Summing Consecutive Counting Numbers

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Figure 1 contains indicated sums that have been generated by adding sets of consecutive counting numbers and then skipping sets of consecutive counting numbers.

Figure 1

1) 1 + 2
2) 4 + 5 + 6
3) 9 + 10 + 11 + 12
4) 16 + 17 + 18 + 19 + 20
5) 25 + 26 + 27 + 28 + 29 + 30
6) 36 + 37 + 38 + 39 + 40 + 41 + 42
7) 49 + 50 + 51 + 52 + 53 + 54 + 55 + 56

Observe that the first indicated sum contains the first two counting numbers. Then, one counting number is skipped. The second indicated sum contains three consecutive counting numbers, and then two are skipped. The third indicated sum contains four consecutive counting members, and then three are skipped.

The skipped numbers are shown in Figure 2.

Figure 2

1) 3
2) 7, 8
3) 13, 14, 15
4) 21, 22, 23, 24
5) 31, 32, 33, 34, 35
6) 43, 44, 45, 46, 47, 48
7) 57, 58, 59, 60, 61, 62, 63

Activity 1

Find the indicated sums of Figure 1. Then sum the skipped numbers as well. Now compare the results of your computations.

The sums of Figure 1 and Figure 2 are both 3, 15, 42, 90, 165, 273 and 420. Since the two sets of sums are equal, we may write:

$$
\begin{array}{rcl}
1 + 2 & = & 3 \\
4 + 5 + 6 & = & 7 + 8 \\
9 + 10 + 11 + 12 & = & 13 + 14 + 15 \\
16 + 17 + 18 + 19 + 20 & = & 21 + 22 + 23 + 24 \\
25 + 26 + 27 + 28 + 29 + 30 & = & 31 + 32 + 33 + 34 + 35 \\
36 + 37 + 38 + 39 + 40 + 41 + 42 & = & 43 + 44 + 45 + 46 + 47 + 48 \\
49 + 50 + 51 + 52 + 53 + 54 + 55 + 56 & = & 57 + 58 + 59 + 60 + 61 + 62 + 63 \\
 & \vdots &
\end{array}
$$

Observe that the first term of each row is a perfect square and the last term of each row is one less than the next perfect square. Also, the number of terms to the left of the equal sign is one more than the number of terms to the right.

Activity 2

Find the differences of the sums of Figure 1. After how many subtractions will a constant emerge?

3			
	> 12		
15		> 15	
	> 27		> 6
42		> 21	
	> 48		> 6
90		> 27	
	> 75		> 6
165		> 33	
	> 108		> 6
273		> 39	
	> 147		
420			
•			
•			
•			

Note that after three subtractions the constant 6 emerges.

Activity 3

Write the first seven squares and subtract them from the sums of Figure 1. Now find differences. Will a constant emerge?

```
  3 -  1 =   2
               >  9
 15 -  4 =  11        > 13
               > 22        > 6
 42 -  9 =  33        > 19
               > 41        > 6
 90 - 16 =  74        > 25
               > 66        > 6
165 - 25 = 140        > 31
               > 97        > 6
273 - 36 = 237        > 37
               > 134
420 - 49 = 371
```

The constant 6 appears after three subtractions.

Activity 4

This time write the first seven cubes and subtract them from the sums of Figure 1. Again find differences. Does a constant appear?

```
  3 -   1 =  2
              >  5
 15 -   8 =  7      > 3
              >  8
 42 -  27 = 15      > 3
              > 11
 90 -  64 = 26      > 3
              > 14
165 - 125 = 40      > 3
              > 17
273 - 216 = 57      > 3
              > 20
420 - 343 = 77
```

After only two subtractions, the constant 3 appears.

Activity 5

Now write the first seven fourth powers. Then from the fourth powers subtract the sums of Figure 1. Find the constant differences. How many subtractions are needed to find the constant?

2401	−	420	=	1981				
					> 958			
1296	−	273	=	1023		> 395		
					> 563		> 126	
625	−	165	=	460		> 269		> 24
					> 294		> 102	
256	−	90	=	166		> 167		> 24
					> 127		> 78	
81	−	42	=	39		> 89		> 24
					> 38		> 54	
16	−	15	=	1		> 35		
					> 3			
1	−	3	=	−2				

The constant 24 appears after four differences are computed.

Challenge

Consider summing sets of consecutive odd numbers or consecutive even numbers. What patterns can you generate?