Escher Revisited: Modeling Gradual Deformations Using Logo

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Introduction

M.C. Escher was a graphic artist who had little interest in traditional mathematics. However, the complexity of his compositions has interested many mathematicians. Escher used repeating patterns extensively and integrated them into very interesting drawings. The study of these geometric patterns is called "Escher Mathematics."

Many of Escher's drawings are composed using figures called "tessellations." A tessellating shape is one that, when repeated across a plane, will cover the plane entirely without leaving any space. It takes very little experimentation to discover that, when a square is repeated along both axes of a plane, the plane will be covered entirely. Hexagons and triangles are also tessellating shapes.

Escher had the ability to take basic tessellating shapes and modify them according to certain rules so that they become artistically fascinating, yet retain their properties of tessellation. Ranucci and Teeters (1977) explain Escher's rules in detail.

Logo plays an important role in the modeling of Escher-type drawings. Because Logo makes is possible to draw lines using few commands, it is a convenient tool in formulating computer simulations of Escher's work. Perhaps one of the greatest advantages of using Logo to model tessellations is that the mathematical component of the drawing becomes apparent while constructing the simulated drawing. Figure 1 shows three gradual deformations, adapted from Hofstadter 1983.

Rationale

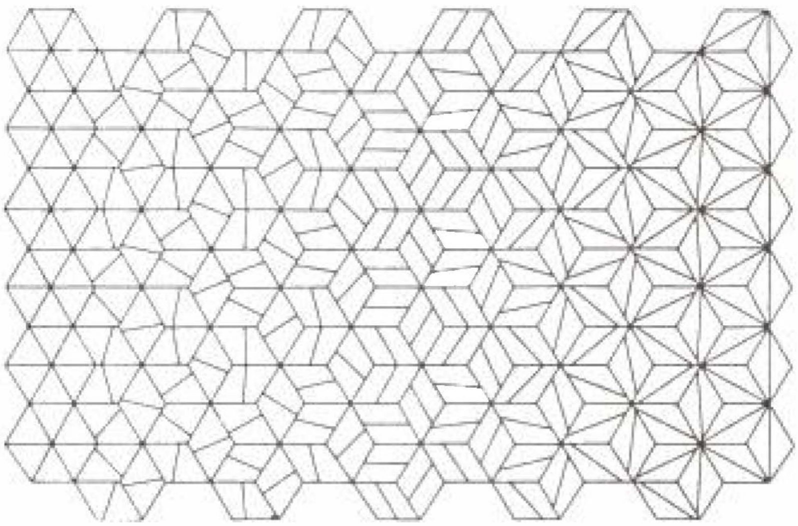
I undertook this research to explore some of the mathematics of gradual deformations in a simple form. Gradual deformations were implemented by Escher in many of his line drawings. I felt that the interesting mathematics of Escher-type drawings would make examining the principles involved an enlightening experience. Additionally, I hoped that the topic might provide enrichment material for advanced or gifted high school students. Having encountered students who are not challenged by the standard curriculum, I am always looking for interesting and demanding activities. As a result, one of my objectives was to evaluate the topic of gradual deformation to see if it could be implemented.

The project was unique; whereas tessellations have received considerable attention since the advent of Logo, gradual deformations have been largely ignored. Therefore, the project provided a takeoff point for studying Escher Mathematics further.

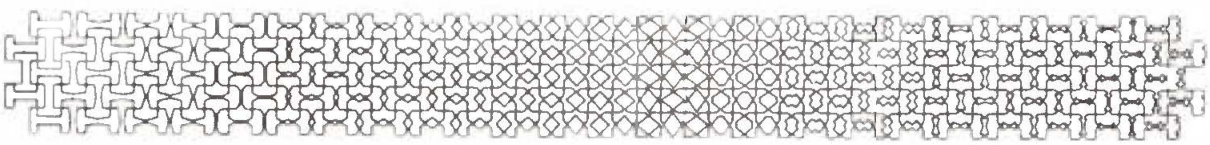
The Problem

The project was to take a square and deform it by having the midpoints of the top and right sides migrate directly away from the centre of the square. The mathematics of this deformation is easier to understand if the problem is broken down and the top side looked at first. The same principles can then be applied to the other axis.

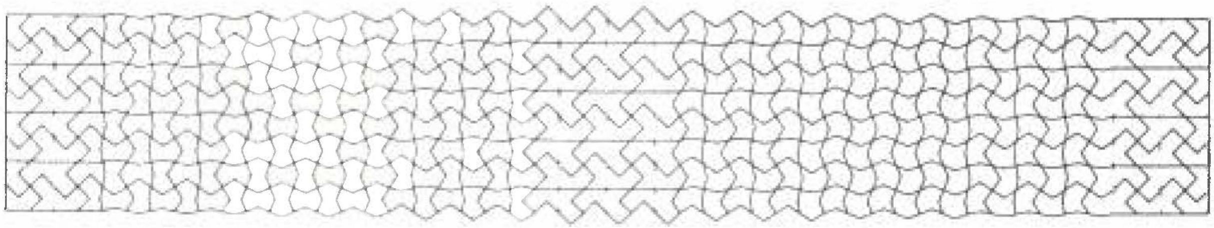
Figure 1. Three Gradual Deformations



Consternation



Razor Blades

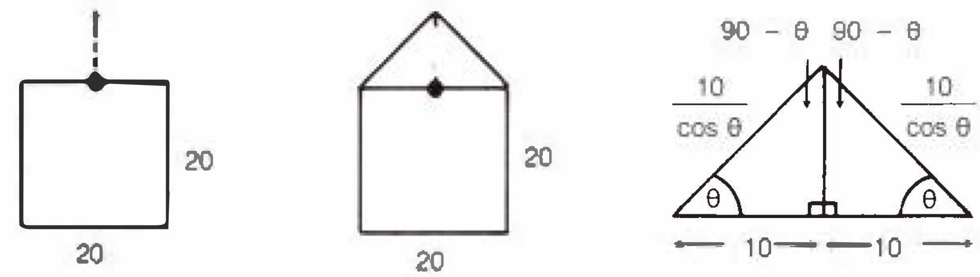


Parquet Deformation Titled Fylfot Flipflop

The result is the "addition" of a triangle on top of the square (Figure 2). Interesting relationships are established if this triangle is divided into two equal parts.

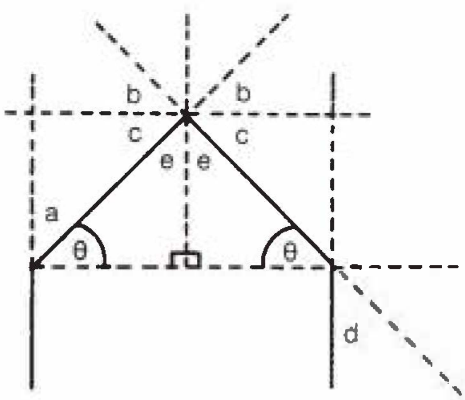
There are two important points on the path that the turtle must take in tracing the deformed shape. The angle the turtle turns at each corner and the distance it travels between corners must be found. Both values can be calculated with trigonometry (Figure 2). Turning the turtle right at the upper left corner of the square at an angle of $90 - \theta$ will point the turtle along the "right" edge of the triangle.

Figure 2. Gradual Deformation of the Top of a Square



The turtle will then travel 10/cos θ units in that direction until it reaches the next corner, which is the apex of the triangle.

Figure 3. Similar Triangles Arising from the Deformation



At this point, a construction of several similar triangles can be used to determine how much the turtle must turn. Figure 3 illustrates the relationships of several triangles. The measure of angle e is 90 − θ, since θ + e = 90. Applying the principles of similar triangles establishes that angles a and e are equal. As well, angle b has the same measure as θ. Angles b and c are vertically opposite, so they must be equal. Finally, angle d can also be proven to be 90 − θ. The result is as follows:

angle a = 90 − θ
angle b = θ
angle c = θ
angle d = 90 − θ
angle e = 90 − θ

Now, it is relatively simple to determine that, at the apex of the triangle, the turtle must turn right at an angle of 2 × θ, move a distance of 10/cos θ and turn right 90 − θ. Turtle will then be pointing straight down, ready to draw the next side.

The Solution

Programming the solution in Logo entailed a minimum of "dirty work." Figure 4 lists the completed program.

To begin with, an arbitrary square size of 20 by 20 units was chosen, as well as an arbitrary grid size of 8 squares by 10 squares. Three short routines were then written to draw a grid, starting at the lower left corner. FILLTILES fills the screen with 10 strips of squares (Figure 4), and STRIPTILES is the procedure that draws each strip. SQUARE is a procedure that draws out each individual distorted square. The program, when executed, draws an undistorted square in the lower left corner of the screen and gradually deforms the shape until the upper right square is drawn. This corner would be the most deformed in both dimensions. The result of running the program with the command FILLTILES 8 6 can be seen in Figure 5.

In programming the solution, I introduced a number of variables and counters to keep track of the angles and increments required for the gradual deformation. XDEG and YDEG are representations of

Figure 4. The Procedures FILLTILES, STRIPTILES and SQUARE

```
TO FILLTILES :XINC :YINC
CS
PU
SETPOS [-100 -60]
PD
MAKE "XDEG 0
REPEAT 10 [STRIPTILES :XDEG :YINC RT 90 FD 20 LT 90 MAKE "XDEG :XDEG + :XINC]
END

TO STRIPTILES :XDEG :YINC
MAKE "YDEG 0
REPEAT 8 [SQUARE :XDEG FD 20 MAKE "YDEG :YDEG + :YINC]
BK 160
END

TO SQUARE :XDEG :YDEG
FD 20
RT 90 - :YDEG
FD 10 / ( COS :YDEG )
RT 2 * :YDEG
FD 10 / ( COS :YDEG )
RT 90 - :YDEG - :XDEG
FD 10 / ( COS :XDEG )
RT 2 * :XDEG
FD 10 / ( COS :XDEG )
RT 90 - :XDEG
FD 20
RT 90
END
```

θ (Figure 2), one for each of the X and Y axes. XINC and YINC are variables that hold the increment of θ for the gradual deformation along each axis.

Implications

The most striking observation was that Logo programming language played a very minor role in the project. I paid much more attention to solving the actual mathematical problem, a result that pleased me because I had intended to concentrate on mathematics rather than on computer programming.

I was somewhat disturbed that the relationship between this type of gradual deformation and trigonometry was not immediately apparent. I did not realize the connection until I had examined the problem more closely. (Perhaps this indicates a general lack of mathematical awareness in society.) Once discovered, the connection is obvious, and I wonder how long it would take someone with a weaker background to recognize the relationship.

Before, during and after the investigation, I considered implementing this and similar problems in a class. The problem would make an interesting and challenging enrichment exercise, implemented in the framework of the mathematics curriculum. Such an activity would do equally well as a culminating activity, to wrap up a unit on trigonometry and similar triangles. Whether this type of activity can be used for direct instruction needs to be examined.

rather than midpoints, are chosen? Also, what happens when the direction of the migration is altered? When the point migrates at X degrees to the right (or left) of perpendicular? And what would happen if both ideas were combined?

The ultimate investigation of this problem would involve three dimensions and repeating all of the above in the context of X, Y and Z axes. A cube would replace the square, and new variables would be introduced. Concepts such as stellate polyhedra could be examined, as well as many other topics. I can't imagine where this might lead.

There exists a vast unexplored area of mathematics. Rich rewards await those prepared to press forward and push their personal limits to the edge of what we now understand. See you there!

References

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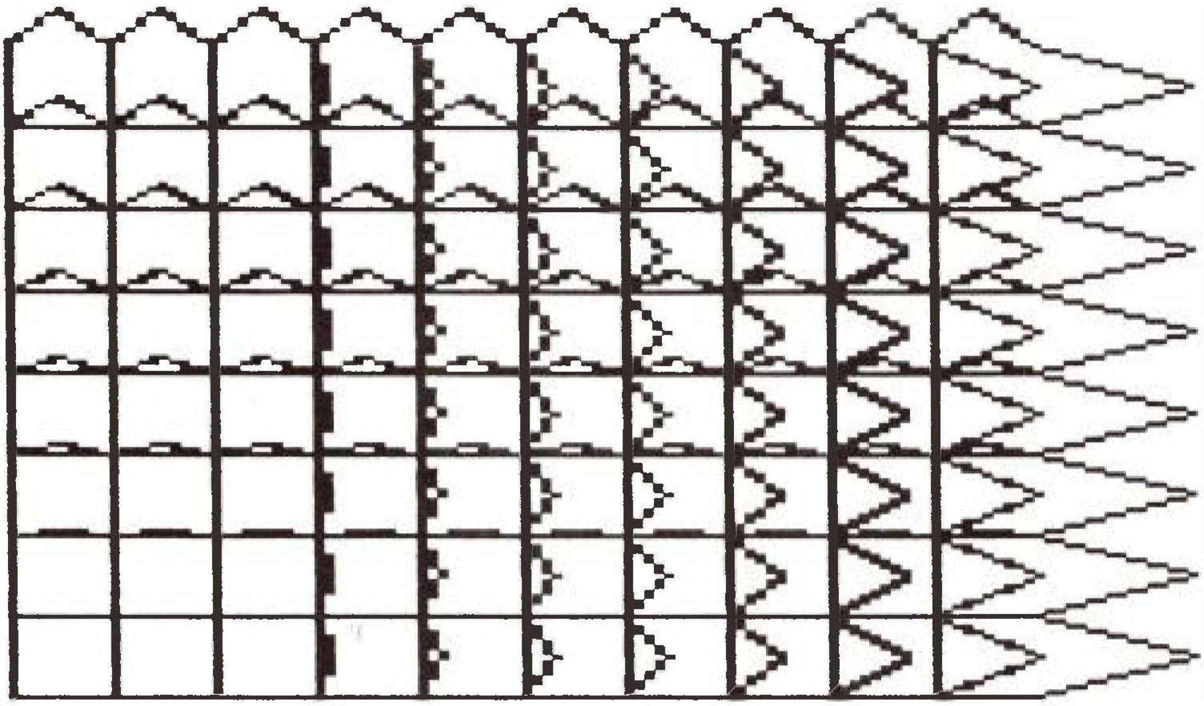
Figure 5. The Result of FILLTILES 8 6



A number of questions come to mind. For example, wouldn't it be interesting to begin with a square in the middle of the screen and to deform all four sides toward the perimeter of the screen? Or to examine the limits of deformation? How much can the square be distorted? What happens when negative numbers are used? Are there values that will not work? Why don't they work?

What happens when one performs similar deformations of figures such as triangles, pentagons and hexagons? Is there a difference in the behavior of tessellating and non-tessellating polygons? What special problems are encountered? Are there cases in which the deformation of one shape in a particular way gives rise to a new shape? Under what conditions does this occur? These and similar questions can be posed and conclusions drawn from the answers.

Other, advanced topics could include the deformation of irregular and curved shapes. The programming would involve a fair amount of dirty work, but the mathematics would indeed be interesting.

This project involved deformations in which the midpoint of a side migrates perpendicular to, and away from, the side on which it is located. It would be interesting to examine the behavior of deformations when a point other than the midpoint is chosen. Does something special happen when corners,