Overcoming Conceptual Obstacles

Barry Onslow

Dr. Barry Onslow is an assistant professor of education at the University of Western Ontario. This paper was presented in 1986 at the NCTM Canadian Conference in Edmonton, Alberta. At the time, Barry was a member of the MCATA executive and vice-principal at Ralston DND School. He taught mathematics to students in Grades 6 to 9.

Introduction

Evidence now shows that on some fundamental concepts, children make little progress, even after a substantial amount of mathematics teaching (Hart 1981). For example, on the concept illustrated in the following problem, a study by Hart showed that students failed to develop greater understanding even after years of instruction.

Problem

Circle the one that gives the BIGGER answer:

a) 8 x 4 or 8 ÷ 4

b) 8 x 0.4 or 8 ÷ 0.4

c) 0.8 x 0.4 or 0.8 ÷ 0.4

Results

Response percentages by year

	12 yrs	13 yrs	14 yrs	15 yrs
x ÷ ÷	13	8	15	18
x x x	50	58	47	30

(Hart 1981, 54)

In this article I will examine some of the misconceptions that hinder students' mathematical progress and will propose an alternative teaching strategy that has been beneficial in assisting students to recognize and overcome these conceptual obstacles.

The Mathematics Curriculum

The mathematics curriculum contains three elements: skills, concepts and problem solving (Figure 1).

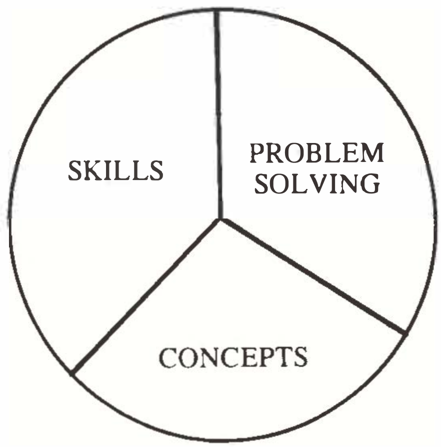

Figure 1
Components of the Mathematics Curriculum

This diagram can be treated as a topological chart insomuch as the three distinct components can be enlarged or diminished as appropriate. However, each of these components must be examined many times during the course of the year, and each requires different techniques for overcoming student difficulties.

A great deal of time and energy has recently been spent in demonstrating that it is the strategies underlying problem solving that are important, rather than the actual answers. Many valuable resources are currently available. However, little has been said about the changes needed in the teaching of skills and concepts. In the majority of classrooms, both are generally taught in a similar fashion, that is, exposition followed by practice. Ausubel, writing about mathematics teaching in the United States during the 1960s, commented that the style of teaching has progressed very little during this century:

> [Mathematics teaching] still relies heavily on rote learning of formulas and procedural steps, on recognition of stereotyped "type-problems" and on the manipulation of symbols.
> (Ausubel 1964, 241)

This might well be adequate for some aspects of skill development, but it does not appear to assist children in overcoming conceptual obstacles. Recent research has indicated that children's mathematical understanding is not what might have been expected, and their errors are generally neither random nor careless. Students have specific incorrect strategies for approaching problems. Moreover, these strategies are so deep-rooted that they often remain with the students throughout their schooling and later life (Hart 1981; Rees and Barr 1984). Other intuitive strategies, which students use correctly in simple situations, often prove to be inadequate for more complex questions (Bell, Swan and Taylor 1981).

The Student's Conceptual Understanding

Two misconceptions typically held by junior high school students are that multiplication always makes bigger and that in division, the larger number is always divided by the smaller one. These misconceptions appear to be trivial for junior high school students; thus, it is not surprising that many teachers are a little skeptical of the seriousness of the problems and of the difficulty their students might have in overcoming them. To determine which misconceptions your students have, present them with the questions in Appendix A. The majority of the students' misconceptions will correspond with the responses presented in Appendix B.

Having tested the students and analyzed the results, employ your usual teaching strategies to overcome these misconceptions and re-test approximately two weeks later. Ensure that students do not receive any feedback regarding their answers until after the retest. Do not be surprised if there is little or no change from the students' original responses.

Why are these misconceptions difficult to eradicate? The most probable reasons are that these strategies have been correct in the past and remain appropriate for many situations. Using the elementary model of repeated addition for multiplication, multiplication does indeed make bigger. Very few verbal division problems presented in textbooks, even at the junior high level, require students to divide the small numeral by the larger one. Also, depending on how the child models division, $10 \div \frac{1}{2}$ will often mean 10 shared in half (similar to $10 \div 2$ meaning 10 shared in/by two) rather than "How many halves in 10?"

The Diagnostic Teaching Strategy

To allow the development of superior strategies, children must be presented with situations that demand the use of such strategies. Misconceptions are like beliefs—only when it is obvious that faulty methods have been employed will students see the necessity for change. This is the premise upon which diagnostic teaching is based.

The key to this change of style is a situation in which students begin analyzing answers and discussing their correctness. The teacher no longer provides the students with an appropriate strategy by demonstrating the correct procedure for solving an example. Rather, the teacher provides a situation in which students attempt (either on their own or in groups) what they consider to be the correct strategies. The focal point of the lesson is the analysis of these answers. They are reviewed by the class, and a discussion takes place to examine the strategies involved in solving the problem and to decide whether or not specific strategies are valid. It is anticipated from Piagetian theory, that opposing strategies will arouse conflict, that the students will try to eradicate the conflict and that a meaningful learning situation will be established. Only when the limits of their primitive frameworks are demonstrated to them, do students see the necessity for change. Once this has been accomplished, they should be able to recognize that an inconsistency exists and that there is a need for reorganization and rethinking. This will require modifications to students' conceptual frameworks if they are to successfully handle the task.

Many children have a fear of failure, and this fear can often be reinforced either by adults or other children. Yet students must be willing to experiment and make mistakes if they are to accomplish worthwhile goals. To overcome this fear and enable diagnostic teaching to be effective, the climate in the classroom must be one of mutual respect. Children must value the opinions of others even though they might disagree with them. In Carl Rogers' words, the teacher has the opportunity to develop a climate in which "threat to the self is minimized, (and) the individual makes use of opportunities to learn in order to enhance himself" (1969, 162).

Under such conditions it is possible to establish an atmosphere of cooperative learning. Once pupils feel confident about expressing their ideas, whether correct or incorrect, they can develop a better appreciation of their thoughts and contributions. This might even be more relevant for girls, who tend to perform better than boys in language arts but worse in mathematics. Girls need equal opportunities to participate in discussions that clarify their ideas and understanding and to learn mathematics satisfactorily (Cockcroft 1982, 63).

Children's approaches to mathematics are important. Allardice and Ginsburg exemplify this with their description of two boys who came to their class (1983, 343). Whereas one child thought that reasoning about mathematics was a sensible activity and learned quite quickly, the other child considered that "going over and over it" was the only acceptable approach—he learned nothing. Allowing students to reflect on their answers and justify their conclusions, rather than simply seek an algorithm to generate the correct answer, will encourage a deeper awareness of the mathematical structure underlying a problem. It will also provide time for the teacher to reflect more carefully upon what the students *really* know, rather than what it is often *assumed* they know. Students' original, and sometimes creative, methods may often go unnoticed if they are never shared with others, especially when children are engaged in individualized teaching programs and mark their own work (Erlwanger 1973). Although, at times, the students' techniques may be correct, at other times they may be severely deficient. Under either condition, an airing of the students' views is important. It is the lack of discussion amongst older pupils, in favor of a more formal approach that often leads to further failure (Cockcroft 1982, 142).

Even the more able students memorize routines find it difficult to explain the processes they put into practice when solving problems. Understanding the structure underlying a problem is imperative if children are to develop their mathematical capabilities.

> Being a mathematician is no more definable as "knowing" a mathematical set of facts than being a poet is definable as "knowing" a set of linguistic facts. (Papert 1972, 249)

In the diagnostic teaching approach, each lesson is usually divided into three component parts: the opening activity, the conflict discussion and consolidation exercises. Each is described below.

The Opening Activity

The opening activity is designed to familiarize students with the problem and prepare the way for a conflict discussion by presenting material that will provoke errors.

A child adopts an incorrect strategy usually because it has been beneficial in an alternate situation, but its limits have not been recognized. For example, "multiplication makes bigger" is a perfectly adequate concept when dealing with the natural number system, but it becomes inadequate when rational numbers are introduced. Once children recognize that this approach is inappropriate, they may comprehend why it is essential to examine a new strategy.

When teaching in this manner, it is not desirable to shield the children from errors. Instead, children are encouraged to face mistakes in a positive manner so as to become more adept at discovering errors for themselves. With the teacher's help, they can make constructive use of the errors. In this regard, teachers have a most important role, both in explaining how errors can be utilized profitably and in changing students' attitudes so that the benefits of both reflecting on and discussing their work become obvious. This will be a different and difficult experience for many students who often desire only the correct answer. The advantages need to be discussed in detail with most classes before they are able to recognize the true worth of exploring mathematics in this way.

Providing the correct level of activity is also crucial since a question that is easily solved provides little challenge or need for exploration, whereas one that is too difficult merely arouses confusion. Cloutier and Goldschmid (1978, 138) recommend that if a question is to provoke discussion it should be difficult as well as interesting, but not so difficult as to be insurmountable by most students.

The Conflict Discussion

Once students have had an opportunity to discuss the opening activity either in pairs or small groups, they are brought together for a class discussion. Situations are purposely contrived so that students have conflicting views on the topic. This allows for beneficial interchange.

> A clash of convictions among children can readily cause an awareness of different points of view. Other children at similar cognitive levels can often help the child more than the adult can to move out of his egocentricity. (Kamii 1974, 200)

Interaction in this manner allows students to share their interpretations of a concept and permits the clarification of new ideas, provided that the question is within the limits of their conceptual frameworks. When this is not the case, discussion resorts to little more than the sharing of ignorance, prejudice, preconceptions and vague generalities. Little benefit will be gained from the latter situation, so it is important to define the attributes of a profitable discussion.

Students are expected to possess the background knowledge that enables them to examine problems in an informed and intelligible manner. Problems are designed so that, while some students may not realize there is conflict, a majority of the students finds the situation challenging yet manageable. Students are encouraged to share their ideas. Although this may be stressful at first for some children, a climate of mutual respect for each other's opinions will reduce this state of anxiety. Hesitant children should be permitted to listen to the more outgoing members of the class during the initial phases and to gradually participate in the verbal exchange. Internalized conflict, aroused by listening to others, can be valuable when developing new structures; and hearing one's own thoughts discussed by others can often provide assistance in confirming or refuting a particular conjecture. Teachers who have tried this approach have often witnessed improvement in students' listening skills because students tend to pay more careful attention to each other's arguments when they want to participate in the discussion.

Discussion can also help to clarify a student's own thoughts. Occasions have arisen when a student has been attempting to explain why an incorrect strategy is the right one. While justifying the strategy, sometimes a puzzled expression suddenly appears on the student's face. It happens when the student realizes that he or she has, in fact, been explaining why the strategy is false. The memory of this "eureka" effect is likely to remain with the student. On such occasions, talking is more powerful than listening, since it is doubtful that such clarity could have been realized within the silence of individual thought.

The concern, expressed by some educators, that children who possess correct concepts may adopt incorrect strategies if exposed to them was not substantiated in studies I've conducted or in the research of others (Silverman and Geiringer 1973). In all instances, children who demonstrated correct conceptual understanding on the pre-test, displayed at least the same level of competency on the post- and delayed post-tests.

The teacher's role during the class discussion should be that of chairperson. Even the location of the teacher can influence the type of discussion that takes place; if he or she is at the front "directing" the lesson, students will wait for the correct strategy to be explained. However, if the teacher changes location slightly, moving to the side of the classroom, he or she becomes part of the class, and students are more willing to participate.

Generating and maintaining a discussion is not a simple matter, and good discussions do not merely "happen." A Socratic approach, from which children can come to the correct conclusion on their own, is most beneficial. Being aware of verbal or nonverbal signaling is crucial; using words such as "good" or nodding the head in approval can inhibit other children from expressing alternative points of view. When students' answers are discussed, it is more profitable to discuss incorrect strategies first. If incorrect answers are investigated before the correct solution is provided, the children are more inclined to discuss why they have chosen a specific strategy. This rarely occurs when a good explanation of the correct solution is given initially. Encouraging students' active involvement in the situation and presenting them with the opportunity to decide amongst themselves upon the benefits or insufficiencies of differing strategies are key ingredients in the development and retention of new ideas (Piaget 1970). Passive acceptance seldom brings new insights; yet, it is under these circumstances, during deductive explanations of new principles by the teacher, that students are often expected to acquire concepts.

Thus, discussion provides the means for students to develop rational arguments and to recognize the strengths and deficiencies of the contributions of others. Students have the opportunity to be actively involved in the communication process rather than simply be passive receivers of information. The children's ability to say what they mean and mean what they say will be greatly enhanced.

At this point in the lesson, students should be aware of their misconceptions and have, perhaps, partly resolved them. It is very doubtful that a class discussion will completely correct a misconception unless it is very close to being resolved in the first place. What is more likely is that the discussion will bring the misconception to the surface where it can be examined more profitably. Following the discussion, correct resolutions will need to be summarized in as concise a form as possible. This leads to the final part of the lesson.

The Consolidation Exercises

These activities are designed to provide students with a deeper understanding of the concept and to provide feedback. Although the opening activity and the conflict discussion might produce a positive change in the way children respond, this is likely to be short-lived unless the children have an opportunity to reflect on these experiences in a meaningful manner. Therefore, feedback is often built into the consolidation exercises, enabling students to reflect on problems given in the opening activity.

Brownell stated that a problem was not truly solved until the student understood what he had done and could explain why he had done it (1972, 155). Consolidation exercises differ from the usual textbook format insofar as they do not consist of a large number of similar questions. Rather, they examine a few examples from different perspectives in anticipation that these will provide students with a firmer grasp of the concept. The children can use this knowledge to explain why their original errors were incorrect.

During much of the work, it is best for students to work in mixed ability friendship groups or at least in pairs. The rationale for this follows.

Working in Groups

Expressing an opinion in class can cause anxiety for some students. The anxiety can be reduced if students first meet in small groups in which they are likely to feel more at ease while expressing an opinion. Small-group settings also allow everyone to participate—a situation that is not always possible in larger groups.

When students, as members of a small group, have settled on a decision following a discussion, they are more likely to support or reject the hypothetical statements of others than if they have arrived at the decision on their own. In an individual situation, students do not have the support of their associates, nor have they committed their ideas to others. They may, therefore, acquiesce to others rather than challenge their opinions. Conforming to the wishes of others without belief is unlikely to promote any permanent change to students' conceptual frameworks.

One concern about group situations is that some students will allow others to undertake an entire task and attempt very little themselves. This can be partly overcome by assigning a different spokesperson for the group each day. The spokesperson is responsible to report the findings of the group to the other members of the class during the large-group discussion and to ensure that each group member contributes to the solution, or solutions, if no consensus is reached.

Situations will arise when it is best for students to work in pairs or on their own, particularly during some of the consolidation exercises. However, cooperation in overcoming the misconceptions is a fundamental principle underlying the diagnostic teaching methodology.

It is not the intent of this article to suggest that diagnostic teaching is the panacea that will overcome all misconceptions or that it should be the sole method of instruction. Teachers need to utilize an eclectic approach, employing the strategy that is most suitable for the occasion. Diagnostic teaching is simply one strategy that has proven to be most effective in overcoming deeply embedded conceptual obstacles.

References

Allardice, D.S., and H.P. Ginsburg. "Children's Psychological Difficulties in Mathematics." In H.P. Ginsburg (ed.) *The Development of Mathematical Thinking.* New York: Academic Press, 1983.

Ausubel, D.P. "Some Psychological Aspects of the Structure of Knowledge." In the Fifth Phi Delta Kappa Symposium on Educational Research. *Education and the Structure of Knowledge.* Chicago: Rand McNally and Co., 1964.

Bell, A.W., M. Swan and G.T. Taylor. "Choice of Operation in Verbal Problems with Decimal Numbers." *Educational Studies in Mathematics* 12 (1981): 399-420.

Brownell, W.A. "The Place of Meaning in Mathematics Instruction: Selected Papers of William A. Brownell." In J.F. Weaver and J. Kilpatrick (ed.) *Studies in Mathematics Series,* Vol. 21. Stanford, Calif.: School Mathematics Study Group, 1972.

Cloutier, R., and M.L. Goldschmid. "Training Proportionality Through Peer Interaction." *Instructional Science* 7 (1978): 127-42.

Cockcroft, W.H. *Mathematics Counts.* Report of the Committee of Inquiry into the Teaching of Mathematics in Schools. London: H.M.S.O., 1982.

Erlwanger, S.H. "Benny's Conception of Rules and Answers." *Journal of Children's Mathematical Behaviour* 1, no. 2 (1973): 7-26

Hart, K. (ed.) *Children's Understanding of Mathematics.* London: Murray, 1981.

Kamii, C. "Pedagogical Principles Derived from Piaget's Theory: Relevance for Educational Practice." In M. Schwebel and J. Raph (ed.) *Piaget in the Classroom.* London: Routledge and Kegan Paul, 1974.

Papert, S. "Teaching Children To Be Mathematicians Versus Teaching About Mathematics." *International Journal of Mathematical Education in Science and Technology* 3 (1972): 249-62.

Piaget, J. "Piaget's Theory." In P.J. Mussen (ed.) *Carmichael's Manual of Child Psychology,* Vol. 1, Chap. 9. New York: Wiley and Sons, 1970.

Rees, R., and G. Barr. *Diagnosis and Prescription—Some Common Math Problems.* London: Harper and Row, 1984.

Rogers, C.R. *Freedom to Learn.* Columbus, Ohio: Charles E. Merrill, 1969.

Silverman, I.W., and E. Geiringer. "Dyadic Interaction and Conservation Induction: A Test of Piaget's Equilibration Model." *Child Development* 44 (1973): 815-20.

APPENDIX A

A Sampling of Questions Used to Uncover Some Deep-Rooted Numerical Misconceptions

Section A

Work out the answers to the following questions. If you think that it CANNOT BE DONE then put CD.

a. $8\sqrt{4} =$
b. $88 \div 4 =$
c. $3 \div 30 =$
d. $3\sqrt{21} =$
e. $0.4 \times 0.4 =$
f. $0.3 \times 0.3 =$
g. $9 \div 9 =$
h. $0.5 \div 0.5 =$
i. $10 \div \frac{1}{2} =$

Section B

Circle the calculation that will give the larger answer. If the answers are the same, circle SAME.

a.	$3 \div 24$	$24 \div 3$	SAME
b.	$3\sqrt{15}$	$15\sqrt{3}$	SAME
c.	7.5×0.8	$7.5 \div 0.8$	SAME
d.	$6 \div 18$	$6\sqrt{18}$	SAME

Section C

Answer each of the following as True (T), False (F) or Unsure (?)

a.	21.4×0.65	more than 21.4	________
		less than 21.4	________
b.	$36.8 \div 0.57$	more than 36.8	________
		less than 36.8	________

Section D

Circle the biggest of the three numbers: 0.6 0.75 0.425

How can you tell it is the biggest? ____

____

APPENDIX B

Usual Incorrect Responses to the Questions Presented in Appendix A

Section A—Usual Responses

a. Cannot be done *or* 2. (Big number divided by small)

b. The correct answer, 22, is given usually.

c. Cannot be done *or* 10. (Big number divided by small)

d. The correct answer, 7, is given usually.

e. The correct answer, 0.16, is given usually.

f. 0.9 (Questions *e* and *f* appear very similar, yet most students answer *e* correctly and *f* incorrectly. Effective questions are paramount to uncovering misconceptions.)

g. Correct answer usually given; sometimes zero is given as the answer.

h. 0.1 (Since both are decimals.)

i. 5 (Although division of fractions is not covered until Grade 8, ask any Grade 6 student how many halves in 10 whole ones, and most will respond "20" very quickly.)

Section B

a. and *b.* Students are usually more successful on *a* than on *b.*

c. 7.5 x 0.8 (Multiplication makes bigger, division makes smaller.)

d. SAME (Students interpret the signs as being synonymous, they divide the big number by the small number for both.)

Section C

Students usually think multiplication makes bigger and division makes smaller.

Section D

Many students will circle 0.425, selecting the largest numeral and ignoring the decimal point, but a surprisingly large number may choose 0.6 because "tenths are bigger than hundredths or thousandths" or because they have confused decimals with fractional numbers such as ½ or ¼. They seem to think "smaller numerals, so bigger pieces."