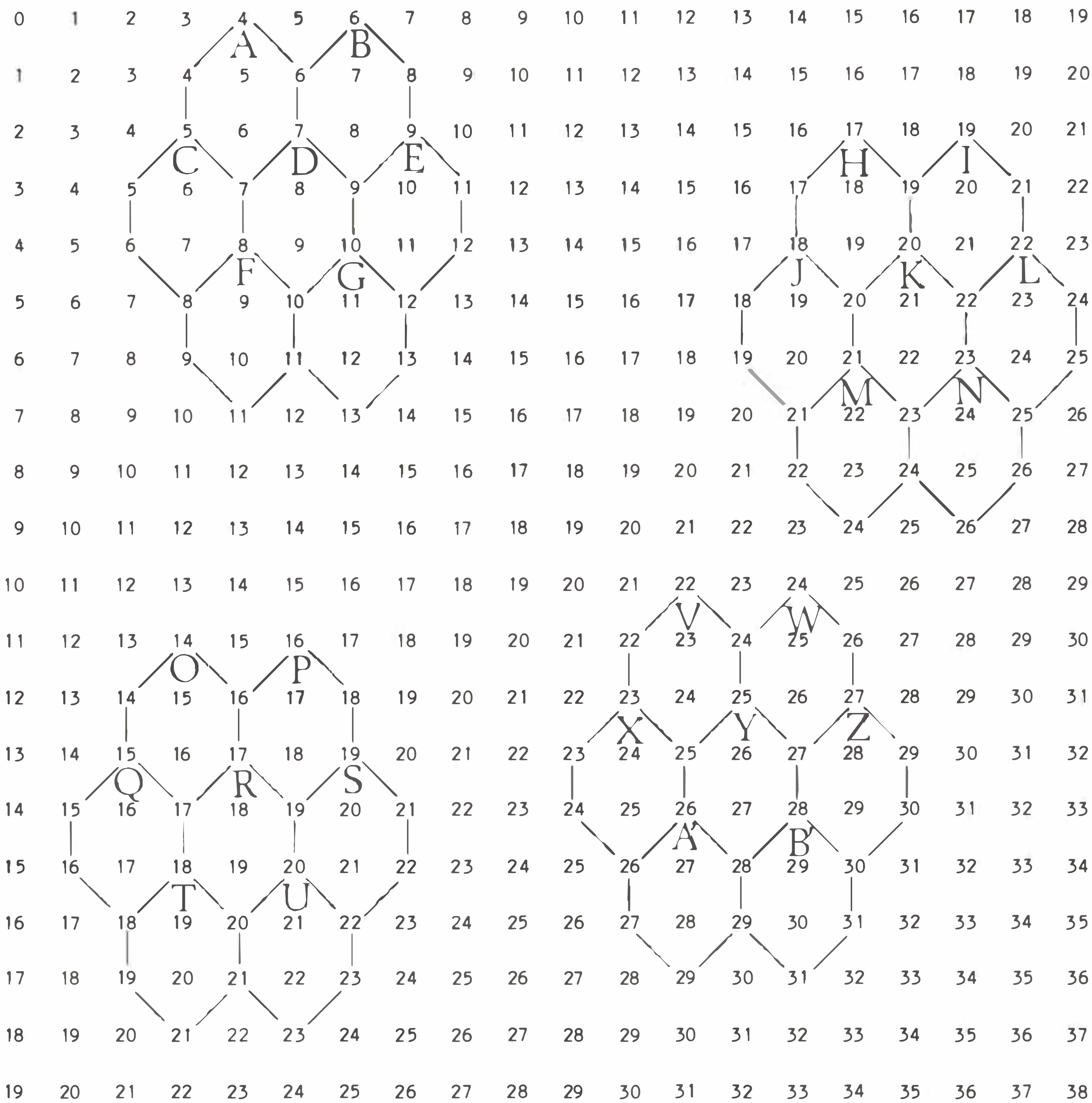


FIGURE 2.

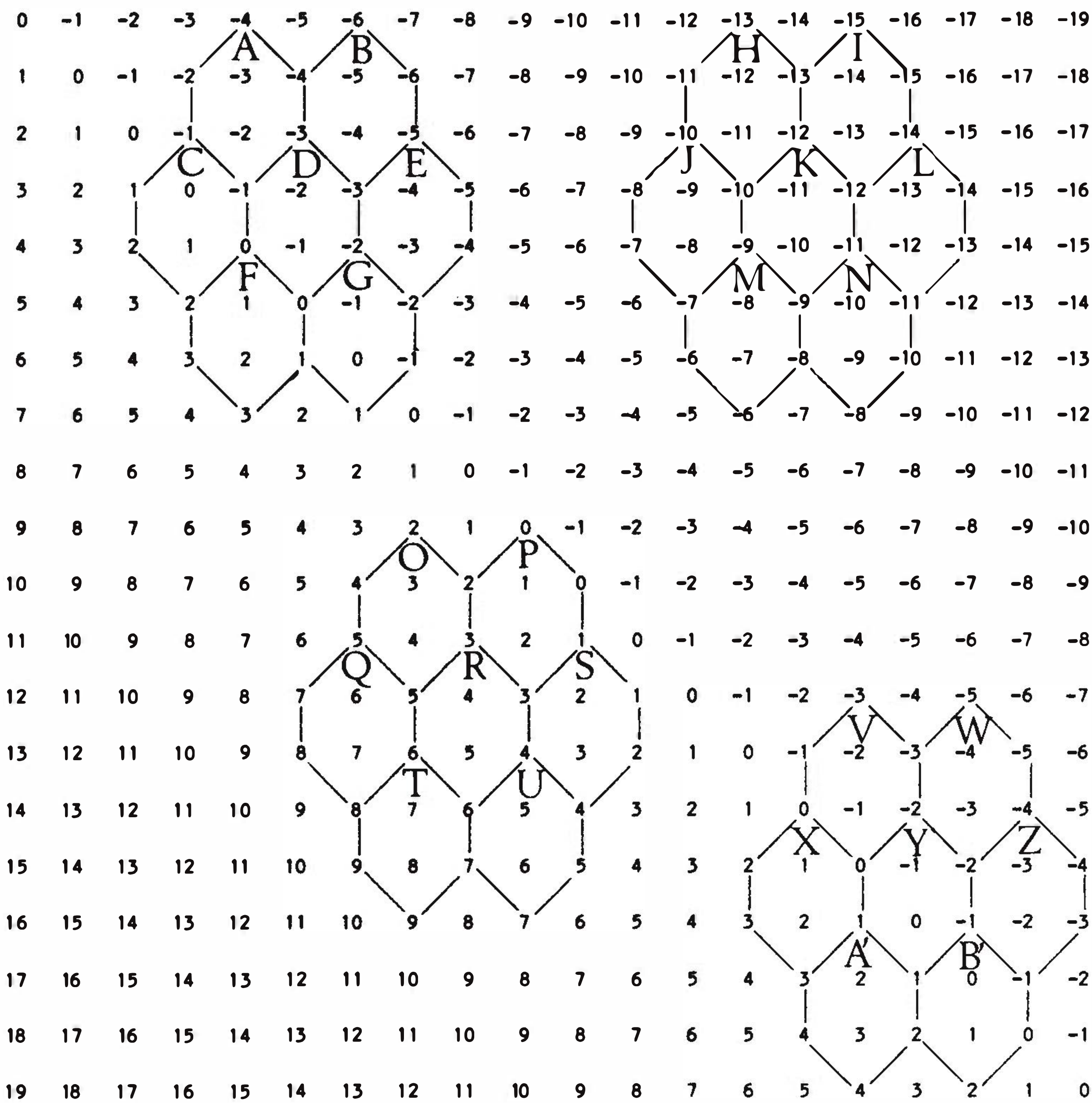

0 -1 -2 -3 -4 -5 -6 -7 -8 -9 -10 -11 -12 -13 -14 -15 -16 -17 -18 -19
1 0 -1 -2 -3 -4 -5 -6 -7 -8 -9 -10 -11 -12 -13 -14 -15 -16 -17 -18
2 1 0 -1 -2 -3 -4 -5 -6 -7 -8 -9 -10 -11 -12 -13 -14 -15 -16 -17
3 2 1 0 -1 -2 -3 -4 -5 -6 -7 -8 -9 -10 -11 -12 -13 -14 -15 -16
4 3 2 1 0 -1 -2 -3 -4 -5 -6 -7 -8 -9 -10 -11 -12 -13 -14 -15
5 4 3 2 1 0 -1 -2 -3 -4 -5 -6 -7 -8 -9 -10 -11 -12 -13 -14
6 5 4 3 2 1 0 -1 -2 -3 -4 -5 -6 -7 -8 -9 -10 -11 -12 -13
7 6 5 4 3 2 1 0 -1 -2 -3 -4 -5 -6 -7 -8 -9 -10 -11 -12
8 7 6 5 4 3 2 1 0 -1 -2 -3 -4 -5 -6 -7 -8 -9 -10 -11
9 8 7 6 5 4 3 2 1 0 -1 -2 -3 -4 -5 -6 -7 -8 -9 -10
10 9 8 7 6 5 4 3 2 1 0 -1 -2 -3 -4 -5 -6 -7 -8 -9
11 10 9 8 7 6 5 4 3 2 1 0 -1 -2 -3 -4 -5 -6 -7 -8
12 11 10 9 8 7 6 5 4 3 2 1 0 -1 -2 -3 -4 -5 -6 -7
13 12 11 10 9 8 7 6 5 4 3 2 1 0 -1 -2 -3 -4 -5 -6
14 13 12 11 10 9 8 7 6 5 4 3 2 1 0 -1 -2 -3 -4 -5
15 14 13 12 11 10 9 8 7 6 5 4 3 2 1 0 -1 -2 -3 -4
16 15 14 13 12 11 10 9 8 7 6 5 4 3 2 1 0 -1 -2 -3
17 16 15 14 13 12 11 10 9 8 7 6 5 4 3 2 1 0 -1 -2
18 17 16 15 14 13 12 11 10 9 8 7 6 5 4 3 2 1 0 -1
19 18 17 16 15 14 13 12 11 10 9 8 7 6 5 4 3 2 1 0
A B C D E F G
H I J K L M N
O P Q R S T U
V W X Y Z A′ B′

FIGURE 3.

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
0	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38
0	3	6	9	12	15	18	21	24	27	30	33	36	39	42	45	48	51	54	57
0	4	8	12	16	20	24	28	32	36	40	44	48	52	56	60	64	68	72	76
0	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90	95
0	6	12	18	24	30	36	42	48	54	60	66	72	78	84	90	96	102	108	114
0	7	14	21	28	35	42	49	56	63	70	77	84	91	98	105	112	119	126	133
0	8	16	24	32	40	48	56	64	72	80	88	96	104	112	120	128	136	144	152
0	9	18	27	36	45	54	63	72	81	90	99	108	117	126	135	144	153	162	171
0	10	20	30	40	50	60	70	80	90	100	110	120	130	140	150	160	170	180	190
0	11	22	33	44	55	66	77	88	99	110	121	132	143	154	165	176	187	198	209
0	12	24	36	48	60	72	84	96	108	120	132	144	156	168	180	192	204	216	228
0	13	26	39	52	65	78	91	104	117	130	143	156	169	182	195	208	221	234	247
0	14	28	42	56	70	84	98	112	126	140	154	168	182	196	210	224	238	252	266
0	15	30	45	60	75	90	105	120	135	150	165	180	195	210	225	240	255	270	285
0	16	32	48	64	80	96	112	128	144	160	176	192	208	224	240	256	272	288	304
0	17	34	51	68	85	102	119	136	153	170	187	204	221	238	255	272	289	306	323
0	18	36	54	72	90	108	126	144	162	180	198	216	234	252	270	288	306	324	342
0	19	38	57	76	95	114	133	152	171	190	209	228	247	266	285	304	323	342	361

FIGURE 4.

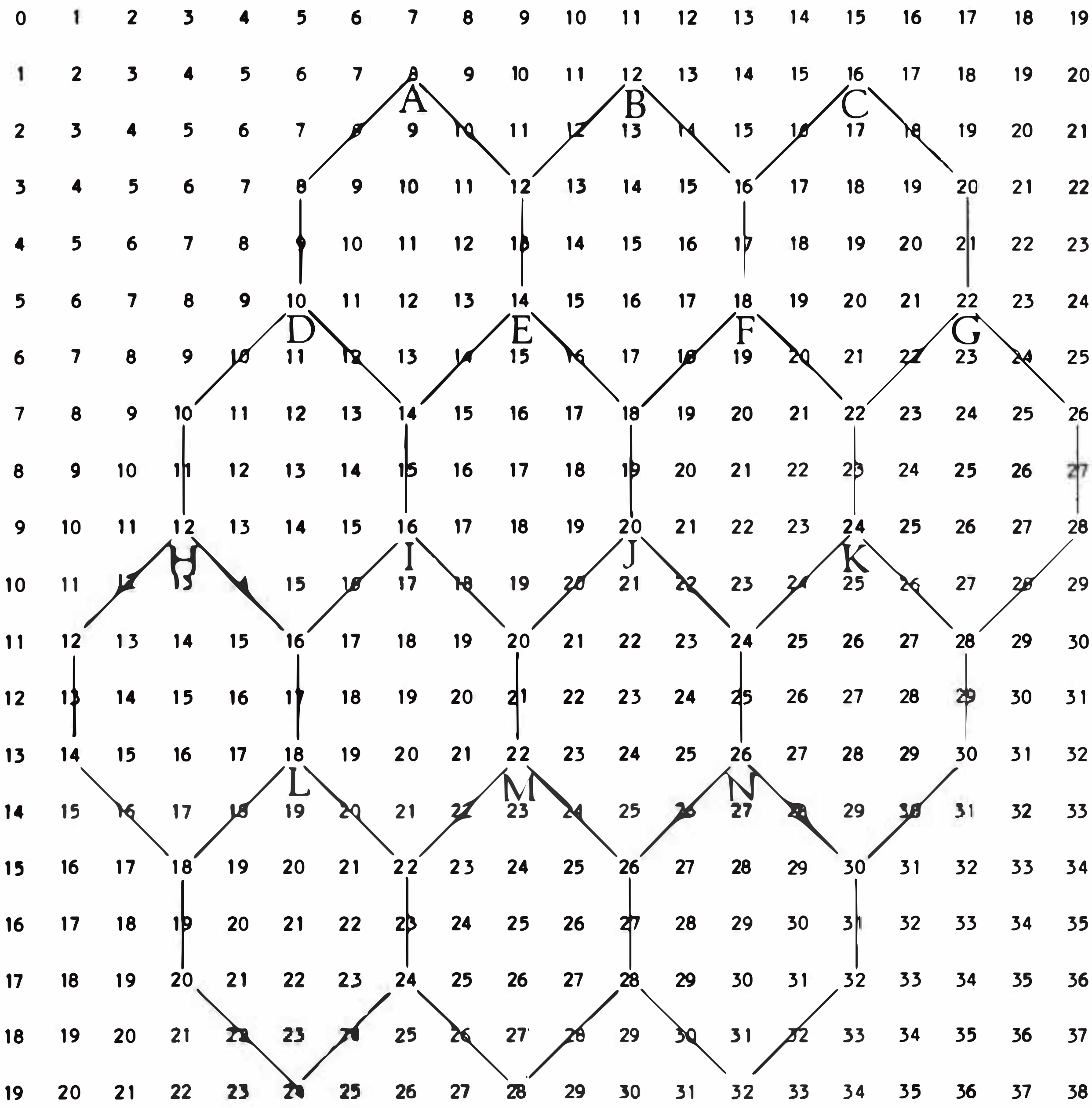
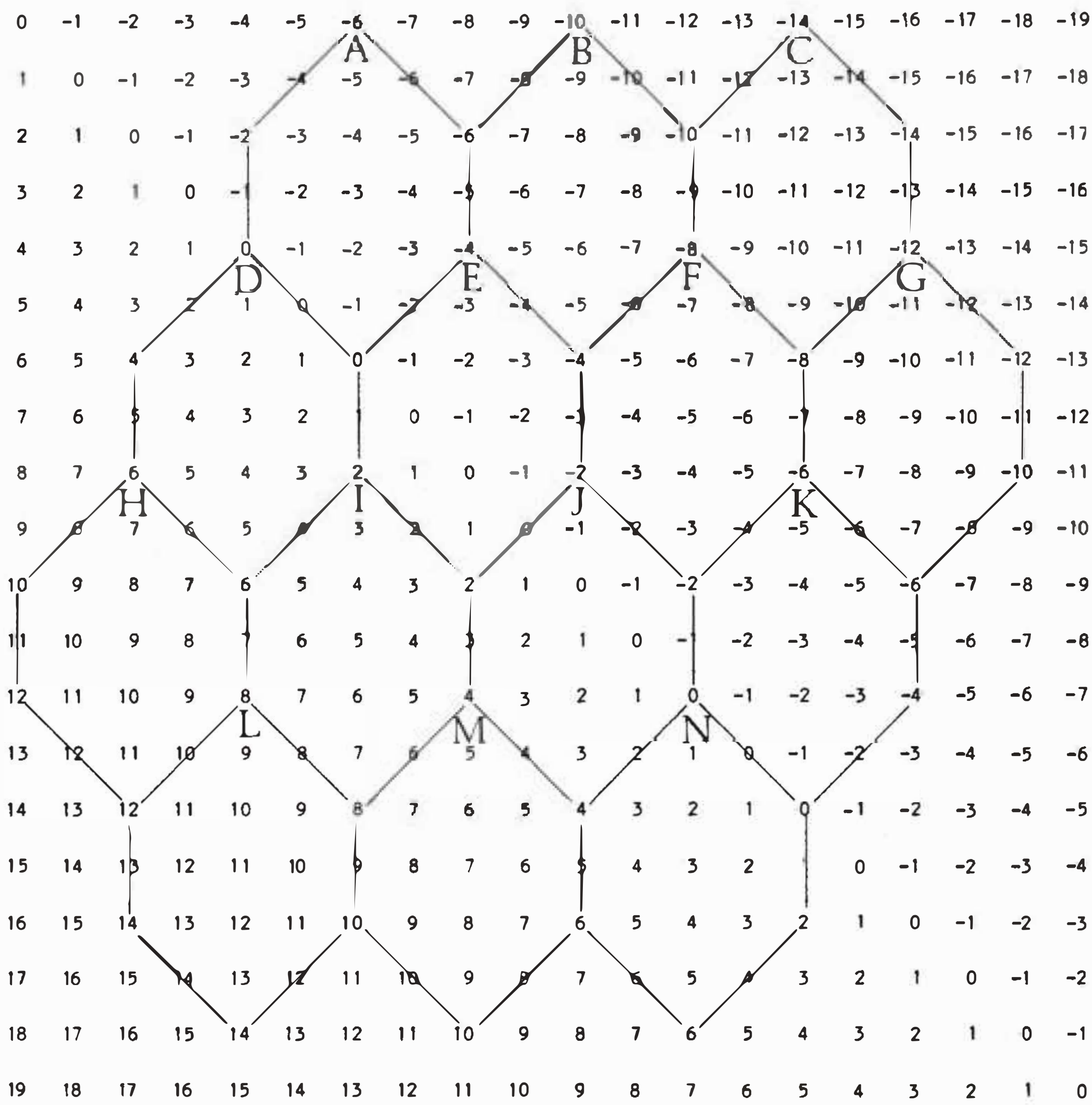


FIGURE 5.



0 -1 -2 -3 -4 -5 -6 -7 -8 -9 -10 -11 -12 -13 -14 -15 -16 -17 -18 -19
1 0 -1 -2 -3 -4 -5 -6 -7 -8 -9 -10 -11 -12 -13 -14 -15 -16 -17 -18
2 1 0 -1 -2 -3 -4 -5 -6 -7 -8 -9 -10 -11 -12 -13 -14 -15 -16 -17
3 2 1 0 -1 -2 -3 -4 -5 -6 -7 -8 -9 -10 -11 -12 -13 -14 -15 -16
4 3 2 1 0 -1 -2 -3 -4 -5 -6 -7 -8 -9 -10 -11 -12 -13 -14 -15
5 4 3 2 1 0 -1 -2 -3 -4 -5 -6 -7 -8 -9 -10 -11 -12 -13 -14
6 5 4 3 2 1 0 -1 -2 -3 -4 -5 -6 -7 -8 -9 -10 -11 -12 -13
7 6 5 4 3 2 1 0 -1 -2 -3 -4 -5 -6 -7 -8 -9 -10 -11 -12
8 7 6 5 4 3 2 1 0 -1 -2 -3 -4 -5 -6 -7 -8 -9 -10 -11
9 8 7 6 5 4 3 2 1 0 -1 -2 -3 -4 -5 -6 -7 -8 -9 -10
10 9 8 7 6 5 4 3 2 1 0 -1 -2 -3 -4 -5 -6 -7 -8 -9
11 10 9 8 7 6 5 4 3 2 1 0 -1 -2 -3 -4 -5 -6 -7 -8
12 11 10 9 8 7 6 5 4 3 2 1 0 -1 -2 -3 -4 -5 -6 -7
13 12 11 10 9 8 7 6 5 4 3 2 1 0 -1 -2 -3 -4 -5 -6
14 13 12 11 10 9 8 7 6 5 4 3 2 1 0 -1 -2 -3 -4 -5
15 14 13 12 11 10 9 8 7 6 5 4 3 2 1 0 -1 -2 -3 -4
16 15 14 13 12 11 10 9 8 7 6 5 4 3 2 1 0 -1 -2 -3
17 16 15 14 13 12 11 10 9 8 7 6 5 4 3 2 1 0 -1 -2
18 17 16 15 14 13 12 11 10 9 8 7 6 5 4 3 2 1 0 -1
19 18 17 16 15 14 13 12 11 10 9 8 7 6 5 4 3 2 1 0
A B C D E F G H I J K L M N

FIGURE 6.

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
0	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38
0	3	6	9	12	15	18	21	24	27	30	33	36	39	42	45	48	51	54	57
0	4	8	12	16	20	24	28	32	36	40	44	48	52	56	60	64	68	72	76
0	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90	95
0	6	12	18	24	30	36	42	48	54	60	66	72	78	84	90	96	102	108	114
0	7	14	21	28	35	42	49	56	63	70	77	84	91	98	105	112	119	126	133
0	8	16	24	32	40	48	56	64	72	80	88	96	104	112	120	128	136	144	152
0	9	18	27	36	45	54	63	72	81	90	99	108	117	126	135	144	153	162	171
0	10	20	30	40	50	60	70	80	90	100	110	120	130	140	150	160	170	180	190
0	11	22	33	44	55	66	77	88	99	110	121	132	143	154	165	176	187	198	209
0	12	24	36	48	60	72	84	96	108	120	132	144	156	168	180	192	204	216	228
0	13	26	39	52	65	78	91	104	117	130	143	156	169	182	195	208	221	234	247
0	14	28	42	56	70	84	98	112	126	140	154	168	182	196	210	224	238	252	266
0	15	30	45	60	75	90	105	120	135	150	165	180	195	210	225	240	255	270	285
0	16	32	48	64	80	96	112	128	144	160	176	192	208	224	240	256	272	288	304
0	17	34	51	68	85	102	119	136	153	170	187	204	221	238	255	272	289	306	323
0	18	36	54	72	90	108	126	144	162	180	198	216	234	252	270	288	306	324	342
0	19	38	57	76	95	114	133	152	171	190	209	228	247	266	285	304	323	342	361

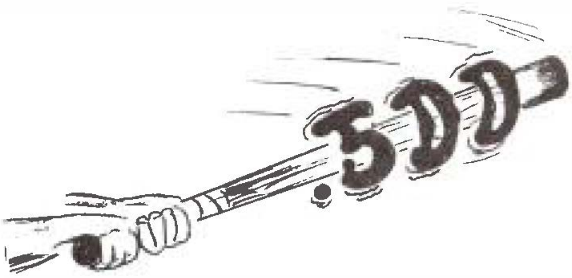
Probability without Formulas and Equations

Irvin K. Burbank

Dr. Burbank is an associate professor of education at the University of Victoria, where his primary responsibility is mathematics curriculum and instruction courses. He is active in inservice education and presented a paper to the NCTM Canadian Conference in Edmonton. Dr. Burbank is coauthor of Houghton Mifflin's Mathematics *series, authorized for use in Alberta.*

The girl by the Probability Machine, watching the balls drop and form an approximate normal distribution, may find it easy to agree with Laplace, who stated that "the theory of probability is nothing more than good sense confirmed by calculation." It is also easy to agree with Laplace when one considers the everyday probability statements on the following page.

Probability Machine from the IBM MATHEMATICA exhibition at the Chicago Museum of Science and Industry. Photo by Office of Charles and Ray Eames.



The batting champion is hitting 500.

"Four out of five doctors recommend sugarless gum for those patients who chew gum."

Insurance records show that very few people die between the ages of 100 and 120.

"Our group used Suds toothpaste and we had 49 percent fewer cavities."

"Seventy-five percent chance of rain."

Artwork by I.K. Burbank.

However, many students who have had formal exposure to probability may find it difficult to agree with Laplace if their introduction to probability was in terms of sophisticated equations, technical terminology, and meaningless tables of value. Many would say they did not have a chance to build their knowledge and understanding of probability on a "good sense" basis because they were busy memorizing definitions and equations.

The purpose of this article is to outline and discuss some basic probability topics that can be taught to students without the use of equations and technical terminology. The topics to be discussed are:

1. Probability--The Science of Chance
2. The Range of Probability
3. Applying the Theory
4. A Picture of Probability Outcomes.

The Science of Chance

GOAL: Students will experience making assessments based on given data.

PROCEDURES:

Place 12 blue cubes and 2 yellow cubes in a pail and follow these steps:

Step 1: Ask the students, "What is in the pail?" They will not be able to answer because of the lack of data.

Step 2: Shake the pail and ask, "What do you think is in the pail?" and "Are the objects hard or soft, round or square?" The students will be able to make some assessment from the information they obtain from sound.

Step 3: Let some students feel the objects (without looking) and ask them, "What do the objects feel like--are they smooth, rough, sharp?" "About how many are in the pail?" and "What color are the objects?" Now that the students have more data by feeling, they are able to give a more accurate assessment of the items in the pail. However, to state the color of the objects, they will have to look, which brings us to the next step.

Step 4: Have a student take out a cube. However, before he or she looks at it, ask: "What color do you think it is?" "Can you guess the color?" "Are you very sure you are right?" and "How much will you bet that you are right?" At this point, the student is not very confident he or she knows the right color. Allow the cube to be shown, then ask, "Do you think all the cubes in the pail are that color?" Have the student replace the cube. Let three or four other students select one cube, and ask them the same questions. Then allow a student to take three cubes, but before opening his or her hand ask, "What color are the cubes in your hand?" "Are they all blue?" "How much are you willing to bet they are all blue?" and "How much would you bet that one is blue?" As more students take turns selecting and replacing cubes, the confidence that a blue cube will be selected increases.

Step 5: Inform the students that the pail contains 14 cubes. The cubes are either blue or yellow. Have students estimate the number of blue cubes and yellow cubes.

In this simple activity, students can experience making "probability" assessments from given data, and it will help them appreciate the association of probability and chance.

The Range of Probability

Once the students understand that probability is the science of chance, they are ready to experience how the probability of an event is determined.

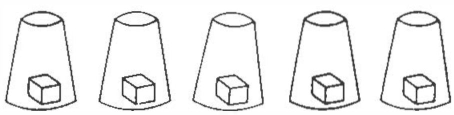
GOAL: Students will experience an activity that illustrates how probability ranges from 0 to 1.

PROCEDURE:

Place 5 cubes under 5 cups, and have students determine the chance of selecting a cup with a cube in each of the following cases. In Cases 2 through 6, replace a cube with a ball, as shown in the diagram.

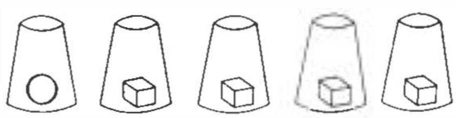
Note: P(cube) is short for "the Probability of selecing a cube."

Case 1: With 5 cubes under 5 cups, what is the probability of selecting a cup with a cube under it?



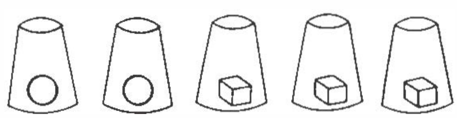
Number of choices 5
Number of cubes 5
Chance of selecting a cup with a cube is 5 out of 5 or 5/5
P(cube) = **5/5 or 1**

Case 2: Replace a cube with a ball. What is the probability of selecting a cube [P(cube)]?



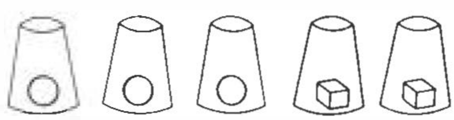
Number of choices ______
Number of cubes ______
P(cube) = ______

Case 3: What is the probability of selecting a cube in this case?



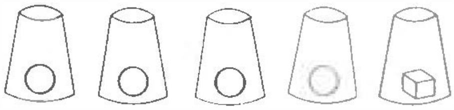
Number of choices ______
Number of cubes ______
P(cube) = ______

Case 4: P(cube)?



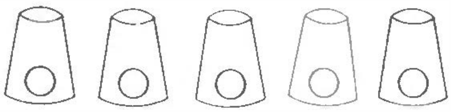
Number of choices ______
Number of cubes ______
P(cube) = ______

Case 5: P(cube)?



Number of choices ______
Number of cubes ______
P(cube) = ______

Case 6: P(cube)?



Number of choices ______
Number of cubes ______
P(cube) = ______

In this activity, the student can conclude that the outcome of an event is known for sure if probability is 0 or 1, and that all other values range between these two extremes when the outcome is not certain. All this can be learned by the student, and the only skill required is to observe and count.

The fact that the probability that an event occurs plus the probability that the event does **not** occur equals one can be illustrated in this activity by having the students fill out the following tables from their observation of cubes, balls, and cups.

	P(Cube)	P(Not a Cube)		P(Cube) + P(Not a Cube)				
Case 1:	5/5	0/5	→	5/5	+	0/5	=	1
Case 2:	4/5	1/5	→	4/5	+	1/5	=	1
Case 3:	3/5	2/5	→	3/5	+	2/5	=	1
Case 4:	2/5	3/5	→	2/5	+	3/5	=	1
Case 5:	1/5	4/5	→	1/5	+	4/5	=	1
Case 6:	0/5	5/5	→	0/5	+	5/5	=	1

In filling out this table, note that the probability of not selecting a cube is the same as selecting a ball.

Applying the Theory

Now that the student knows that the probability of an event is the ratio of the number of times the event occurs over the total number of possible events, the next step is to have them apply the theory.

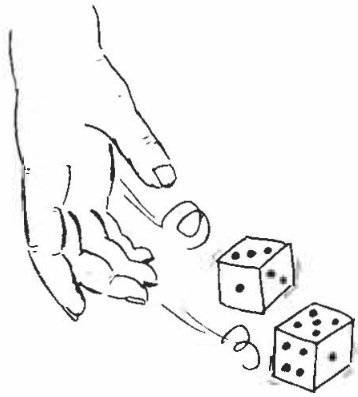
GOAL: The students will experience a probability problem in a game setting.

PROCEDURE:

Provide each pair of students with a pair of dice. Let the students decide who is to be Player A and Player B, based on the following rule:

- Player A scores a point if the sum on the pair of dice is 9, 8, 7, 6, 5.
- Player B scores a point if the sum on the dice is 2, 3, 4, 10, 11, 12.
  Have one student in each pair roll the dice 30 to 40 times and the partner record the scores for A and B.

For example:



Player A	Player B

3 + 5 = 8

Player A scores a point.

Although Player A has 5 sums (9, 8, 7, 6, 5) and Player B has six sums (2, 3, 4, 10, 11, 12), the students soon realize that there is something strange about the outcome of the game. When they share results, they will say, "Player A won almost twice as often as Player B. Why?" In answer to this question, have the students complete the following table by circling the sums that scored a point for A and putting a box around the sums that scored a point for B. Then have them respond to the questions.

Red Die

White Die	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

How many sums does Player A have? ________

How many sums does Player B have? ________

What is the total number of sums possible in the roll of two dice? ________

What is the probability of A's winning? ________

What is the probability of B's winning? ________

The probability of A's winning is 24/36, and the probability of B's winning is 12/36. Theoretically, Player A should win twice as many times as Player B. Compare this theoretical probability with the students' tallies for this game by having each pair of students record their scores on the chalkboard.

For example, assume four pairs had these results:

	Player "A"	Player "B"
Joe, Mary	39	21
Bill, Ed	38	22
Sue, Jill	40	18
Dick, Anne	41	19
TOTAL	158	80

The students will note that the actual outcome of 158 to 80 was close to the theoretical outcome in which Player A wins twice as often as Player B.

From the following table of values, many additional questions can be answered by observing and counting.

		Red Die					
		1	2	3	4	5	6
White Die	1	2	3	4	5	6	7
	2	3	4	5	6	7	8
	3	4	5	6	7	8	9
	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

Total number of sums = ________

Most frequent sum = ________

Least frequent sum = ________

P(sum of 7) = ________

P(sum of 12) = ________

P(of an even sum) = ________

P(of an odd sum) = ________

Finally, the students are ready to complete the probability of each sum from 2 to 12 and identify patterns.

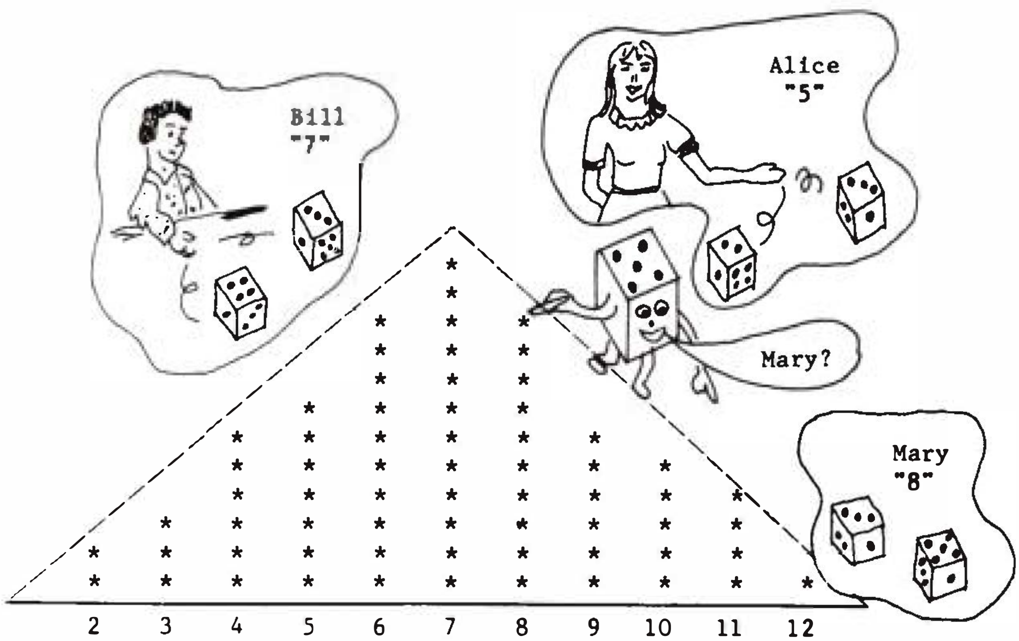
A Picture of Probable Outcomes

From the previous activities, students will realize that certain sums occur more frequently than others. The next activity will build on this perception.

GOAL: The students will experience an activity which illustrates the concept of a triangular distribution.

PROCEDURE:

Have each student roll a pair of dice 4 or 5 times and record their sums on the chalkboard in the following manner.



After the results are recorded, ask the following questions:

1. What do you notice about the sums?
2. In a throw of a pair of dice, what sums will likely show up?
3. What sums are the least likely to show up?
4. Count the number of sums less than 7 and greater than 7. How do they compare?
5. What shape does the dotted line form?
6. How many numbers are between 2 and 7?
7. How many numbers are between 7 and 12?
8. Make a statement about 7.

This activity can be extended to illustrate the concept of a normal distribution. Have each student roll 4 dice a number of times and record the sums on the board. The sums will range from a 4 (four ones) to 24 (four sixes).

In this article, probability topics such as chance, sampling, ratio, range, $P(A) + P(A') = 1$, distribution, and sample space have been illustrated without the use of equations and technical terminology. The major skills required to work through the activities were observation and counting. It is hoped that the ideas in this article will be of help to teachers so that their students may also conclude that "The theory of probabilities is nothing more than good sense, confirmed by calculation" (Laplace).

Evaluation through Problem Solving

Marie Hauk

Marie Hauk received her bachelor of education and master of education degrees at the University of Alberta, and is presently a doctoral candidate in the field of elementary mathematics education. Marie has given lectures on mathematics curriculum and instruction courses. The following article is based on a work session she presented at the NCTM Canadian Conference held in Edmonton, October 16-18, 1986.

This article will present the concept of evaluation **through** problem solving rather than evaluation **of** problem solving. These two concerns are related, but the latter, which is of no lesser importance, would require a different interpretation than will be focused on here.

This article is based on two of my personal beliefs:

1. Learning or demonstrating knowledge of a concept or skill in isolation or minimal context is of limited, if any, value to the learner.
2. Learning of subskills or concepts, which are developed in a linear or hierarchical order, does not necessarily result in holistic understanding of these concepts or skills for the purpose of application in any but the most routine of situations.

An illustration of the first point is the method of teaching the associative property of addition. It is often presented in symbolic form,

$$(a + b) + c = a + (b + c)$$

preceded and/or followed with examples such as,

$$(2 + 5) + 3 = 2 + (5 + 3)$$
$$(7) + 3 = 2 + (8)$$
$$10 = 10$$

A number of practice exercises may follow:

$$(10 + \_\_) + 2 = 10 + (6 + 2)$$
$$3 + (9 + 11) = (3 + 9) + \_\_$$
$$6 + (\_\_ + 8) = 6 + (4 + 8)$$
$$(7 + 4) + 2 = \_\_ + 6$$
$$15 + 3 = 7 + (\_\_ + 3)$$

After learning this, children may verbalize the property as "You can add the same numbers in any order and still get the same sum," yet have little sensitivity to personal application. For example, suppose the children were selling cookies at recess as a class money-raising project. The cookies cost 10¢ each and, at the end of recess, the day's sellers' recorded their sales for 1, 8, 9, 7, 3, and 2 cookies, respectively:

10
80
90
70
30
20

When adding to obtain the total amount, they would probably use the standard algorithmic method (that is, add the ones column for a total of zero), which would be recorded, and then add the tens column (1 + 8 = 9 + 9 = 8 + 7 = 25 + 3 = 28 + 2 = 30), which would be recorded for a total of 300.

Taking a moment to view this addition holistically, one may see sums of 100 and quickly add mentally (10 + 90, 80 + 20, and 70 + 30, for a total of 300).

The associative property can also be used in conjunction with estimation. For example, when going through the express lane at the grocery store, I usually estimate the total price of my purchases. Suppose I had five items, for $1.49, $.33, $1.19, $.67, and $2.79. I could group them on the counter as follows: $1.19 and $2.79 for about $4, plus $.67 and $.33 for about $1, plus $1.49 for about $1.50, making a quickly estimated total of $6.50.

For an illustration of the second point, consider the following problem. At Westvale Elementary School, a total of 189 students joined the art club and a total of 153 students joined the science club. Some of these students are in both clubs. If there are 456 students in all, how many did not join the art club? To solve this problem, one must subtract 189 from 456. Note that using the popular "decomposition" algorithm,

$$\begin{array}{r} 456 \\ \underline{-189} \end{array}$$

"regrouping" or "borrowing" over both tens and hundreds is required. As the algorithm increases in difficulty in the textbook--that is, has larger numbers and requires more regroupings (including the dreaded zeros!)--problems that require each level of subtraction are presented. This means that Jimmy, who has difficulty when regrouping over hundreds is required, would probably not solve (or be expected to solve) this problem. The fact of the matter is, though, that the algorithm used is irrelevant to the problem. It could be solved using the "equal additions" or the "left to right" algorithm. What is important is that Jimmy should be given the opportunity (and he well may be able) to attempt to solve the problem--to cut through the extraneous data and know why subtraction is appropriate. Being a whiz at computation will not serve this purpose for him.

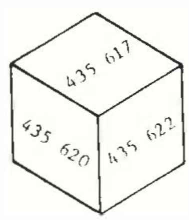
Now consider an example of what evaluation of problem solving is **not**, but often is accepted as such in textbooks. You probably can find a page in the textbook you are using that fits the following description. About 35 multiplication exercises which range from 1 digit by 2 digits to 3 digits by 4 digits. Following this are perhaps six word problems that require the application of multiplication for solution and each one is of the form, "There are n objects in each of m sets. How many objects in all?" Is there any doubt about how to solve these "problems," given the schema aroused by the 35 exercises and the fact that each problem has only two numbers? The children need not read these "problems" to solve them. In this context, they are not problems; they require only trivial application of multiplication. A problem is a situation in which a person does not know immediately what to do; it requires thought and decision making.

Consider the following problems. Solve each one and record the concepts or skills you used.

1. If you can place a 2, 4, or 7 in each box, how many different numerals can you write that are less than 35 000?

   3 □ 2 □ 5

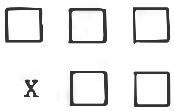
2. The numbers on the faces of this cube are in consecutive order. What numbers are on the unseen faces?



3. Three of the following four numbers were added, and the sum was about 90 000. Which addends were used?

41 195
56 308
19 687
26 429

4. Arrange these digits 6, 2, 5, 7, and 4 to give the greatest product.



5. Use the associative property to help you add these numbers on your calculator.

6 957 843 152
9 834 279 947
3 679 982 158
7 465 815 279

6. Without using a marked ruler, draw a triangle with sides of 4 cm, 5 cm, and 10 cm. Verify your answer, using a ruler.

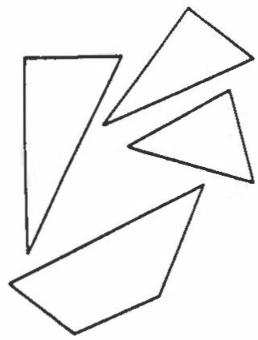
7. How long (to the second) will it be until the minute and hour hands are in a straight line?

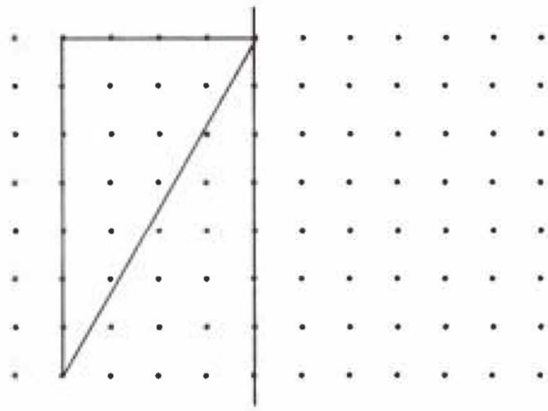
8. The perimeter of a rectangle is 14 cm.
(a) Draw such a rectangle.
(b) Find the area of your rectangle.

9. Marge bought three of the following items. If she got $5.60 change from a $10 bill, what did she buy?

ice cream cone	$ .65
sundae	$1.50
parfait	$1.95
banana split	$1.75
1 litre of ice cream	$2.25

10. Place these shapes so that they form the flip image of the triangle.





When you have finished solving the problems, compare them with the problems in the following exercises. Each objective is taken from the <u>Alberta Elementary Mathematics Curriculum Guide</u> (1982) and is followed by a typical textbook evaluation question. Each of the exercises is matched with the problem having the same number.

1. OBJECTIVE: Identifies and names place value of digits (to 999 999).

TEST ITEM: What is the place value of the 5 in each numeral?
(a) 45 203
(b) 351 462
(c) 698 520

2. OBJECTIVE: Reads, writes, and orders whole numbers to 999 999.

TEST ITEM: List from smallest to largest:
326 951
26 591
326 519
362 591

3. OBJECTIVES: Rounds whole numbers (up to nearest 10 000). Estimates sums.

TEST ITEM: Round each number to the nearest ten thousand and add to estimate the sum.
53 162
29 579
15 406
72 658

4. OBJECTIVES: Multiplies whole numbers using one-, two-, and three-digit multipliers. Estimates products.

TEST ITEMS: Multiply:

256	3412	795
x 79	x 6	x 184

5. OBJECTIVE: Understands the associative property of addition.

TEST ITEM: (195 + 623 + 439) + (866 + 219) = (623 + 219 + 439) + (195 +___)

6. OBJECTIVE: Estimates and uses standard units of length.

TEST ITEM: Write your estimate of the length of this line in cm and then measure to the nearest cm.

7. OBJECTIVE: Reads and writes time to seconds.

TEST ITEM:
(a) Write the time shown.
(b) What time will it be 2h 36 min 10s later?

8. OBJECTIVES: Finds perimeter of polygons without using formulas. Finds area of polygons without using formulas. Uses standard units of linear measure.

TEST ITEM: Measure the sides of this rectangle in cm. Find the perimeter and the area.

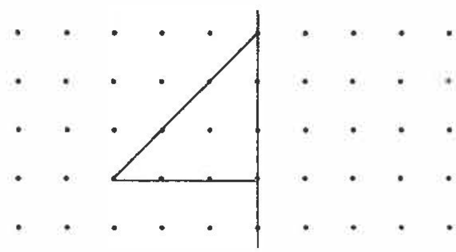
9. OBJECTIVE: Uses money (coins and bills) for purchasing and making change.

TEST ITEM: Sundaes:
lge. 65¢ med. 50¢ sm. 35¢

Joe gave the clerk $10 for 4 large, 5 medium, and 3 small sundaes. How much change should he get?

10. OBJECTIVE: Identifies and draws flips of 2-dimensional figures.

TEST ITEM: Draw the flip image of the triangle.



After you have compared the problems with the test items, consider the following comments.

- The test items tell students what to do. Applying the concept or skill routinely is all that is required.

The problems, which require students to apply the same concepts or skills, also require students to think about how the concepts or skills can be used.

- Test items usually have one answer. Problems may have several possible correct answers, as in problems 5 and 8.

- Test items usually have an answer! Problem 6 is impossible. Children must learn to be critical of data given and trust their own judgment. Explaining why a problem has no answer is a solution.

- Test items usually require only pencil and paper. Problems may require concrete manipulation, as in problem 10.

- Test items usually require manual computation. A calculator is required to complete problem 5, and it could be useful for problems 4 and 9 as well.

- Test items usually focus on one concept or skill, whereas problems may combine concepts nonroutinely. Problem 2 requires knowledge about properties of a cube as well as of numeration.

The reader may not agree with all of these ideas. In fact, it could be beneficial to do the problems and test items with a colleague and discuss interpretations. Through collaboration, a bank of suitable problems for evaluation could be developed.

The intention of this article was to stimulate reflection on the purpose of evaluation in mathematics education. There are no absolute responses to this concern, but none that has been given should be taken for granted.

Problem Solving: Teacher Profile

Gary Flewelling

Mr. Flewelling is a mathematics consultant for the Wellington County Board of Education, Guelph, Ontario. *The following pretested document was presented and discussed by Mr. Flewelling at the NCTM Canadian Conference held in Edmonton, October 16-18, 1986.*

Your response to the 20 statements below will give you an indication of the degree to which you, the teacher, bring problem solving into your classroom.

		(Running) TOTAL
. "Learning to solve problems is the principal reason for studying mathematics."	disagree 0 1 2 3 4 5 agree	
. "I teach my students general problem-solving strategies."	never 0 1 2 3 4 5 routinely	
. "I allow my students to work on problems in small groups."	never 0 1 2 3 4 5 routinely	
. "I introduce math topics through problems."	never 0 1 2 3 4 5 routinely	
. "I encourage students to write out their solution 'plan'."	never 0 1 2 3 4 5 routinely	
. "I permit the use of calculators in problem-solving activities."	never 0 1 2 3 4 5 routinely	
. "I stress the process of solving a problem rather than getting the answer to a problem."	never 0 1 2 3 4 5 routinely	
. "I assign problems that have extraneous or missing information."	never 0 1 2 3 4 5 routinely	
. "I assign ambiguous problems or problems that require an assumption to be made before a solution can be found."	never 0 1 2 3 4 5 routinely	

• "I assign problems that permit more than one method of solution."	never 0 1 2 3 4 5 routinely	☐
• "I assign problems that have more than one (or no) solution."	never 0 1 2 3 4 5 routinely	☐
• "The solving of a problem usually leads to my, or my students, asking, 'What if...?'"	never 0 1 2 3 4 5 routinely	☐
• "I have my students practise 'rule selection'."	never 0 1 2 3 4 5 routinely	☐
• "I don't have enough time to teach the course and teach problem solving too."	agree 0 1 2 3 4 5 disagree	☐
• "My students solve problems of their own creation."	never 0 1 2 3 4 5 routinely	☐
• "Problem solving represents a significant part of my students' marks."	disagree 0 1 2 3 4 5 agree	☐
• "I integrate problem solving throughout my courses."	never 0 1 2 3 4 5 routinely	☐
• "Problem solving is for every-one."	disagree 0 1 2 3 4 5 agree	☐
• "I enjoy solving problems with my students."	disagree 0 1 2 3 4 5 agree	☐
• "My students enjoy problem solving."	disagree 0 1 2 3 4 5 agree	☐

PROBLEM SOLVING SCALE

INEFFECTIVE					HEALTHY					EXEMPLARY
0	10	20	30	40	50	60	70	80	90	100

Teaching Problem Solving Using Core Topics in Mathematics

Marshall P. Bye and Robert Midyette

Marshall Bye, a former MCATA president, is an author of mathematics texts for Holt, Rinehart, and Winston. Robert Midyette has taught junior and senior high school and has served as a consultant with the Calgary Math Team. Presently, he is on leave of absence and is developing materials for Holt, Rinehart, and Winston. The following paper was presented at the NCTM Canadian Conference held in Edmonton, October 16-18, 1986.

An old proverb has been paraphrased: "Teach people a fact and they will know that fact; teach people to think and they can learn all the facts they'll ever need and be able to use them."*

Today, teachers are challenged to teach students how to be critical and creative thinkers. Mathematics programs across the nation are being revamped to provide greater dimensions to achieve this goal. One such dimension is problem solving. However, concerns are being voiced about making problem solving a mandatory part of the math curriculum: "How can problem solving be worked into a program that is already filled, and which already takes up all the time available?" "How can we get students to use the recommended problem-solving method?"

The authors contend that these two issues can be addressed by incorporating the four-step problem-solving model into the teaching of core topics in mathematics. Four distinct reasons for using this approach while developing the core concepts are outlined explicitly:

1. The four-step model can be developed without requiring additional time or cutting of topics.
2. It reassures students that the process is a bona fide one that should be used regularly.
3. It reassures students that the process is an essential part of mathematics.
4. It helps to ensure that the model becomes a natural part of the students' operational mode.

Just how the four-step model can be used while developing the core of the mathematics program will be illustrated in the balance of this paper. While the method can be used in many ways and many places within the mainstream of the program, examples drawn from the following three aspects of the core program are provided:

1. reviewing previous topics,
2. developing new topics, and
3. solving traditional word problems and applying solutions.

The examples have been written to illustrate some of the clues that teachers might provide to encourage a class to start with what they already know and proceed to build new concepts and skills. The sample questions are generally open-ended and are models of what the students might ask themselves when working alone. Since it is desirable that the students choose the

*John D. Long, Administrator of Special Training for General Motors of Canada.

direction to take in solving a problem, it is recommended that, after introducing the model, teachers resist the temptation to provide too much structure and direction. Students should and can be creative in choosing how, and in which direction, to proceed. This will require teachers to be creative in structuring the questions and directions they give students.

Reviewing Previous Topics

OBJECTIVE:
Calculate the surface area of cylinders.

PROBLEM:
Calculate the surface area of a cylinder with a radius of 14 cm and a height of 23 cm.

Understanding the Problem

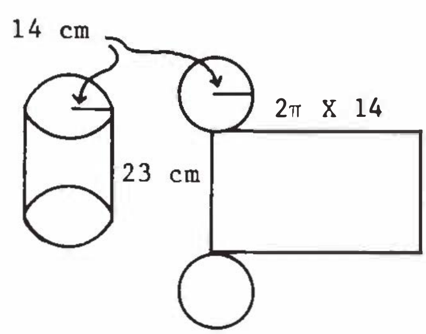
Can I simulate the situation to help me understand the problem?
Will making a drawing help me to understand the problem?

Developing a Plan

Can I simulate the problem?
Can I draw a diagram?
Can I partition the problem and solve each part?

Carrying Out the Plan

Where should I start in carrying out the plan?
Use a tomato juice tin.
Make a cylinder and two circles from paper.
Draw a diagram.



How can I break the problem into **simpler** parts?
For which three regions should I find the area?

How can I find the area of the two circles and the rectangle?
The area of the two circles and the rectangle is
$2\pi(14)^2 + 2\pi(14)(23)$
or 3255 cm^2.

Looking Back

What is the answer to the question?
The surface area is 3255 cm^2.

Can I generalize the solution?
In words:
The surface area of the cylinder is the sum of the areas of the two ends and the lateral surface **area.**
By formula:
$S.A. = 2\pi r^2 + 2\ rh$

Can I simplify?
$S.A. = 2\pi r(r + h)$

Can I make and solve a similar or related problem?
Find the surface area of
- rectangular solids
- pyramid.

Developing New Topics

OBJECTIVE:
Derive and apply: $S_n = n(a_1 + a_n)/2$

PROBLEM:
Find the sum of the arithmetic **series** 1 + 4 + 7 + . . . to 100 terms.

Understanding the Problem

Do I know the meaning of the key words sum, arithmetic series, and terms?
Is there sufficient information to solve the problem?
What information do I know already that would help me with this problem?
Is there a pattern to the problem?
Is this problem like any that I have seen before?

Developing a Plan

Can I solve a simpler problem by using smaller numbers?
Can I apply the discovered patterns?

Carrying Out the Plan

Can I solve a simpler problem?

(a) 1 + 4 + 7 = ?
What is the average of the first and last term?

(b) 1 + 4 + 7 + 10 = ?
What is the average of the first and last term? What is the average of the second and second last term? How many averages are there?

(c) 1 + 4 + 7 + 10 + 13 = ?
What is the average of the first and last term? What is the average of the second and second last term? How many averages are there?

(d) Repeat part (c) as necessary for 6, 7 . . . terms until students see the pattern of the first and last terms divided by 2 gives the average and there are n of these averages.

What would I get by applying the pattern?
The sum of five terms is found by taking the product of the average of the first and last terms and 5.

Can I apply this to the original question?
The sum of 100 terms is found by taking the product of the average of the first and last terms and 100.

Do I have enough information to find this product?
I do not know the last term.

How can I find that term?
I can apply the formula I used when I studied arithmetic sequences yesterday, $a_n = a_1 + (n - 1)d$. The 100th term is 1 + (99)3 or 298.

Can I find the sum of 100 terms now?
Yes, it is the product of the average of the first term (1) and the last term (298) and 100, which is 14 950.

Looking Back

What is the answer to the question?
The sum of 100 terms of the series is 14 950.

Can I generalize the answer?
$S_n = n(a_1 + a_n)/2$

What other methods are there of doing the problem?

(a) Find the sum of the series and the reverse of the series; that is,

$$\begin{array}{r} 1 + 4 + 7 + \ldots + 295 + 298 \\ 298 + 295 + 293 + \ldots + 4 + 7 \\ \hline 299 + 299 + \ldots \quad + 299 \end{array}$$

(b) Use symbols for the terms; that is, $a_1 + d\ldots$, $a_1 + (n - 1)d$, and find the sum in the manner in (a) above.

Solving Traditional Word Problems and Applying Solutions

OBJECTIVE:
To solve word problems.

PROBLEM:
Sara is 4a older than her brother Hanif. The sum of their ages is 7/10 that the their mother's age. In 12a time, the sum of their ages will equal their mother's age. Find Sara's, Hanif's, and their mother's present age.

Understanding the Problem

Have I read and reread the problem carefully?
What am I being asked?

Is there a pattern to the problem that will help me to understand it?
Do I understand the relationships in the problem; that is, who is older, how the mother's age compares to the children's, and the comparison of ages in 12a?

Developing a Plan

Can I see the relationships in the problem by using a chart?
Can I formulate equations to show the relationships?
Can I make an intelligent guess, refine, and recheck?

Carrying Out the Plan

Where do I start in carrying out the plan?

A relationship between the ages of the children and the age of their mother is given both at the present time and in 12a. Summarize the information in chart form.

	Hanif	Sara	Mother
Present Ages	x	$x+4$	$\frac{10(x+x+4)}{7}$
Ages in 12a	$x+12$	$x+16$	$\frac{10(2x+4)}{7}+12$

How do I use the chart to consolidate the information?

Write an equation using the ages in 12a and solve the equation for x, which will give us Hanif's present age.

$$(x+12)+(x+16) = 10/7(2x+4)+12$$

$$2x+28 = \frac{20}{7}x+\frac{40}{7}+12$$

$$14x+196 = 20x+40+84$$

$$72 = 6x$$

$$x = 12$$

Looking Back

Have I answered the question asked in the problem?

Since x represents Hanif's present age, Hanif is 12, Sara is 16, and their mother is 40.

Is the answer reasonable?

It is reasonable for the children to be 12 and 16 when the mother is 40.

Have I checked the answer?

At present, the sum of Hanif's and Sara's ages is 28, which is 7/10 of their mother's age. In 12a, Hanif will be 24, Sara will be 28, and their mother will be 52, which is the sum of Sara's and Hanif's ages.

Can I solve the problem in another way?

I could make a reasonable guess, check, and refine the answer.

Can I make and solve a similar or related problem?

For any similar age problem, I could make a chart showing the relationships of the ages, and then write and solve the equations.

Can I solve similar problems such as distance, money, or mixture problems?

By setting up a chart to summarize the problem and then writing equations to show the relationships in the chart, any problem of this type can be solved.

In Conclusion

Topics in which the four-step model is used should be selected carefully. Each example used should have a distinct purpose, such as to illustrate or to practise a particular strategy, to illustrate how the model can be used for a particular purpose, or simply to illustrate its usefulness in solving a problem.

It is not recommended that this method be the only one used in the classroom, but rather that it be added to the variety of developmental teaching strategies already employed to meet the needs of all students.

With appropriate use of the four-step problem-solving method, teachers can not only cover the topics prescribed in the program, but can also achieve the objective of getting students to be critical and creative thinkers so they, too, can "learn all the facts they'll ever need and be able to use them."

WHOLE NUMBER CHALLENGE

Addition, Subtraction, Multiplication, Division, and Order of Operations

PREPARATION

Prepare fraction cards: two each of 0/12 - 12/12, 0/6 - 6/6, 0/4 - 4/4, 0/3 - 3/3, 0/2 - 2/2 (64 cards). One die or spinner.

HOW TO PLAY
(2 to 6 players)

Each player is dealt five cards. One player rolls the die. All players may use two or more of their cards and any of the operations of addition, subtraction, multiplication, or division to write a number sentence that equals the number rolled on the die.

EXAMPLE: cards 7/12, 3/4, 1/6, 1/2, 1/12
die rolled 1

$$\frac{1}{6} \div [\frac{3}{4} - (\frac{7}{12} + \frac{1}{12})] \times \frac{1}{2}$$

SCORE:
1 point for every card used
1 point for each different operation used
3 bonus points for using all five cards
3 bonus points for using all four operations

Contributed by Karen Gibling, Elboya Elementary-Junior High School, Calgary.

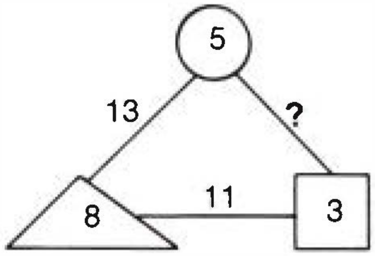
Problem-Solving Relationships in Algebra

John B. Percevault

John Percevault is associated with the Faculty of Education at the University of Lethbridge. Although he is retired, his interest in problem solving continues.

Many problems prepared for upper-elementary grades may also be used as algebra readiness problems for junior high school students.

One problem that may be used in this manner is illustrated.

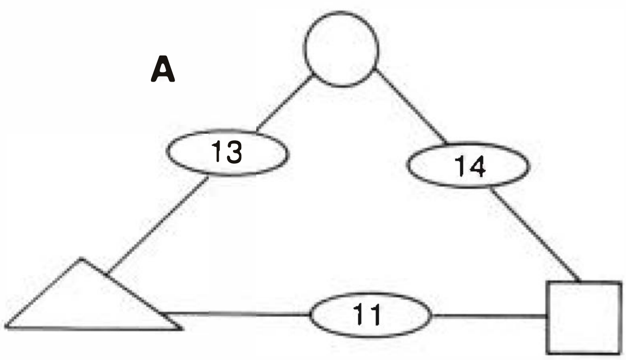


The problem solver did not complete the puzzle. How many relationships can you find? What number should replace the question mark? Why? What relationship did you use?

Set of Problems

The relationship that was found among the numbers in the sample problem applies in the following problems. Given the sums, the solver is asked to find the addends.

A



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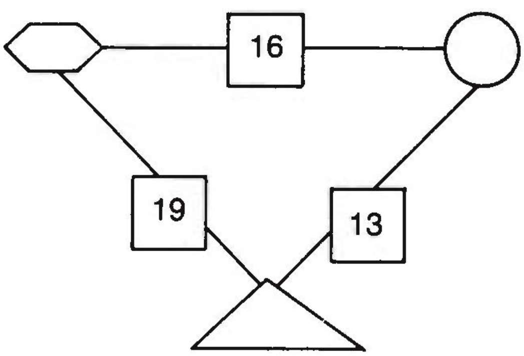
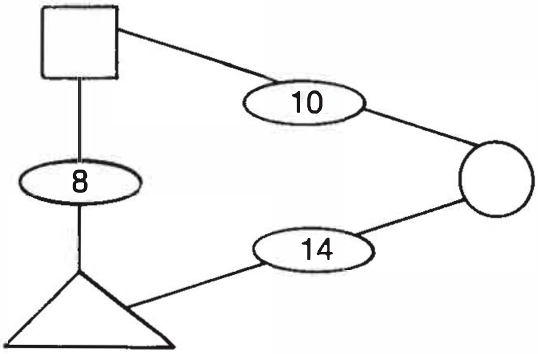
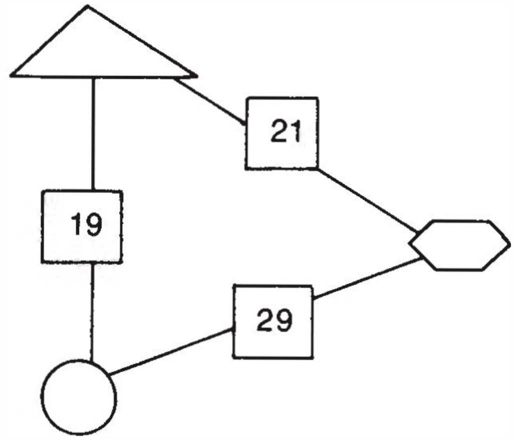


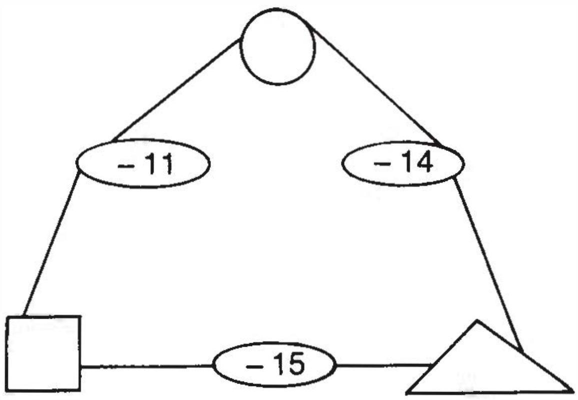
C



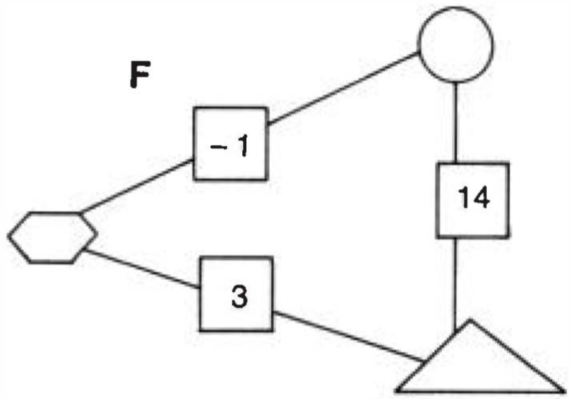
D



E



F

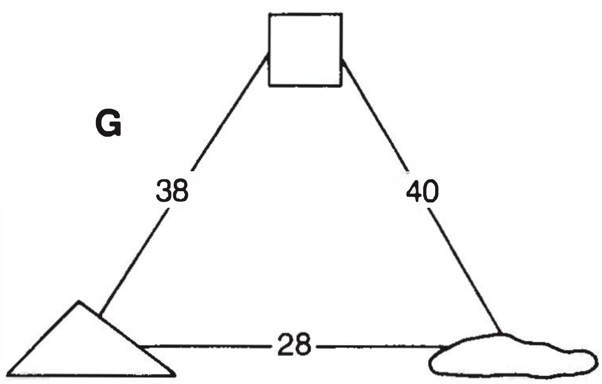


An analysis of students' approaches to solving the problems reveals that the initial strategy is guess-check-refine. Others may use an organized list of addends for each given sum. Others seem to solve the problem intuitively. Questioning will reveal that this group has analyzed the problem and determined an algebraic relationship.

Exploring Algebraic Relationships

A RELATIONSHIP

G



After students have completed the sample problems and others developed by you and them, ask them to compare the sum of the sums to the sum of the addends. Encourage the students to generalize. The relationship that should be evident is that the sum of the sum is twice the sum of the addends.

Consider the problem and express the addition facts with the symbols provided.

□ + △ = 38

⬭ + △ = 28

□ + ⬭ = 40

---

2 □ + 2 □ + 2 ⬭ = 106 OR

2 (□ + △ + ⬭) = 106

Determine that □ + △ + ⬭ = 53

But you also know that □ + △ = 38

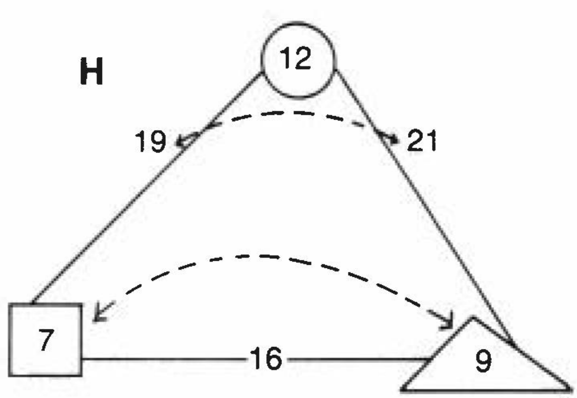
Do you know what ⬭ is?

Substitute value for ⬭ into the two equations in which ⬭ is an addend to find the value of △ and □ .

Verify in the equation □ + △ = 38.

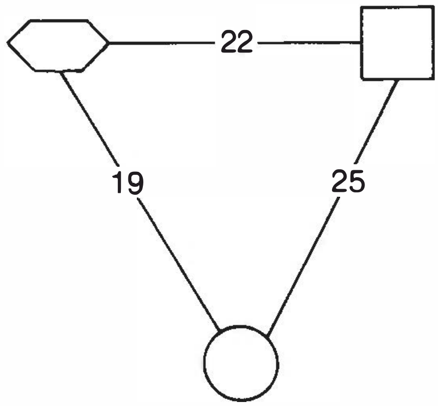
Another Relationship

Encourage students to reexamine the solved problems to discover other relationships that exist. Consider the following solved problem.



The dotted arrows suggest that the reader focus on the relationship between two sums and two addends. A difference of "2" is constant. Explore other sets of two addends and two sums. Is there a constant difference?

I



○ + □ = 25

○ + ⬡ = 19

□ − ⬡ = 6

SUBTRACT AND OBTAIN □ = ⬡ + 6

Now the value of ⬡ and □ can be determined by substituting into

⬡ + □ = 22.

Students will formulate the generalization that the difference between the sums is the difference between the two related addends. To return to the sample problem, the students can generalize that 22 is the sum of a number and that number increased by 6, or $n + (n + 6) = 22$.

Exploration

Refer to pr›blem H.

⬡ + □ = 22

⬡ + ○ = 19

Add 2⬡ + □ + ○ = 41

But we know that □ + ○ = 25

Can we find the value of ⬡ ?

Is the problem solve1? Can the student verify by substitution? Finally, almost, take the two equations:

⬡ + □ = 22

and

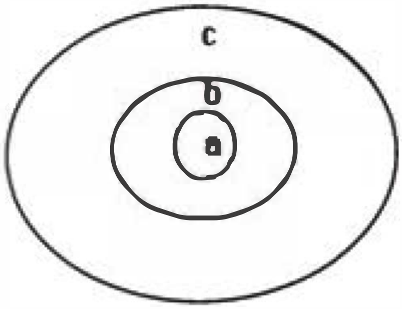
⬡ + ○ = 19

Subtract to obtain □ − ○ = 3. Use the equation, □ + ○ = 25 and substitute.

Use any other process or any conclusions.

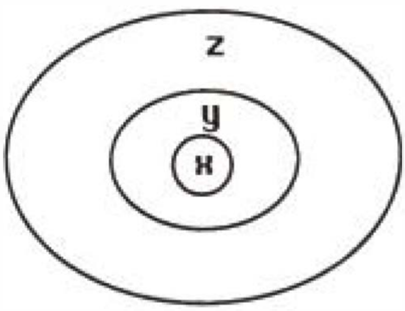
A Final Problem

Two little space travellers escape from the planet Rood. Skol really escapes. Rood is allowed to use the planet's name. Anyway, Skol and Rood land on Earth and are immediately lured to an arcade. Skol is attracted to an electronic deal game (EDG) whose score board is illustrated.



After a few random shots with his space gun, Skol realizes that his score is recorded after two shots. He determines that when he hits a and b, the computer records a score of 15. A hit on a and c is recorded by a score of 13, while hits on b and c show a score of 8.

Meanwhile, Rood is attracted to a different EDG whose score board is illustrated. After a few experimental shots, Rood determines that the same rule applies. Rood determines that:



$x + z = 15$
$x + y = 7$
$z + y = y + z = 2$

Post Mortem

Well! What do you do now? It's up to you to determine the questions you could pose. Change the rule? Change the variable? Others?

Intersecting Lines: Problem Solving in Geometry and Algebra

Mary-Jo Maas

Mary-Jo Maas teaches at G.R. Davis School in Fort Macleod and continues to serve as secretary for the MCATA.

The 1980s have been, and continue to be, the decade in which problem solving has come to the forefront. One has only to look at a convention program to realize the impact that problem solving has had on the field of mathematics. As teachers, you have been overwhelmed with a blizzard of problem-solving sessions, workshops, articles, and books. The probability of there being a mathematics teacher who hasn't been exposed to this storm is slim, indeed.

As the 1980s draw to a close, we are faced not with the task of exposing teachers to more problem-solving exercises, but with that of helping them to use problem-solving more effectively as an integral part of their lessons. All teachers have a nice set of exercises that they use to teach problem solving. Now teachers must take these exercises and extend, modify, and redesign them so that they complement the curriculum and can be used at a variety of grade levels. Problem solving was not meant to be a totally separate topic, but rather a strand of the curriculum that is to be found in all topics.

The following is a problem that can be extended to complement the regular curriculum:

PROBLEM:

> Seven line segments intersect a circular area to produce the maximal number of areas.

or

> Using only 7 straight cuts, what is the maximal number of pieces of pizza that can be obtained?

or

> Given a square pigpen and 7 straight fences, what is the maximal number of pigs that can be placed in this pen?

This problem is generally given to upper elementary or junior high students as an enrichment problem. It could be just as easily used in the geometry unit during a discussion of intersecting lines and could also be used during a study of number patterns.

The beauty of this problem is the diversity of ways in which it can be solved. Students can create simpler problems to solve it. They can also make toothpick models, draw pictures, create a chart, organize their data, look for patterns, hypothesize, or use a combination of any of these. All of these methods are problem-solving skills and strategies.

A possible solution employing several strategies might look like this:

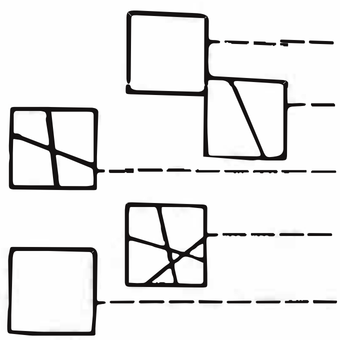


Diagram	# of Lines	Maximal Areas
	0	1
	1	2
	2	4
	3	7
	4	?

From a chart such as this, students are encouraged to explore the number patterns they see.

Intersecting Lines	Maximal Areas	Number Pattern
0	1	1
1	2	1+1
2	4	1+1+2
3	7	1+1+2+3
4	11	1+1+2+3+4
?	?	?

Students might then hypothesize how many spaces would exist for 10 lines, or 12 lines, or n lines.

This problem, however, is not just for the junior high student. At the senior high level, this problem could be presented as an application question in the study of arithmetic sequences and series. The students would use all of the previously mentioned problem-solving skills and strategies, as well as formulas such as the sum of an arithmetic series. The students would discover that given n lines, the maximum number of areas obtained would be $1 + S_n$.

I hope that, after reading this article, you have become more aware of the possibility of using problem-solving exercises as a means to explore and apply new mathematical concepts and topics. It is important that we look at our list of problem-solving exercises not as extras to use if time permits, but as teaching aids in our regular lesson plans.

These exercises should also be reviewed to see if they can be expanded and used at a variety of grade levels instead of earmarked for a particular grade. It is time that the blizzard of problem-solving exercises, workshops, and articles be turned into useful tools to truly capture the spirit of NCTM's Agenda for Action.

STUDENT PROBLEM CORNER

Students are encouraged to examine the problems presented below. Send explanations or solutions to:

The Editor
delta-K
c/o 2510 - 22 Avenue S
Lethbridge, Alberta
T1K 1J5

delta-K will publish the names of students who successfully solve the problems.

Final Clearance

Kevin J. Sherratt

Kevin Sherratt is a free-lance writer from London, Ontario.

Near closing time on the last day of a sporting goods liquidation, only $800 worth of equipment was yet to be sold: 1 canoe @ $160, 3 tents @ $80, 5 sleeping bags @ $40, 6 camp stoves @ $20, and 8 bush knives @ $10.

The next five customers each spent $160, clearing out the last pieces of equipment. From the given clues, find the items that each customer bought.

CLUES:

- Brad picked up neither bush knives nor tents.
- Doris bought at least one piece of four different kinds of equipment.
- Andy and Brad each bought 5 items.
- Carla bought at least one knife.

Tutors

Karen M. Gibling

Karen Gibling teaches at Elboya Elementary-Junior High School in Calgary.

From the following clues, determine on which day of the week each student tutors.

CLUES:

- Jane tutors later in the week than Tony does.
- Jane's day is earlier in the week than Bob's day.
- Frank will tutor on a day that is later in the week than Cathy's.
- Frank will tutor earlier than will Bob.
- Frank, Cathy, and Bob will not tutor on Monday.
- Frank cannot tutor on Thursday.

FRACTION RUMMY

Equivalent Fractions and Consecutive Fractions

PREPARATION

Prepare fraction cards: two each of 0/12 - 12/12, 0/6 - 6/6, 0/4 - 4/4, 0/3 - 3/3, and 0/2 - 2/2 (64 cards). Place each set of cards in an envelope.

HOW TO PLAY
(2 to 4 players)

Each player is dealt seven cards (held so that other players cannot see them).

One card is turned face up to form the beginning of a discard stack. The remaining cards are placed face down in the playing stack.

The player with the largest denominator starts the game. If there is a tie, the player with both the largest denominator and the largest numerator starts.

The player takes one card from the playing stack or the complete discard stack, and may play as follows:

- three or more equivalent fractions may be laid down;
- three or more consecutive fractions may be laid down (remember, the player may convert to twelfths to get consecutive fractions);
- one or more cards may be placed on any of the other cards the player has laid down.

A player ends each turn by placing a card face up on the discard stack.

The first player to lay down all of his/her cards ends the game. Each player receives one point for each card laid down and loses one point for each card remaining in his/her hand.

Contributed by Karen Gibling, Elboya Elementary-Junior High School, Calgary.

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