Casino Gambling: The Best Strategy

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Though the old problem of "Gambler's ruin" has long been solved (see W. Feller's "Introduction to Probability Theory"), a new study of it offers a strategy for the best odds in gambling at a casino.

We enter a casino with \$n and agree to play (repeatedly) at some game (craps, roulette, 21, etc.) for \$1 a play initially—where the probability of winning \$1 on one play is p and the probability of losing \$1 on one play is q, $(p + q = 1)$—until we either end up with \$N or with \$0 (ruined!).

The question that must be answered is, "What is the probability of being ruined (in terms of n, N, p and q)?"

Let R_n denote the probability of being ruined when you have \$n (n = 0,1,2, . . ., N). Clearly, $R_0 = 1$, for if we have \$0, ruin is certain. $R_N = 0$, for if we have \$N, we leave the casino.

With \$n, we make a play and we could be ruined in two mutually exclusive ways: we could "win the play and then be ruined" *or* we could "lose the play and then be ruined." It follows that

$$R_n = pR_n + 1 + q\,R_n - 1$$

The solution to this equation (with the boundary conditions $R_0 = 1$ and $R_N = 0$) is

$$R_n = 1 - n/N \quad \text{if } p = q = \tfrac{1}{2}$$

$$= \frac{(q/p)^N - (q/p)^n}{(q/p)^N - 1} \quad \text{if } p \neq q.$$

While it is not easy to come up with these solutions, unless we are familiar with difference equations, it is a simple matter to verify that they are indeed solutions to the given problem.

The effect of halving the bet is to replace n and N by 2n and 2N so that the probability of ruin with bets halved is equal to:

$$\frac{(q/p)^{2N} - (q/p)^{2n}}{(q/p)^2 - 1} = R_n \frac{(q/p)^N + (q/p)^n}{(q/p)^N + 1}$$

If $q > p$, then $(q/p)^n > 1$ and the right hand side is larger than R_n.

That is, if $q > p$, as is the case in a casino, the probability of being ruined increases if we halve the bets. It is immediately clear that we should bet big to lessen our chances of ruin.

Note that the probability of ruin remains constant if $p = q$, since $R_n = 1 - 2n/2N$.

The following is a list of some examples of various bets:

Example F1. Enter Fairyland Casino (where $p = q = ½$ for all games) with $9,000 and agree to leave when you have $10,000 or when you are ruined.

Bets	n	N	P(Ruin)	Expected Gain	Expected Duration
$1	9,000	10,000	0.1	0	27 yrs @ 15 hrs/day
$10	900	1000	0.1	0	100 days @ 15 hrs/day
$100	90	100	0.1	0	15 hours
$1000	9	10	0.1	0	9 minutes (9 plays)

Example F2. Enter Fairyland Casino with $100 and agree to leave when you have $200 or when you are ruined.

Bets	n	N	P(Ruin)	Expected Gain	Expected Duration
$1	100	200	0.5	$0	11 days @ 15 hrs/day
$10	10	20	0.5	$0	100 minutes
$100	1	2	0.5	$0	1 minute (1 play)

Example F3. Enter Fairyland Casino with $1 and agree to leave when you have $100 or when you are ruined.

Bets	n	N	P(Ruin)	Expected Gain	Expected Duration
$1	1	100	0.99	$0	99 minutes

Summary for gambling in Fairyland Casinos:

1. The expected gain is zero for all games in Fairyland!
2. The size of the bet (the "action") has no effect on the probability of ruin or on the expected gain.
3. The closer n is to N, the lower the probability of ruin.
4. The expected (average) duration is inversely proportional to the square of the action. The expected duration is determined by solving another difference equation (in another paper). For further discussion of the Classical Ruin Problem, see Chapter 14 of W. Feller's *An Introduction to Probability Theory and Applications,* published by John Wiley and Sons.

The following is a list of examples of playing the game of craps in a real casino:

Example R1. Enter any casino with $9,000 and agree to leave when you have $10,000 or you are ruined. For the game of craps: $p = 0.4929\bar{2}9$.

Bets	n	N	P(ruin)	Expected Gain	Expected Duration
$1	9,000	10,000	$.5 \times 10^{10}$	−$9000	2 years @ 15 hours/day
$10	900	1,000	0.94089	−$8409	66 days @ 15 hours/day
$100	90	100	0.262	−$1620	19 hours
$200	45	50	0.174	−$740	4.37 hours
$500	18	20	0.127	−$270	38.6 minutes
$1000	9	10	0.113	−$130	9.3 minutes
Ultimate Strategy			0.1053	−$53	1.9 minutes
Fairyland			0.1000	$0	9 minutes

Example R2. Enter any casino with $100 and agree to leave when you have $200 or when you are ruined. Play the game of craps.

Bets	n	N	P(ruin)	Expected Gain	Expected Duration
$1	100	200	0.9442	−$88.62	104.70 hours
$10	10	20	0.5702	−$14.04	99.34 minutes
$50	2	4	0.5141	−$2.82	4 minutes
$100	1	2	0.507	−$1.42	1 minute

Example R3. Enter with $150 and agree to leave with eight $200 or $0.

Bets	n	N	P(ruin)	Expected Gain	Expected Duration
$1	150	200	0.7595	−$101.90	5 days @ 15 hours/day
$10	15	20	0.3053	−$11.06	78.14 minutes
$50	3	4	0.2607	−$2.50	3.03 minutes
Ultimate Strategy			0.25712	−$1.60	1.50707 minutes

Final Example R4. Enter a real casino with $1 and play on the craps table (''come line'') until you have $128 or you are ruined!

Action	n	N	P(ruin)	Expected Gain	Expected Duration
$1	1	128	0.9992	−89.90 cents	63.55 minutes
Ultimate Strategy			0.9929	−9.12 cents	2.17 minutes

The Ultimate Strategy

The Ultimate Strategy is simply to bet the maximum that your resources will allow, to achieve your stated goal.

For Example R1: Bet #1: $1000. If you win, you leave; if you lose then
bet #2: $2000. If you win, you leave; if you lose then
bet #3: $4000. If you win, you leave; if you lose then
bet #4: $2000. (All the cash you have left. A stiff upper lip is needed!)
If you lose, you leave; if you win then
bet #5: $4000. And so on.

For Example R2, the Ultimate Strategy is to simply bet the $100. You then either leave with $200 or leave ruined.

For Example R3, the Ultimate Strategy is simple again, comprising at most two bets. Bet #1: $50. If you win, you leave; if you lose, then bet #2: $100. You either leave with $200 or you leave ruined.

For Example R4, the Ultimate Strategy is to bet $1, then $2, then $4, then $8, then $16 and so on, as long as you keep winning, until you reach your preset goal of $128. You must win seven times running to avoid ruin.

Appendix A

The Game of CRAPS (Betting on the "Come Line")

Throw two dice; if the sum is 7 or 11, you win (even money); if the sum is 2, 3 or 12, you lose. If you shoot 4, 5, 6, 8, 9 or 10, this becomes your "mark" and you continue to throw the two dice until you shoot your "mark" and win (even money). If you shoot 7, you lose.

This is the very best (no decisions) game in any casino in that the probability of winning any one game is $p = 0.4929$; the probability of losing any one game is $q = 0.507$. This is as close to Fairyland odds ($p = q = 0.50$) as you can get for a "no-decisions game." Blackjack or 21 are "decisions games," and p can exceed 0.50 if you make the correct decisions (see E.O. Thorp's *Beat the Dealer*). The expected duration for a game of craps is 3.3757 throws.