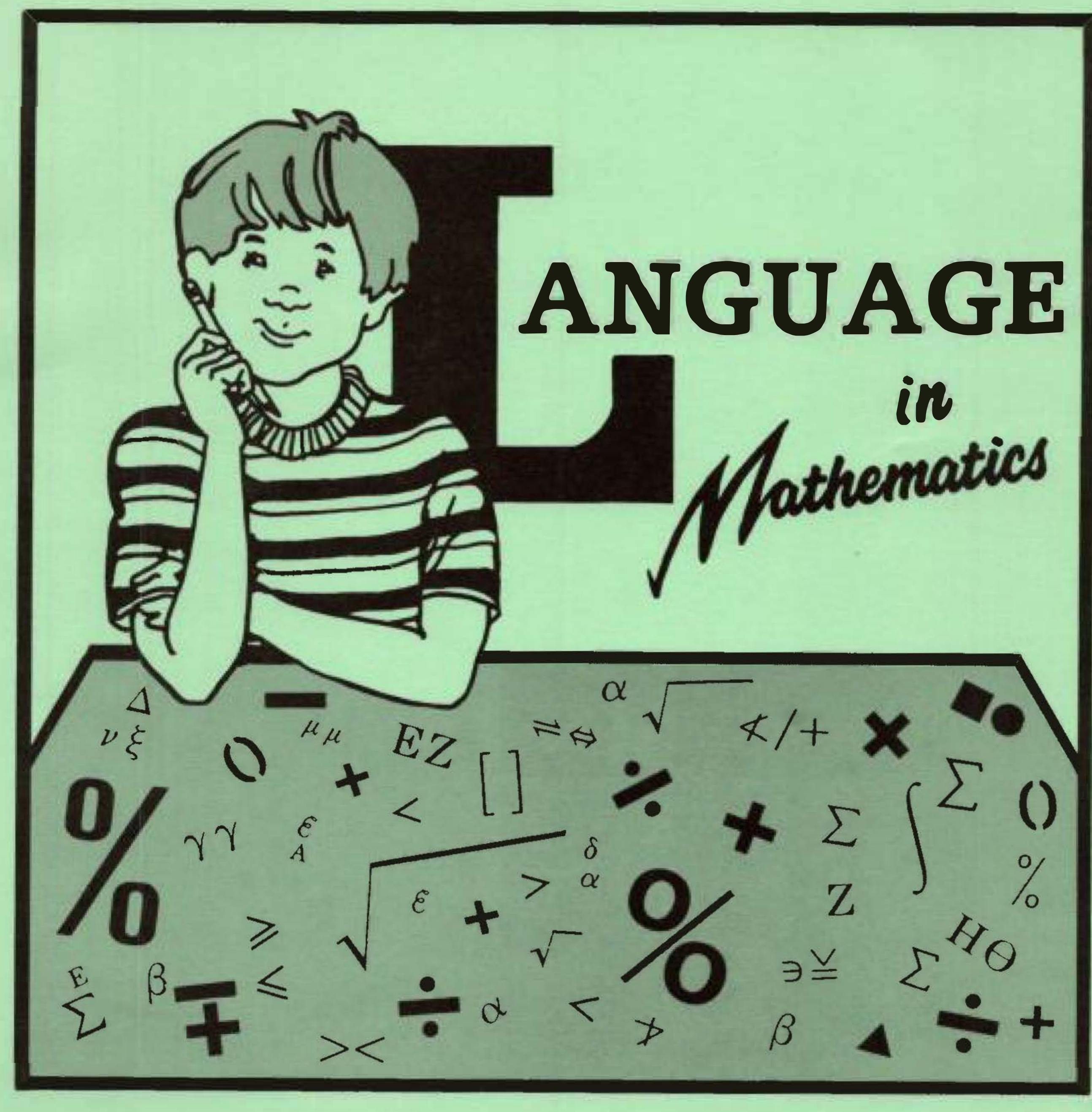
delta-k

JOURNAL OF THE
MATHEMATICS COUNCIL
OF THE ALBERTA
TEACHERS' ASSOCIATION

Volume 27, Number 2 September 1988



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EDPRESS

delta-k is published by The Alberta Teachers' Association (ATA), for the Mathematics Council (MCATA). Editor: John B. Percevault, 2510 22 Avenue South, Lethbridge, Alberta T1K 1J5. Editorial and production services by ATA Central Word Services staff. Please address all correspondence regarding this publication to the editor. *delta-k* is indexed in the Canadian Education Index.

Language and Mathematics

Mathematics has been defined as a language by D.A. Johnson and G.R. Rising in *Guidelines for Teaching Mathematics,* 2nd ed. (Belmont, Calif.: Wadsworth, 1972). Technical words and definitions abound. Many words in common usage have a distinct mathematical meaning and, occasionally, more than one mathematical meaning. Above all, mathematics is dependent upon symbols that allow us to compute, to solve and to show relationships. The problem of language and its symbols is complicated. Many symbols have more than one meaning, and many symbols may be used interchangeably to connote the same mathematical meaning. As teachers of mathematics, we cannot merely assume that the language of mathematics will be apparent to students. Both encoding and decoding it must be taught in our instructional strategies.

Mary Jo Maas advocates ''Groups of Four'' as a method of instruction. The group provides a nonthreatening environment in which vocabulary may be developed. Barry Onslow summarizes research that illustrates student development as a result of group discussion. Language development is an important desired outcome at all grade levels. Marilyn Burgoyne shares ideas appropriate for junior high school teachers.

Readability of mathematics texts should be an important consideration in deciding whether to adopt a text. Yvette d'Entremont analyzes a text being considered for possible adoption. Bernard Yvon and Jane Zaitz encourage teachers to have students develop mathematical stories and books and to read their stories to other students. Irene Eizen and Arlene Dowshen state that students' writing in mathematics must focus on content. They include samples of students' writing in their article. The concept of integration is of current interest in Alberta schools. George Calder shares ideas on how to teach translation in beginning algebra courses.

Two articles are included that do not directly support the theme of language in mathematics. Bonnie Litwiller and David Duncan offer another investigation of counting numbers. They use a problem-solving format. Dennis Connolly discusses the probability of winning in gambling games.

And, finally, in the Student Problem Corner, Craig Loewen turns poet and presents a problem in a snooker game.

—John B. Percevault

Small-Group Learning in Mathematics

Mary Jo Maas

Mary Jo Maas teaches mathematics at F.P. Walshe High School. Mary Jo was seconded to the University of Lethbridge where she taught curriculum and instruction in mathematics. Mary Jo is the secretary of the MCATA.

Scene I

Setting: A Math 10 class that has just been taught the concept of addition and subtraction of polynomials.

Mike: *"Mrs. M., come here. Do I have to change the signs in this one (polynomial)?"*

(Mrs. M looks at the text and sees that the question reads "sum," not "difference," as this student is suggesting.)

Mrs. M: (Pointing at the word "sum.") *"What does this word mean?"*

Mike: *"The answer."*

Mrs. M: *"And how do you obtain this answer? By doing what?"*

(Silence.)

(Other leading questions are asked, only to be met with silence.)

Mrs. M.: *"Sum means to add. Are you going to have to change the signs then?"*

Mike: *"No. I didn't know what the word meant, so I didn't know what to do. Now I can do the work."*

Oh, that this scene was simply from a book of plays rather than from our very real classroom experiences. Somehow, somewhere, our students have not been provided with sufficient experiences to read math. The students can say the words, but they can't provide a meaning for them, much less give a synonym for a word. What then can be done to provide students with the skills necessary to read successfully in math?

This article provides some suggestions on what to do to help your students. Since word problems seem to show up a student's weakness in reading, it is within this realm that this article will be developed.

A great deal of success has been shown to occur when small-group learning is incorporated into a math classroom. The technique that I use and am most familiar with is called "Groups of Four." It is a classroom technique that allows four students to sit together, share ideas and assist each other. Using Groups of Four and the directed reading process, the teacher has a chance to teach students how to read a word problem with understanding and to develop better reading and math skills. Since this is done within the safety of a group setting, the pressure is off individual students to always know the correct answer.

The first lesson using Groups of Four is to simply have the groups read a word problem and state the question in their own words, that is, tell what they are to find. Using a very simple problem such as "Tom has four cats, Sally has three cats. How many cats do they have altogether?" one child could quietly read the problem to the group, and another child could restate the problem. By having a variety of problems ready, all of the students in the group can have a chance to assume each of the roles. Once a group is comfortable with that process, the members can create their own problems to share with other groups, then repeat the initial process.

During the initial lesson, synonyms for words such as "equal," "sum" and "difference" should be used in the problems. Exposing students to these words forces them to decide what the words mean and to learn appropriate synonyms for them. The objective at this point is to get the students talking math, not solving it.

If appropriate, groups can also be asked to act out various problems, as one member restates the problem. Again, understanding the words and the concepts that they refer to is the objective.

The second lesson will expand upon the first and will encourage students to state what quantities are involved and what information is given in the problem. It is important that, at this stage, the groups be given problems that contain:

a) only sufficient information
b) insufficient information
c) extraneous information
d) information that has to be recalled or inferred such as, for example, the formula for perimeter
e) information that has to be obtained from a graph or diagram
f) information that needs to be researched

Too often, the problems that students encounter in their textbook have exactly the right amount of information and correspond exactly to what has just been studied. Thus, students are never truly forced to read the problems; they merely pick out the numbers and perform the correct operation on them, as was described in the lesson.

As the groups encounter the various types of problems, encourage the students to talk out the problems, draw pictures, act them out, and, in general, do whatever is needed to understand the problem. As the groups do this, their math, reading and language skills will be expanded. If a question requires them to do some research in the library, let them. The students need this experience, and the groups allow the students to explore in a "safe environment."

Once the students are proficient at this, encourage them to create their own problems for others to solve. The problems should require research and/or include extraneous information.

The third lesson involves having the group members discuss what process(es) is (are) to be used. Once a process is chosen, the group must be willing to support its choice. Group members may begin to observe that there are key words that, when identified, help to determine the process(es) to be used.

The nice thing about having students in groups when completing this lesson is that often there is more than one process that may be used to solve a problem. Being in the group allows the students to see a variety of thinking strategies and processes, and, thus, individual students begin to realize that there isn't just one correct way to solve a problem.

Up to this point, there has been nothing mentioned about solving the problem. It is only after lessons one, two and three have been completed satisfactorily that you should begin having the students solve the problems. It is at this point that the students have the necessary tools to read the problem with understanding, decide what is being asked for, what quantities are involved and how to solve it.

When students have reached the fourth lesson—solving the problem—you know that you have encouraged a great deal of growth in the students' vocabularies, reading development and math skills, and that you have provided them with a comfortable setting in which to do so. (See Appendix A.)

Now, let's revisit our classroom and see how things have changed.

Scene II

Setting: Math 10 class in which groups of four are used.

Topic: Addition and Subtraction of Polynomials.

Mike: *"I see that this question says to find the sum of the polynomials. I can't remember what sum means."*

Carla: *"I remember that. Sum is the answer when you add things."*

Mike: *"Oh, ya! Now I remember. That means that the signs won't change inside the brackets. Hey, this is easy."*

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APPENDIX A

Word Problems and Directed Reading Processes

1. What is the question? What are we to find?

   Have students read the question, then restate it in their own words or tell the question to a friend.

2. What quantities are involved? What is the information given?

   It is suggested that students encounter problem situations with the following limitations.

Limitation	Directions to Student
• Only sufficient information	State, tell, write
• Insufficient information	Identify what is needed Supply missing information
• Extraneous information	Cross out extraneous information Reread and omit extraneous matter
• Information that has to be recalled or inferred	Supply formula Recall or infer
• Information that has to be obtained from a diagram or graph	Interpret diagram, graph or picture
• Information that needs to be researched	Use library to find information

3. What process(es) is (are) to be used?

   Have students support their choice(s) of process(es). Cue words (more, less, in total, etc.) may be identified. Students may also be asked to develop related problems that are easier, similar or more difficult. Students may draw a diagram or use a more abstract representation such as a number line.

4. Can you solve the problem?

   Each step of the directed reading procedure may be used as the basis of a lesson.

Overcoming Conceptual Obstacles

Barry Onslow

Dr. Barry Onslow is an assistant professor of education at the University of Western Ontario. This paper was presented in 1986 at the NCTM Canadian Conference in Edmonton, Alberta. At the time, Barry was a member of the MCATA executive and vice-principal at Ralston DND School. He taught mathematics to students in Grades 6 to 9.

Introduction

Evidence now shows that on some fundamental concepts, children make little progress, even after a substantial amount of mathematics teaching (Hart 1981). For example, on the concept illustrated in the following problem, a study by Hart showed that students failed to develop greater understanding even after years of instruction.

Problem

Circle the one that gives the BIGGER answer:

a) 8 x 4 or 8 ÷ 4

b) 8 x 0.4 or 8 ÷ 0.4

c) 0.8 x 0.4 or 0.8 ÷ 0.4

Results

Response percentages by year

	12 yrs	13 yrs	14 yrs	15 yrs
x ÷ ÷	13	8	15	18
x x x	50	58	47	30

(Hart 1981, 54)

In this article I will examine some of the misconceptions that hinder students' mathematical progress and will propose an alternative teaching strategy that has been beneficial in assisting students to recognize and overcome these conceptual obstacles.

The Mathematics Curriculum

The mathematics curriculum contains three elements: skills, concepts and problem solving (Figure 1).

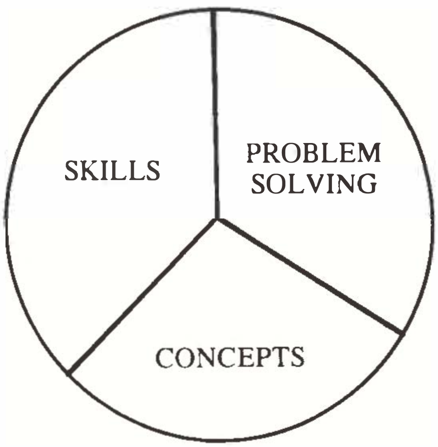

Figure 1
Components of the Mathematics Curriculum

This diagram can be treated as a topological chart insomuch as the three distinct components can be enlarged or diminished as appropriate. However, each of these components must be examined many times during the course of the year, and each requires different techniques for overcoming student difficulties.

A great deal of time and energy has recently been spent in demonstrating that it is the strategies underlying problem solving that are important, rather than the actual answers. Many valuable resources are currently available. However, little has been said about the changes needed in the teaching of skills and concepts. In the majority of classrooms, both are generally taught in a similar fashion, that is, exposition followed by practice. Ausubel, writing about mathematics teaching in the United States during the 1960s, commented that the style of teaching has progressed very little during this century:

> [Mathematics teaching] still relies heavily on rote learning of formulas and procedural steps, on recognition of stereotyped "type-problems" and on the manipulation of symbols.
> (Ausubel 1964, 241)

This might well be adequate for some aspects of skill development, but it does not appear to assist children in overcoming conceptual obstacles. Recent research has indicated that children's mathematical understanding is not what might have been expected, and their errors are generally neither random nor careless. Students have specific incorrect strategies for approaching problems. Moreover, these strategies are so deep-rooted that they often remain with the students throughout their schooling and later life (Hart 1981; Rees and Barr 1984). Other intuitive strategies, which students use correctly in simple situations, often prove to be inadequate for more complex questions (Bell, Swan and Taylor 1981).

The Student's Conceptual Understanding

Two misconceptions typically held by junior high school students are that multiplication always makes bigger and that in division, the larger number is always divided by the smaller one. These misconceptions appear to be trivial for junior high school students; thus, it is not surprising that many teachers are a little skeptical of the seriousness of the problems and of the difficulty their students might have in overcoming them. To determine which misconceptions your students have, present them with the questions in Appendix A. The majority of the students' misconceptions will correspond with the responses presented in Appendix B.

Having tested the students and analyzed the results, employ your usual teaching strategies to overcome these misconceptions and re-test approximately two weeks later. Ensure that students do not receive any feedback regarding their answers until after the retest. Do not be surprised if there is little or no change from the students' original responses.

Why are these misconceptions difficult to eradicate? The most probable reasons are that these strategies have been correct in the past and remain appropriate for many situations. Using the elementary model of repeated addition for multiplication, multiplication does indeed make bigger. Very few verbal division problems presented in textbooks, even at the junior high level, require students to divide the small numeral by the larger one. Also, depending on how the child models division, $10 \div \frac{1}{2}$ will often mean 10 shared in half (similar to $10 \div 2$ meaning 10 shared in/by two) rather than "How many halves in 10?"

The Diagnostic Teaching Strategy

To allow the development of superior strategies, children must be presented with situations that demand the use of such strategies. Misconceptions are like beliefs—only when it is obvious that faulty methods have been employed will students see the necessity for change. This is the premise upon which diagnostic teaching is based.

The key to this change of style is a situation in which students begin analyzing answers and discussing their correctness. The teacher no longer provides the students with an appropriate strategy by demonstrating the correct procedure for solving an example. Rather, the teacher provides a situation in which students attempt (either on their own or in groups) what they consider to be the correct strategies. The focal point of the lesson is the analysis of these answers. They are reviewed by the class, and a discussion takes place to examine the strategies involved in solving the problem and to decide whether or not specific strategies are valid. It is anticipated from Piagetian theory, that opposing strategies will arouse conflict, that the students will try to eradicate the conflict and that a meaningful learning situation will be established. Only when the limits of their primitive frameworks are demonstrated to them, do students see the necessity for change. Once this has been accomplished, they should be able to recognize that an inconsistency exists and that there is a need for reorganization and rethinking. This will require modifications to students' conceptual frameworks if they are to successfully handle the task.

Many children have a fear of failure, and this fear can often be reinforced either by adults or other children. Yet students must be willing to experiment and make mistakes if they are to accomplish worthwhile goals. To overcome this fear and enable diagnostic teaching to be effective, the climate in the classroom must be one of mutual respect. Children must value the opinions of others even though they might disagree with them. In Carl Rogers' words, the teacher has the opportunity to develop a climate in which "threat to the self is minimized, (and) the individual makes use of opportunities to learn in order to enhance himself" (1969, 162).

Under such conditions it is possible to establish an atmosphere of cooperative learning. Once pupils feel confident about expressing their ideas, whether correct or incorrect, they can develop a better appreciation of their thoughts and contributions. This might even be more relevant for girls, who tend to perform better than boys in language arts but worse in mathematics. Girls need equal opportunities to participate in discussions that clarify their ideas and understanding and to learn mathematics satisfactorily (Cockcroft 1982, 63).

Children's approaches to mathematics are important. Allardice and Ginsburg exemplify this with their description of two boys who came to their class (1983, 343). Whereas one child thought that reasoning about mathematics was a sensible activity and learned quite quickly, the other child considered that "going over and over it" was the only acceptable approach—he learned nothing. Allowing students to reflect on their answers and justify their conclusions, rather than simply seek an algorithm to generate the correct answer, will encourage a deeper awareness of the mathematical structure underlying a problem. It will also provide time for the teacher to reflect more carefully upon what the students *really* know, rather than what it is often *assumed* they know. Students' original, and sometimes creative, methods may often go unnoticed if they are never shared with others, especially when children are engaged in individualized teaching programs and mark their own work (Erlwanger 1973). Although, at times, the students' techniques may be correct, at other times they may be severely deficient. Under either condition, an airing of the students' views is important. It is the lack of discussion amongst older pupils, in favor of a more formal approach that often leads to further failure (Cockcroft 1982, 142).

Even the more able students memorize routines find it difficult to explain the processes they put into practice when solving problems. Understanding the structure underlying a problem is imperative if children are to develop their mathematical capabilities.

Being a mathematician is no more definable as "knowing" a mathematical set of facts than being a poet is definable as "knowing" a set of linguistic facts. (Papert 1972, 249)

In the diagnostic teaching approach, each lesson is usually divided into three component parts: the opening activity, the conflict discussion and consolidation exercises. Each is described below.

The Opening Activity

The opening activity is designed to familiarize students with the problem and prepare the way for a conflict discussion by presenting material that will provoke errors.

A child adopts an incorrect strategy usually because it has been beneficial in an alternate situation, but its limits have not been recognized. For example, "multiplication makes bigger" is a perfectly adequate concept when dealing with the natural number system, but it becomes inadequate when rational numbers are introduced. Once children recognize that this approach is inappropriate, they may comprehend why it is essential to examine a new strategy.

When teaching in this manner, it is not desirable to shield the children from errors. Instead, children are encouraged to face mistakes in a positive manner so as to become more adept at discovering errors for themselves. With the teacher's help, they can make constructive use of the errors. In this regard, teachers have a most important role, both in explaining how errors can be utilized profitably and in changing students' attitudes so that the benefits of both reflecting on and discussing their work become obvious. This will be a different and difficult experience for many students who often desire only the correct answer. The advantages need to be discussed in detail with most classes before they are able to recognize the true worth of exploring mathematics in this way.

Providing the correct level of activity is also crucial since a question that is easily solved provides little challenge or need for exploration, whereas one that is too difficult merely arouses confusion. Cloutier and Goldschmid (1978, 138) recommend that if a question is to provoke discussion it should be difficult as well as interesting, but not so difficult as to be insurmountable by most students.

The Conflict Discussion

Once students have had an opportunity to discuss the opening activity either in pairs or small groups, they are brought together for a class discussion. Situations are purposely contrived so that students have conflicting views on the topic. This allows for beneficial interchange.

> A clash of convictions among children can readily cause an awareness of different points of view. Other children at similar cognitive levels can often help the child more than the adult can to move out of his egocentricity. (Kamii 1974, 200)

Interaction in this manner allows students to share their interpretations of a concept and permits the clarification of new ideas, provided that the question is within the limits of their conceptual frameworks. When this is not the case, discussion resorts to little more than the sharing of ignorance, prejudice, preconceptions and vague generalities. Little benefit will be gained from the latter situation, so it is important to define the attributes of a profitable discussion.

Students are expected to possess the background knowledge that enables them to examine problems in an informed and intelligible manner. Problems are designed so that, while some students may not realize there is conflict, a majority of the students finds the situation challenging yet manageable. Students are encouraged to share their ideas. Although this may be stressful at first for some children, a climate of mutual respect for each other's opinions will reduce this state of anxiety. Hesitant children should be permitted to listen to the more outgoing members of the class during the initial phases and to gradually participate in the verbal exchange. Internalized conflict, aroused by listening to others, can be valuable when developing new structures; and hearing one's own thoughts discussed by others can often provide assistance in confirming or refuting a particular conjecture. Teachers who have tried this approach have often witnessed improvement in students' listening skills because students tend to pay more careful attention to each other's arguments when they want to participate in the discussion.

Discussion can also help to clarify a student's own thoughts. Occasions have arisen when a student has been attempting to explain why an incorrect strategy is the right one. While justifying the strategy, sometimes a puzzled expression suddenly appears on the student's face. It happens when the student realizes that he or she has, in fact, been explaining why the strategy is false. The memory of this "eureka" effect is likely to remain with the student. On such occasions, talking is more powerful than listening, since it is doubtful that such clarity could have been realized within the silence of individual thought.

The concern, expressed by some educators, that children who possess correct concepts may adopt incorrect strategies if exposed to them was not substantiated in studies I've conducted or in the research of others (Silverman and Geiringer 1973). In all instances, children who demonstrated correct conceptual understanding on the pre-test, displayed at least the same level of competency on the post- and delayed post-tests.

The teacher's role during the class discussion should be that of chairperson. Even the location of the teacher can influence the type of discussion that takes place; if he or she is at the front "directing" the lesson, students will wait for the correct strategy to be explained. However, if the teacher changes location slightly, moving to the side of the classroom, he or she becomes part of the class, and students are more willing to participate.

Generating and maintaining a discussion is not a simple matter, and good discussions do not merely "happen." A Socratic approach, from which children can come to the correct conclusion on their own, is most beneficial. Being aware of verbal or nonverbal signaling is crucial; using words such as "good" or nodding the head in approval can inhibit other children from expressing alternative points of view. When students' answers are discussed, it is more profitable to discuss incorrect strategies first. If incorrect answers are investigated before the correct solution is provided, the children are more inclined to discuss why they have chosen a specific strategy. This rarely occurs when a good explanation of the correct solution is given initially. Encouraging students' active involvement in the situation and presenting them with the opportunity to decide amongst themselves upon the benefits or insufficiencies of differing strategies are key ingredients in the development and retention of new ideas (Piaget 1970). Passive acceptance seldom brings new insights; yet, it is under these circumstances, during deductive explanations of new principles by the teacher, that students are often expected to acquire concepts.

Thus, discussion provides the means for students to develop rational arguments and to recognize the strengths and deficiencies of the contributions of others. Students have the opportunity to be actively involved in the communication process rather than simply be passive receivers of information. The children's ability to say what they mean and mean what they say will be greatly enhanced.

At this point in the lesson, students should be aware of their misconceptions and have, perhaps, partly resolved them. It is very doubtful that a class discussion will completely correct a misconception unless it is very close to being resolved in the first place. What is more likely is that the discussion will bring the misconception to the surface where it can be examined more profitably. Following the discussion, correct resolutions will need to be summarized in as concise a form as possible. This leads to the final part of the lesson.

The Consolidation Exercises

These activities are designed to provide students with a deeper understanding of the concept and to provide feedback. Although the opening activity and the conflict discussion might produce a positive change in the way children respond, this is likely to be short-lived unless the children have an opportunity to reflect on these experiences in a meaningful manner. Therefore, feedback is often built into the consolidation exercises, enabling students to reflect on problems given in the opening activity.

Brownell stated that a problem was not truly solved until the student understood what he had done and could explain why he had done it (1972, 155). Consolidation exercises differ from the usual textbook format insofar as they do not consist of a large number of similar questions. Rather, they examine a few examples from different perspectives in anticipation that these will provide students with a firmer grasp of the concept. The children can use this knowledge to explain why their original errors were incorrect.

During much of the work, it is best for students to work in mixed ability friendship groups or at least in pairs. The rationale for this follows.

Working in Groups

Expressing an opinion in class can cause anxiety for some students. The anxiety can be reduced if students first meet in small groups in which they are likely to feel more at ease while expressing an opinion. Small-group settings also allow everyone to participate—a situation that is not always possible in larger groups.

When students, as members of a small group, have settled on a decision following a discussion, they are

more likely to support or reject the hypothetical statements of others than if they have arrived at the decision on their own. In an individual situation, students do not have the support of their associates, nor have they committed their ideas to others. They may, therefore, acquiesce to others rather than challenge their opinions. Conforming to the wishes of others without belief is unlikely to promote any permanent change to students' conceptual frameworks.

One concern about group situations is that some students will allow others to undertake an entire task and attempt very little themselves. This can be partly overcome by assigning a different spokesperson for the group each day. The spokesperson is responsible to report the findings of the group to the other members of the class during the large-group discussion and to ensure that each group member contributes to the solution, or solutions, if no consensus is reached.

Situations will arise when it is best for students to work in pairs or on their own, particularly during some of the consolidation exercises. However, cooperation in overcoming the misconceptions is a fundamental principle underlying the diagnostic teaching methodology.

It is not the intent of this article to suggest that diagnostic teaching is the panacea that will overcome all misconceptions or that it should be the sole method of instruction. Teachers need to utilize an eclectic approach, employing the strategy that is most suitable for the occasion. Diagnostic teaching is simply one strategy that has proven to be most effective in overcoming deeply embedded conceptual obstacles.

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APPENDIX A

A Sampling of Questions Used to Uncover Some Deep-Rooted Numerical Misconceptions

Section A

Work out the answers to the following questions. If you think that it CANNOT BE DONE then put CD.

a. $8\sqrt{4} =$
b. $88 \div 4 =$
c. $3 \div 30 =$
d. $3\sqrt{21} =$
e. $0.4 \times 0.4 =$
f. $0.3 \times 0.3 =$
g. $9 \div 9 =$
h. $0.5 \div 0.5 =$
i. $10 \div \frac{1}{2} =$

Section B

Circle the calculation that will give the larger answer. If the answers are the same, circle SAME.

a. $3 \div 24$ $24 \div 3$ SAME
b. $3\sqrt{15}$ $15\sqrt{3}$ SAME
c. 7.5×0.8 $7.5 \div 0.8$ SAME
d. $6 \div 18$ $6\sqrt{18}$ SAME

Section C

Answer each of the following as True (T), False (F) or Unsure (?)

a. 21.4×0.65 more than 21.4 ________
less than 21.4 ________
b. $36.8 \div 0.57$ more than 36.8 ________
less than 36.8 ________

Section D

Circle the biggest of the three numbers: 0.6 0.75 0.425

How can you tell it is the biggest? ____

____

APPENDIX B

Usual Incorrect Responses to the Questions Presented in Appendix A

Section A—Usual Responses

a. Cannot be done *or* 2. (Big number divided by small)

b. The correct answer, 22, is given usually.

c. Cannot be done *or* 10. (Big number divided by small)

d. The correct answer, 7, is given usually.

e. The correct answer, 0.16, is given usually.

f. 0.9 (Questions *e* and *f* appear very similar, yet most students answer *e* correctly and *f* incorrectly. Effective questions are paramount to uncovering misconceptions.)

g. Correct answer usually given; sometimes zero is given as the answer.

h. 0.1 (Since both are decimals.)

i. 5 (Although division of fractions is not covered until Grade 8, ask any Grade 6 student how many halves in 10 whole ones, and most will respond "20" very quickly.)

Section B

a. and *b.* Students are usually more successful on *a* than on *b.*

c. 7.5 x 0.8 (Multiplication makes bigger, division makes smaller.)

d. SAME (Students interpret the signs as being synonymous, they divide the big number by the small number for both.)

Section C

Students usually think multiplication makes bigger and division makes smaller.

Section D

Many students will circle 0.425, selecting the largest numeral and ignoring the decimal point, but a surprisingly large number may choose 0.6 because "tenths are bigger than hundredths or thousandths" or because they have confused decimals with fractional numbers such as ½ or ¼. They seem to think "smaller numerals, so bigger pieces."

Helping Students to Become Literate in Mathematics

Marilyn Burgoyne

Marilyn Burgoyne is mathematics coordinator at Bishop Pinkham School in Calgary. The school has an English and a bilingual program. Mrs. Burgoyne has taught for the Calgary Public School Board for 12 years. She is a member of MCATA and presented two sessions at the 1987 annual conference in Calgary.

The goals of the junior high mathematics program are to enable students to

1. use Polya's four-step problem-solving procedures to deal with new or different situations,
2. use mathematics as a tool to deal with everyday situations,
3. recognize the need for mathematics in various future career options,
4. develop a positive self-concept and a positive attitude toward mathematics.

To fulfill these goals, the needs of the individual student must be met. Students need to develop an understanding of mathematical concepts. To do so, a concrete process-oriented approach is to be used. In this manner, the teacher will guide the student from the concrete to the transitional to the formal stages of cognitive development in the understanding of mathematics.

The Junior High Mathematics Program is divided into six strands:

1. Problem solving
2. Number systems and operations
3. Ratio and proportion
4. Measurement and geometry
5. Data management
6. Algebra

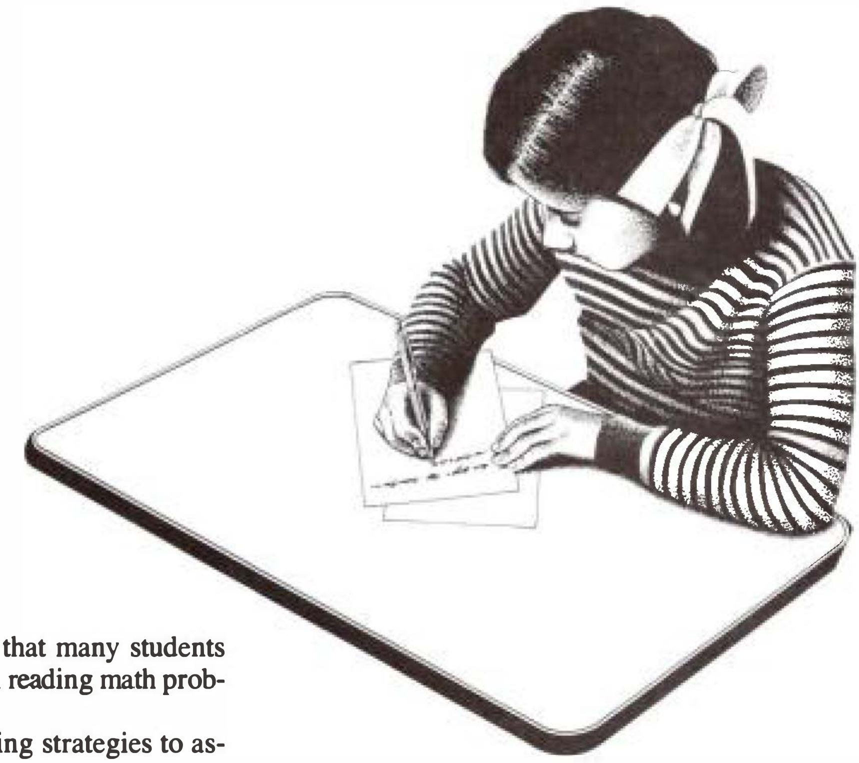
This new program of studies is now at an interim stage. Mandatory implementation will occur in September 1988.

When implementing this curriculum, math teachers must also consider the relationship of math with other subjects such as language arts, social studies, science, art and music. No discipline can or should function in isolation in the school.

When mathematics is taught, the needs of the learner must be respected. Since language is a key element in learning, its effect and use in the math classroom must also be examined. "Language across the curriculum" needs to be addressed.

The mathematics team at each school should consider using some of the following techniques and methods to incoporate language arts into the teaching of math.

1. Teachers should ensure that they use correct spelling, grammar and punctuation in their presentations of math vocabulary and concepts.
2. Teachers should ensure that students' notes have correct spelling, grammar and punctuation.
3. Teachers could have students do "write-ups" to explain math activities and concepts in more detail.
4. Students could use the library to do written reports on mathematical ideas.
5. Students could use computers and word processors to write up mathematical ideas and concepts.
6. Students could do written reports of problem solving activities.
7. Students could present oral reports on math ideas.



8. Teachers should recognize that many students may encounter difficulties in reading math problems and assignments.
9. Teachers should plan teaching strategies to assist students when reading and interpreting technical writing.
10. Math teachers should consult with the language arts department when assigning written work to their students.
11. Teachers should maintain consistency among departments in marking written student work.

In summary, the complexity of the material in most mathematics textbooks stresses the need for good language skills. These skills must be taught. The goal of the math team, consequently, is to teach good language skills when implementing the mathematics curriculum.

Readability: A Factor in Textbook Evaluation

Yvette M. d'Entremont

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Selecting an appropriate text for a school mathematics program is an important task and should not be taken lightly. Because mathematics has a language of its own, one of the factors that should be considered in textbook selection is readability.

Mathematics teachers are aware that reading mathematics material is different from reading other subject matter. Reading in mathematics is more than reading words. It involves decoding the words; decoding and interpreting the various mathematical symbols; and being able to interpret, comprehend and solve mathematical sentences and phrases. The mathematics textbook serves as an aid in developing language in mathematics.

One way in which students acquire skills and knowledge is by reading instructional materials; therefore, they must have textbooks that are easy to read and comprehend. Progress in learning mathematics and the language of mathematics will be achieved if the reading level of the textbooks is appropriate to the grade or course for which the texts are intended.

Departments of education and teachers are faced with an ever-increasing flood of printed materials, which differ widely in content, style and difficulty, and from which selections have to be made. In this situation, readability formulas may help by providing teachers with an additional guide for selecting suitable material. The textbook that is most effective is the one in which the author, through his or her writing style and vocabulary, produces a text with a reading level that is matched with the reading level of the student (Kennedy 1974).

In preparing new textbooks, the readability level of the text is considered by some publishers. In 1982, the readability level of *Starting Points in Math 10* was analyzed. J.E. Freeman, associate program manager for Ginn and Company (publishers of *Starting Points*), stated that it is the company's policy to establish readability levels of textbooks by the Fry Graph but that provincial departments of education that purchase books do not inquire about the readability level of particular texts.

When considering whether to adopt a new mathematics textbook, Alberta Education does not apply readability formulas but pilots or field tests the book. Publishers are invited to submit textbooks that they feel will fit the scope and sequence of the curriculum sent to them by Alberta Education. A committee of teachers from across the province then evaluates the textbooks submitted by the publishers and chooses a number of them to field test. After piloting the textbooks, the teachers come together to discuss the books and the program and how they fit together. They select the books that will be listed as "basic resources." Schools then select the books that they wish to use from the list of basic resources (Jim Neilsen 1988).

Many students experience difficulties in comprehending the explanations and problems found in mathematics textbooks. Concern about this has led me to assess the readability level of the text referred to earlier, *Starting Points in Math 10,* which I use with my Grade 10 students.

The readability level of particular textbooks can be determined by using readability formulas. Applying formulas usually involves selecting a sample from a text, counting some easily identifiable characteristics such as the average number of words per sentence or the proportion of polysyllabic words in the sample, and performing a calculation to produce a score (Gilliland 1972). Thus, readability formulas are based on correlational data, the correlation between sentence length and passage reading difficulty.

My objectives were to ascertain the readability level of the text by using two readability formulas, the Fry Readability Graph and McLaughlin's SMOG Grading Formula, and by administering cloze tests. A readability formula is a formula that is intended to provide quantitative objective estimates of the difficulty of reading (Klare 1963).

Three passages were selected at random from the text:

1. Finding the Equation of a Line, Given Two Points (p. 36)
2. Adding and Subtracting Rational Expressions (p. 288)
3. The Pythagorean Theorem (p. 257).

The Fry Readability Graph, Figure 1, (Fry 1968) and McLaughlin's SMOG Grading Formula were applied on each passage selected.

Figure 1

Grade	Average Number of Sentences per 100 words	Average Number of Syllables per 100 words
1	14.3	120
3	8.6	123
6	5.8	129
9	4.5	149
12	4.0	162

Extracted from Fry's Readability Graph. [From "Reading Level Determination for Selected Texts" by K. Kennedy, *The Science Teacher* 41 (March 1974): 26.]

The Fry Readability Graph uses two factors to determine reading level: the average number of sentences per 100 words and the average number of syllables per 100 words. The intersection point of these two factors on the Fry Graph gives the grade level.

The McLaughlin SMOG Grading Formula, developed in 1969 by G. Harry McLaughlin, is based on only one factor: the number of words having three or more syllables in 30 selected sentences. The grade level is calculated by adding "3" to the nearest appropriate square root of the polysyllabic word count.

The Fry Readability Graph and McLaughlin's SMOG Grading Formula were not designed for use with mathematics materials, but they have been modified to measure the readability of a variety of mathematics books. In applying the formulas to mathematics textbooks, the samples selected should include only sentences. Non-sentence material such as pure computation, equation-solving, geometric proofs, titles of chapters and illustrative problems are not part of the content examined (Johnson 1957).

Readability scores were calculated on the text in question using the above readability formulas. Analysis of the selected passages by use of the Fry Readability Graph produces 163.7 as the average number of syllables per 100 words, and 4.9 as the average number of sentences per 100 words. Plotting the average number of syllables per 100 words and the average number of sentences per 100 words on the Readability Graph results in an average reading level of Grade 12 for the text, *Starting Points in Math 10.*

Analysis of the same selected passages by use of the SMOG Grading Formula produces a polysyllabic word count of 105 in 30 selected sentences. Calculating the square root of 100 (the nearest appropriate figure to 105) and adding 3 to the square root gives a figure of 13. Therefore, according to the SMOG Grading Formula, the reading level of the text in question is Grade 13.

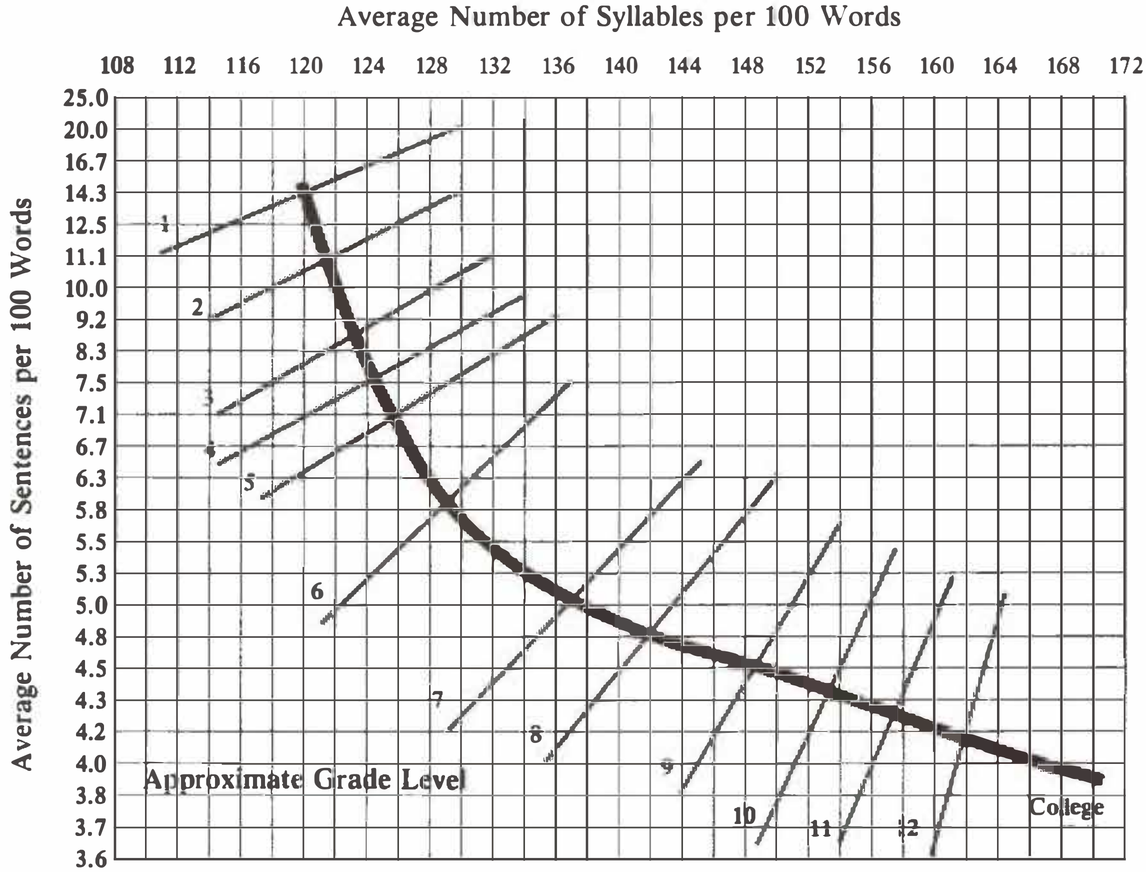
When discussing these results, one must consider the grade level of the intended user of the text as well as the accuracy of the formulas involved. Fry (1968) states that the Readability Graph results are accurate to "probably within a grade level" (p. 514). For the SMOG Grading Formula, the standard error is about 1.5, providing a range of three years (McLaughlin 1969).

The Fry Readability Graph and McLaughlin's SMOG Grading Formula are based on different prediction criteria. The Fry Readability Graph predicts the reading level that a student must have to be able to read the text with 50 to 75 percent comprehension. The SMOG Grading Formula attempts to predict the reading level necessary to read with 90 to 100 percent comprehension.

The results obtained predict that a Grade 12 reading level is required to read *Starting Points in Math 10* with 50 to 75 percent comprehension and that a Grade 13 reading level is required to read the text with 90 to 100 percent comprehension. The results

Figure 2. Fry's Readability Graph.

[From "Reading Level Determination for Selected Texts" by K. Kennedy, *The Science Teacher,* 41 (March 1974): 26.]



suggest that a reading level of Grade 13, which is three levels above the intended grade level of the user, is required to be able to read and fully understand the text.

The text was further tested by evaluating individual students' comprehension. The students were divided into three groups and randomly assigned a cloze test on the same passages selected for the readability formulas. A cloze test is a mutilated passage in which every fifth word or symbol from the passage has been deleted and replaced with a blank. In contructing cloze tests for mathematics texts, not only words but symbols such as >, % and 5 may be deleted. The student is then required to fill each blank with the exact word or symbol according to the original text material. The cloze procedure allows readers to use their knowledge of language patterns and their ability to respond to contextual clues (Malo 1978).

The cloze procedure is advocated as a measure of the readability of "mathematical English" by Hater and Kane (1975). In 1970, they conducted a study to adapt the cloze procedure to the language of mathematics and to assess its behavior as a measure in that language.

The scores obtained by the students on each test were separated into three categories:

0%—43% correct—frustration level
44%—57% correct—instruction level
58%—100% correct—independent level

Bormuth (1968) found that a score of 75 percent on conventional comprehension tests is comparable to a score of 44 percent on a cloze readability test made from the same passage. The three levels listed above have been accepted as a standard when interpreting cloze test results.

The percentage score means achieved on passages from *Starting Points in Math 10* were 57.38, 55.52 and 71.50 for passages 1, 2 and 3, respectively. These scores suggest that passage 3 may have been easier than passages 1 and 2, perhaps due to familiarity with the topic (Pythagorean Theorem). Averaging the percentage score mean of each passage provides a text mean of 61.47. The text mean of 61.47 falls into the independent level, but is only slightly above the instructional level of 44 to 57 percent. Because students were familiar with the content of the cloze tests (which may have affected the scores), one can conclude with a degree of certainty that the text assessed is suitable to be used if instructional support is provided.

One must always keep in mind that readability formulas and cloze tests are tools that can be used to assess the readability level of texts. Readability scores give an approximate grade level for materials and should be used as guides rather than absolute values. Knowing the readability level of a particular text can influence whether one will adopt it for a group of students.

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Combining Literature and Mathematics

Making Math Books and Finding Math Concepts in Books

Bernard R. Yvon and Jane Zaitz

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There is more to teaching math than one textbook or many worksheets. Counting books, geometric shape books and many other math concept books can be made and shared by children. Hundreds of library and classroom books contain math concepts to be discussed, written and even acted out. Factual books can be used to make graphs and charts. Poetry books about numbers can be read, and similar ones can be written by children themselves. Numerous activities and projects, including the making of books can be undertaken to motivate math students and involve them in the everyday world of numbers and literature.

Making Math Books

Books come in all sizes; the "big book" is becoming popular for shared reading experiences in whole language programs. Large groups of children can read in unison from the pages of these books, which are printed with large type. Such reading is fun and noncompetitive. However, instead of buying expensive copies of such books, why not make them? Have each child make one page. The making of a book requires skills in language arts, creative writing, art and, if it is a counting book, math.

A Grade 2 class in one school enjoyed making a book that they entitled "The Colorful Counting Book" as a gift for the kindergarten and Grade 1 classes. The Grade 2 students cooperated and shared ideas while compiling the book. In the first lesson, each child selected a number and object(s) such as balls, trains, lollipops, gifts and balloons, which would attract young children. The students did their drawings on an 8" X 12" (20 cm × 30 cm) paper that had a ruled line across the bottom for the copy.

In big books, the lettering should be as large as possible—½ inch (1 cm) for lower case letters and 1 inch (2 cm) for upper case letters—so that a group of children can read the book from a distance. Some children may need to practise making big letters on separate sheets of paper. After practising lettering, their sentences should be checked for errors in spelling and punctuation.

After students have completed their sentences, they can begin their drawings. Drawings should initially be done on smaller paper, then transferred to the larger sheets that make up the big book. If the book is a counting book, numerals should be written in large squares on the top right corner of each page. Once the drawings and lettering are done, the students can assist the teacher in putting the book together.

Pages should be glued back-to-back, that is, page 1 is glued to page 2; page 3 is glued to page 4, and so on. Each page becomes sturdier and more durable when reinforced with another sheet of paper. If possible, the pages should be laminated to protect them from wear and tear. The front and back covers can be made by students who have finished their pages. Holes should be punched in the sheets, then two large rings inserted to hold the book together.

The fun continues when the children see their finished product and read it among themselves. If they've made the book for another class, they can perform an oral presentation when giving the book. Each of the writers can read his or her own page. To do so will provide each with a sense of accomplishment. The students who receive the book will be very excited. To know the authors and even, at times, have played with them in the schoolyard is exciting.

Counting books are only one of the many different kinds of math books that a class can make. A colorful geometric shape book with two- and three-dimensional shapes was made by students in a Grade 8 class and left as a gift to their successors when they graduated into high school. The procedure described above was used. Students referred to their math text and their teacher while making the more difficult geometric shapes.

A simple sequence book like Maurice Sendak's *Seven Little Monsters*, which shows seven monsters doing different things, could be tried. Any child can create a favorite character and make several pictures of that character performing different, humorous acts. The potential for creativity with a simple math concept is great and can be used to turn young math geniuses loose.

A poetry book like the classic *Over in the Meadow* by Olive Wadsworth creates wonderful sets of 10 directions between mother animals and their offspring in an easily reproduced rhythm pattern. Creative students in Grades 4 through 8 could write similar books of directions from parents to children, or teachers to students. The potential for fantastic illustrations and humor are enormous. The finished product could remain in the classroom or be shared with or acted out for other classes. The whole school will be inspired to make other poetry or math-related books.

A dictionary of difficult mathematical terms could be made, illustrated and shared by upper class students. It would serve as a handy reference book and would help to develop students' math vocabulary. A small binder to which pages are easily added could be used to make this dictionary.

A book showing parts of favorite desserts, cut up into servings, could be helpful in teaching students about fractions. A story about a growing family in which the parents keep dividing food into smaller and smaller portions could place fractions into a humorous account of family life. Metric units and decimals could be explained in a book in which each child selects a unit or term and draws a picture that illustrates the length or size of the unit. Graphing and charting activities can also be a part of this creative process of making math relevant to members of the class. After they start producing books, reading the books of others and making graphs on particular units of study, students will come up with more and more ideas in brainstorming sessions.

Nonfiction

Books of facts such as the *Guiness Book of World Records*, *The Book of Lists* (for the middle and upper grades) and *Do You Know? One Hundred Fascinating Facts* by Random House (for younger children) list intriguing data for children to study and make into graphs and charts. For example, a child who is interested in the speeds that animals can travel can make a graph that illustrates animals, from slowest to fastest moving. Older children can chart or graph information about buildings, sports, populations, speeds of vehicles or other subjects that interest them.

Logic, order and planning all go into making a graph or chart. Handsome finished products can be

displayed, shared and discussed by young mathematicians. Follow-up activities can include the construction of intriguing problems and questions that require the interpretation of data from the charts and graphs. Each student is an authority on his or her graph and can verify others' results with his or her expertise. Positive classroom dynamics are at work; each child is king of his own castle and knight at his neighbor's castle.

The variety of factual material is overwhelming. Teachers can search libraries for books related to science, social studies, history, geography and other areas of interest.

Acting Out Stories with Math Problems

Many stories and nursery rhymes have number and math concepts in them. "The Three Little Pigs," "The Three Bears" and "The Three Little Kittens" all have sets of three. Instead of just reading these stories to younger children, the teacher can organize a reading group to read them or find volunteers to act them out. Puppets are a good way for shy children to begin experiencing drama. In the book *The Teacher Who Could Not Count,* by Craig McKee and Margaret Holland, students teach their teacher to count by acting out each number with their bodies. Games in which numbers are acted out and guessed can be great rainy day activities for children. Remember, Roman numerals need extra cooperation and teamwork.

In many books, such as *It Could Always Be Worse* by Margot Zemach and *Mushroom in the Rain* by Mirra Ginsberg, people or animals are added to the original set. In the Yiddish folktale, *It Could Always Be Worse,* a poor family that lives in an overcrowded house keeps adding, on the advice of the rabbi, more inhabitants (animals) to the house. After the father can no longer stand the overcrowded conditions, the rabbi advises him to return the animals to the shed. Life seems peaceful and pleasant after the animals depart. The equation that corresponds to the story is 6 (children) + 2 (parents) + 1 (grandmother) + 3 (fowl) + 1 (cow)–4 (animals) = 9 (the original number of family members).

Good listening and math skills are demanded of the children so that they can write an equation on the board after listening to a story. With practice, the children can reverse the process and tell a story from a simple addition or subtraction sentence. Their original stories can be acted out as well.

Math and books . . . books and math equal a fun learning experience. With a little imagination and a big desire to relate math to other areas of the curriculum, every teacher and class can make books that excite and motivate. Likewise, abstracting mathematical equations, activities and concepts from books may take some time and planning, but will make math the most exciting class of the day. The bibliography that follows will help you start an integrated math program that combines the world of numbers with the world of children's literature.

Procedure for Making a Big or Little Book with Your Class

1. Have a planning session in which children select a number or a math concept.
2. Have them draw a picture and write a sentence describing the picture, which illustrates the number or math concept. This should be done on draft paper.
3. Have the students practice large printing, if necessary.

4. On large sheets of paper (construction paper works well) draw a box in the top right hand corner, where a number will be written.
5. Check students' lettering and spelling, and make corrections or additions before allowing the students to start their final copy.
6. Make front and back covers. On the front cover, indicate the class, the year and the names of the authors.
7. Laminate or use clear contact paper.
8. Punch holes and place reinforcements around the holes for strength.
9. Use two 2-inch rings to hold the book together.
10. Read and share the book with other classes.
11. Hang the book on a coat hook to store it.
12. Take photographs of the book for the class bulletin board, class journal or newspaper.
13. Have fun!

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A counting book that is beautifully illustrated with country scenes. On the first page, zero is indicated with an empty landscape; the next page has one piece of scenery. As the number increases, so does the number of objects filling the landscape. A fun, natural way to count. The last page talks about early number systems and one-to-one correspondence.

Charlip, Remy, and Terry Joyner. *Thirteen*. New York: Parents' Magazine Press, 1975.

A wordless concept book consisting of 13 picture sequences in which shapes evolve into new forms. Good for developing observation skills among children of all ages. Needs introduction.

Emberley, Ed. *Ed Emberley's Drawing Book of Animals*. Boston: Little Brown, 1970.

A wonderful book that teaches how to draw animals with simple lines, squares, triangles and angles. Has fun art lessons for all ages.

Hillman, Priscilla. *The Merry Mouse Counting and Colors Book*. New York: Doubleday, 1983.

A small, square cardboard book with colorful drawings of mice. Counting up to 10. Written in verse.

Hoban, Tana. *Circles, Triangles and Squares*. New York: Macmillan, 1974.

Beautiful black and white photographs of city scenes and everyday objects show the three most common geometric shapes for children to identify.

Other books by Hoban include *Shape and Things*, *Look Again*, *Push-Pull*, *Empty Full: A Book of Opposites*, *Count* and *See and Round and Round and Round*. Each could be used to promote discussion or provide follow-up activities for young children.

Hutchins, P. *One Hunter*. New York: Greenwillow, 1982.

One hunter meets up with 10 African animals hidden in the forest and walks by each camouflaged set. A humorous account that allows for guessing and counting. (Preschool to Grade 2)

Mathews, Louise. *The Great Take-Away*. New York: Dodd, Mead and Co., 1980.

One hog in a town of pigs steals. In rhyme, with five subtraction problems to solve. (Grades 1 to 3)

McKee, Craig, and Margaret Holland. *The Teacher Who Could Not Count*. School Book Fairs, Inc., 1981.

A story about a mixed-up teacher who makes mistakes in learning to count. Her students act out the numbers with their bodies to teach her properly. Great for rainy days or for number games in physical education.

Merriam, E. *Project 1.2.3.* New York: McGraw-Hill, 1971.

A fascinating book for urban or rural children to learn about life in a huge complex. Has eight pages at the end for observation and counting.

Oxenbury, Helen. *Helen Oxenbury's Numbers of Things.* New York: Heinemann, 1967.

A counting book about a lion. Simply but humorously illustrated. Depicts numbers 1 to 50.

Pienkowski, Jan. *Numbers.* New York: Harvey House, 1975.

Numbers is similar to Pienkowski's books *Colors, Sizes* and *Shapes.* The numbers one through 10 are illustrated with objects in a natural setting. On the opposite page, an abacus shows combinations of 10. (For two- to six-year-olds)

Random House. *Do You Know? One Hundred Fascinating Facts.* New York: Random House, 1979.

Lots of facts about things smallest to largest, from animals to vehicles. Ideal for graph and chart making.

Scarry, Richard. *Richard Scarry's Best Counting Book Ever.* New York: Random House, 1975.

A counting book in which Willy Bunny counts everything he sees. Goes to 100. Ideal for playing such games as "You Find It." Ask a child such questions as "How many firemen have green mops?"

Sendak, Maurice. *Seven Little Monsters.* New York: Harper and Row, 1975.

A simple, short account of seven monsters who get into trouble. The book could provide inspiration to children for making their own sequence books.

Shapiro, A. *Mr. Cuckoo's Clock Shop,* Los Angeles: Intervisual Communications, 1978.

A rhyming story about a clock shop with a large clock that has movable hands. The reader moves the time ahead one hour per page.

Silverstein, S. *The Missing Piece Meets the Big O.* New York: Harper and Row, 1981.

A triangle searches for his whole and meets many disappointments until the Big O tells him to wear off his edges and become a circle. A good introduction for young children to various shapes.

Wadsworth, Olive. *Over in the Meadow: A Counting Out Rhyme.* New York: Viking-Kestrel, 1985.

A counting book of the numbers one to 10. Ideal for ideas when asking a class to make their own books.

Warren, Cathy. *The Ten-Alarm Camp-Out.* New York: Lothrop, Lee and Shepard Books, 1983.

A story about a mother armadillo and her nine babies who like even numbers. They have a strange camping adventure. An enjoyable story with counting practice. (Preschool to Grade 2)

Writing in Mathematics

Irene Eizen and Arlene Dowshen

Mathematics instruction involves the interrelationship of two languages: the native language and the language of mathematics. Just as we communicate our native language via an extensive and comprehensive set of symbols and vocabulary, so too do we communicate the language of mathematics. However, the language of mathematics differs from the native language in that it knows no cultural or geographic boundaries, thus it is unique and universal. Furthermore, vocabulary shared by both languages has a different meaning in a mathematical context than it has in a native language.

There are four developmental phases in language arts: listening, speaking, reading and writing. They are interrelated. Speaking and writing are closely related in that they impart language. They combine with the receptive phases, listening and reading, to facilitate growth in and mastery of language as a whole. The same four phases are essential to learning mathematics, the language and the body of knowledge.

It is recognized that success in mathematics is correlated with the ability to read mathematics. Just as listening and speaking are precursors to speaking, reading is a precursor to writing. Writing is, therefore, the most complex of the four phases of language arts.

This article will focus on writing and how writing should be used to reinforce mathematical content and afford students opportunities to develop higher-order thinking skills through analysis and synthesis of content. Not only will mathematical content be reinforced, but writing, which traditionally is taught and practiced exclusively in English classes, will be freed from the confines of this single subject.

Writing is a powerful learning tool. Yet, while writing occurs in almost all classes, it is frequently in the form of notetaking or other types of informational writing. Usually classwork, homework and other assignments demand of students only rote feedback of information. Text is infrequently constructed, thus mechanical writing seems to be emphasized. Thinking is outside the context of such written expression. The powerful potential of writing cannot thereby be realized. By contrast, writing in imaginative and creative ways involves thinking at higher levels and provides opportunities for students to analyze and synthesize content and to express ideas and relationships in unique forms.

What are some of these forms? How can mathematics teachers use writing in the teaching of mathematics to reinforce mathematical content? Some examples from our teaching experience will be presented in this article.

Clearly, it is important that writing in the mathematics curriculum emphasize content. Stephen Tchudi, talking about content as the focus of the writing process, states:

> Teachers . . . know their discipline well. . . . If they will simply keep a focus on helping students to express ideas in the discipline clearly, matters of content, form, style and even mechanics will be taught in context as appropriate. (Tchudi, p. 24)

Whereas writing in other content areas largely involves the use of native language to express ideas, writing in math allows students to reinforce their facility with the language of mathematics, as well. Writing in various subjects affords teachers a special opportunity to communicate with one another,

Reprinted from Washington Mathematics *32, no. 2, Winter 1987. This article previously appeared in* ATMOPAV Newsletter *28, no. 2, Fall 1987, published by the Association of Teachers of Mathematics of Philadelphia and Vicinity.*

especially with the English teacher who teaches writing as a principal part of the curriculum. Ideally, mathematics teachers who intend to assign creative writing should consult with English teachers prior to doing so to ensure that they are reinforcing the writing standards taught in English class while their students write in a mathematical context. From the same perspective, writing in mathematics should help other subject areas. All lessons require reinforcement to be learned, and writing is no different.

Our first effort in implementing a "writing in mathematics" program in our classes is to motivate our students to want to write. To do this, we must be motivated ourselves. Our attitudes are contagious. If we are involved and excited about writing, our students will share this excitement. Next, the classroom environment must be conducive to mathematics language skill development, that is, our students' use of spoken and written words must be encouraged and reinforced. Verbal exchanges in a brainstorming session are an important prewriting activity.

To create an atmosphere of discovery requires inductive teaching. It is believed that students learn more effectively through self-discovery of ideas, using language to express these ideas, verbally and in written form. The problem-solving process is at work here. This is the antithesis of the common "show-and-tell" method of teaching, which can't lead to true discovery by students. Thus, writing in mathematics must be viewed in conjunction with problem solving. This was the number one recommendation of the National Council of Teachers of Mathematics for mathematics education in this decade.

As with problem solving, writing activities should begin with the group; they should be a class endeavor. This endeavor should include the four stages in the writing process: prewriting, writing, rewriting and editing. Prewriting involves the verbal exchange of ideas. Along with the class, discuss the following: What type of writing shall we do today? Shall we write a paragraph to clarify and show the interrelationship of terms we've learned in each unit? Should we do some imaginative writing, expressing some mathematical content through a poetic form? The prewriting stage is vital to the writing process for it is through the exchange of ideas that plans are made for what to write about (the content focus) and what writing medium will be used.

Once we write, we need to examine what and how we are writing. Rewriting and editing are important activities for achieving a high quality final product. At these stages a variety of questions should be explored: Have we kept content the focus of our writing? Are the relationships clearly expressed? Have we spelled, capitalized and punctuated correctly? How can we build on the strong points of our writing? Do we need to do some research for further writing, to expand that which we have already done? Is there anything we wish to delete? Several drafts may be necessary to achieve the finished product.

Following are several examples of writing that focus on mathematical content. Most of the writing was done by Grade 7 students at Baldi Middle School.

1. **"Splish-Splash, I Was Taking a Bath: A Mathematical Version"** *by Michael Marcus. In limerick form.*

   Archimedes had no clothes or shoes,
   As he ran through the streets of Syracuse.
   He found out, and not too late,
   Displaced water was his weight.
   And he ran to tell the people this news.

2. **Mathaiku** *by Class B7-4*

   The problems in math
   Are real challenges to us
   Be we still solve them.

3. **Hidden Sequence** *by Michael Ganetsky. An example of free verse.*

   Once
   There
   Lived a
   Boy, whose name
   Was Leonardo.
   We know him as Fibonacci.
   He studied the reproduction of rabbits to find
   A sequence of numbers named for him, which
   can be found by counting the syllables in
   Each line of this free verse which can't continue
   indefinitely as does the special sequence of numbers discovered by Fibonacci.

4. **Euclid** *by Jeffrey Jenofsky. A narrative poem with the rhyming pattern* a,a,a,a,b,b,c *and four beats per line.*

   Euclid lived in 300 B.C.
   He wrote 13 books on geometry.
   The king asked, "Euclid," I quote he,
   "Is there an easier way to geometry?"
   Euclid replied with some sense,
   "You must read the *Elements*.
   There is no royal road to geometry."

5. A **mathematical version of "One"** *from* A Chorus Line *by Class B7-4*

   One, mathematical sensation,
   Can be called by many a name.
   One, when used in multiplication,
   The product will remain the same.
   Whole, natural, integer, rational, real, square, too,
   Triangular, positive, cube, just to name a few.
   One, in base ten numeration
   Can be used in any place.
   One, as a power keeps the same base, base,
   Ooh, sigh, give one your attention.
   Do we really have to mention,
   "One's the one!"

6. **Popular song titles**—*mathematical, of course! Contributions made by all students in the class.*

   One More Point by Phil Colinear
   Automorphic by the Seventy-Sixers
   Walk Like a Mathematician by the Angles
   Hold Three Now by the Thompson Triplets
   I'll Be Square the Jackson Twenty-Five
   The Bunny Hop by Leonardo Fibonacci
   Tapestry by M.C. Escher
   Lune River by Hipparchus

7. **The Definition of Confusion**
   Expository writing by Grade 8 math students.

   A *factor* of a number *divides* the number. Therefore, the *factor* is a *divisor* of that number, which in turn, becomes a *multiple* of the *factor*. When the *factor* is *divided* into the *multiple*, another *factor* is produced. The pair of *factors* gives a unique *product*. Since each *factor divides* the *product*, the *product* becomes a *dividend* in a *division* problem and the *factors* become the *quotient* and *divisor*.

8. *Selections from* **An Anthology of Problem-Solving Poems**

   The following two poems were group efforts; the first was led by Michael Marcus and the second by Michael Ganetsky.

   I

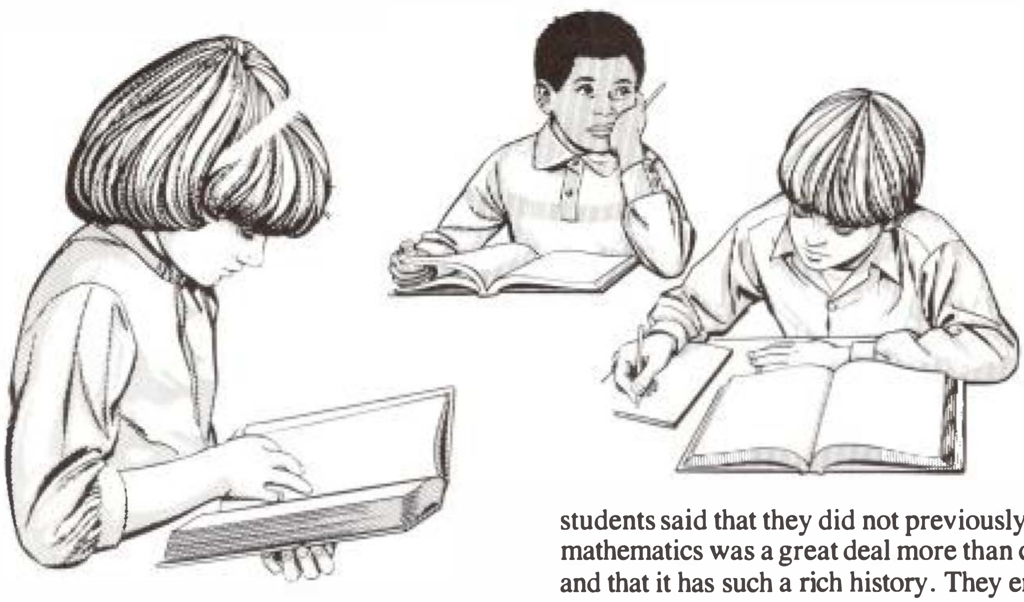
   When we solve a problem
   Here's what we do.
   First, we have to read the problem through.
   Next, we see what we're looking for.
   Then we read the problem once more.
   Now we write the important information,
   And using these facts we look for a relation.
   Next, we pick from the many strategies
   And solve the problem using these.
   Last, we check our answer to see
   If it fits in accurately.

   II

   When you solve a problem
   You use your head.
   But you can't glace
   For a problem must be read.
   Next, you have to know
   What you have to find.
   Read the problem over again
   Then shuffle it in your mind.
   You must note the key facts.
   Then list this important information.
   Next, you decide on a strategy,
   Like writing an equation.
   Another one is make a chart
   Or you can often guess and test.
   Depending on the situation
   Any one could work best.
   You need to check the answer,
   The last step left to do.
   And if you find the answer fits,
   You'll know that you are through.
   But if your answer doesn't check,
   And you did not succeed,
   Go back and read this poem again
   To find out how to proceed.

Several of the examples shown above focus on the history of mathematics. History should be part of the total math program to help students develop an appreciation of mathematics and the interrelationship of its branches. Independent research and writing projects such as students' autobiographies, biographies of famous mathematicians or articles on special historical or technical topics provide students with information to foster further creative writing.

Consistent with curricular planning, of course, most written work by students is done at home (after the classroom motivational sessions mentioned above). Occasional individual conferences can be held, if necessary, to provide comments on revision, editing and final drafting. To encourage and maintain the students' interest, comments should focus on the strengths of each student's work and on the content. The mechanics of writing should be noted but not emphasized.



When the first piece of writing has been completed, display it prominently. Share it with other staff members and pupils. Exhilaration, excitement and the desire to continue to write will likely be evident among your students. Let it be known that writing has been freed from the confines of the English classroom. This opens up a multitude of possibilities, limited only by the imagination!

We have found that students' attitudes have improved and changed this past year. Our students have been exposed to a language arts program in mathematics classes for an entire year; they recently expressed their reactions to this program in a very positive way. In an informal survey, the majority of students said that they did not previously realize that mathematics was a great deal more than computation and that it has such a rich history. They enjoyed writing about this history and other topics. They listened to and read each other's work. They wrote, they learned and they enjoyed!

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Algebra Can Be a Language

George A. Calder

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Effectively teaching word problems in algebra is probably as difficult for teachers as solving the problems is for students. Some textbooks approach word problems as individual sets. That is, age problems are treated in one section, mixture problems in another section and motion problems in a third section. Other textbooks purport to intersperse word problems throughout the text.

Most algebra teachers adapt one of the above treatments to their own teaching styles. As new textbooks are adopted, the teachers may or may not adjust. But regardless of textbook treatments and teacher styles, the topic of word problems in algebra has traditionally been a low point in most algebra classes (for both teachers and students).

One technique for teaching word problems is to treat "Algebra" (with a capital "A") as a language. Indeed, one can make a very good argument that algebra *is* the language of mathematics. What is suggested in this article, however, is that Algebra is a language, like English, Spanish or German, and that word problems can be handled as if one was "translating" from English into Algebra.

The first step in learning a new language is usually to buy a dictionary. No such item exists for Algebra. Thus, we are forced to create our own. Before introducing word problems in an algebra class, a teacher might use a worksheet similar to the one shown here.

Reprinted from Mathematics in Michigan *27, no. 2, Winter 1988.*

DICTIONARIES

English	French	German	Spanish
white	blanc, blanche	weiss	blanco, blanca
the dog	le chien	der Hund	el perro
the house	la maison	das Haus	la casa
is	est	ist	es, esta
closed	fermee	geschlossen	cerrada

Translate the following into English:

Example: Le chien blanc. Answer: The white dog.

1. Der Hund ist weiss.
2. El perro blanco.
3. La maison est fermee.
4. El perro es blanco.
5. Das Haus ist geschlossen.
6. Der weisse Hund.
7. Le chien blanc.
8. Le chien est blanc.
9. Das Hundhaus ist weiss.
10. La casa esta cerrada.

Translate the following into the language indicated:

11. Spanish.....................The house is white.
12. French......................The house is white.
13. German.......................The dog is white.
14. German...............The doghouse is closed.
15. Spanish...........................The white dog.
16. French.........................Der weiss Hund.

An example on the worksheet can be used to point out that, in some languages, the adjectives come after the nouns. Thus, while a word-for-word translation of the examples would be "the dog white," the rules of grammar for English would require "the white dog'' as the correct translation. In the worksheet itself, however, students should be allowed word-for-word translations. The example is good to recall later when a student must translate ''three less than John's age'' as ''j − 3'' rather than word-for-word as ''3 − j'' (a common error among students). In other words, Algebra also has it's own set of ''grammar rules'' when making translations.

After students have completed the worksheet, it can be pointed out that even though they do not know all (or perhaps any) of the languages, they *can be successful* in doing something brand new when given appropriate aids (like a dictionary).

The next day's lesson, then, is spent introducing the concept of a ''dictionary'' for algebra word problems. Simple age problems are ideal for introducing this concept. Here is an example:

Problem

Mary is three times as old as Bob. The sum of their ages is 28. How old is Bob?

DICTIONARY

	Age in years
Mary	3b
Bob	b *

Students are taught to complete the dictionary in the following order:

1. List the ''characters'' of the story (word problem) down the left-hand column. (Mary and Bob)
2. List the ''characteristics'' of those characters across the top row. Be sure to include the measures, i.e., years not months. (Age in years)
3. Put an asterisk in the box you are asked to find. (Bob's age in years)
4. Choose a variable and put it in any box. Usually it will be the box with the asterisk, but not always. Sometimes it will be in the box with the least amount of information given in the problem. Don't always choose ''x'' or ''y,'' but instead, choose a variable that will remind you of what it represents. (b for ''Bob'')
5. Fill in all other boxes using information for the problem.

After the ''dictionary'' is complete, use it to translate the story from English into Algebra. Then solve the resulting algebra equation. Finally (and this is important), translate the Algebra answer back into English by writing an English sentence.

Translation $3b + b = 28$

Solving $4b = 28$

$b = 7$

Answer: Bob is seven years old.

Some students may even discover that the dictionary could be as follows:

ALTERNATE DICTIONARY

	Age in years
Mary	m
Bob	(⅓)m*

Translation $m + (1/3)m = 28$

Solving $(4/3)m = 28$

$m = 21$

$(1/3)m = 7$

Answer: Bob is seven years old.

Notice that in the alternate solution the answer to the equation (21) is *not* the answer to the story problem. This is why the asterisk is placed in the dictionary—to remind the student of what to look for.

After a day's practice with simple age problems, the students can be exposed to more difficult age problems such as follows.

Problem

Kim is four years older than Dean. Eight years ago, Kim was twice as old as Dean was. How old is Kim now?

DICTIONARY

	Age in years	
	now	8 years ago
Kim	d + 4*	d + 4 − 8
Dean	d	d − 8

Translation	$d + 4 - 8 = 2(d - 8)$
Solving	$d - 4 = 2d - 16$
	$-4 = d - 16$
	$12 = d$
	$d + 4 = 16$

Answer: Kim is now 16 years old.

Notice how the concept of "8 years ago" automatically requires the subtraction of eight (− 8) in equations for every character in the story, whether or not that information is later needed. A concept of "10 years from now" would, of course, require "+ 10" in its equations.

Later in the school year, when students have studied two variable systems of equations, the same dictionary can be used, with "d + 4" changed to "k" (for Kim) and "d + 4 − 8" changed to "k − 8."

Age problems can become quite complicated with the addition of more characters and the consideration of their ages at present, in the past and in the future, all at the same time.

The use of such a dictionary in all word problems not only forces the student to read and re-read the problem many times (something many students never

bother to do), but also helps the teacher to pinpoint where a student's reading of the problem is incorrect. Even the simple act of placing an asterisk can show which students do not know what they are supposed to find. How can a teacher expect a student to find something in a problem, if the student's comprehension of the words is so flawed that he or she can't even tell what to look for?

Summing Consecutive Counting Numbers

Bonnie H. Litwiller and David R. Duncan

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Figure 1 contains indicated sums that have been generated by adding sets of consecutive counting numbers and then skipping sets of consecutive counting numbers.

Figure 1

1) 1 + 2
2) 4 + 5 + 6
3) 9 + 10 + 11 + 12
4) 16 + 17 + 18 + 19 + 20
5) 25 + 26 + 27 + 28 + 29 + 30
6) 36 + 37 + 38 + 39 + 40 + 41 + 42
7) 49 + 50 + 51 + 52 + 53 + 54 + 55 + 56

Observe that the first indicated sum contains the first two counting numbers. Then, one counting number is skipped. The second indicated sum contains three consecutive counting numbers, and then two are skipped. The third indicated sum contains four consecutive counting members, and then three are skipped.

The skipped numbers are shown in Figure 2.

Figure 2

1) 3
2) 7, 8
3) 13, 14, 15
4) 21, 22, 23, 24
5) 31, 32, 33, 34, 35
6) 43, 44, 45, 46, 47, 48
7) 57, 58, 59, 60, 61, 62, 63

Activity 1

Find the indicated sums of Figure 1. Then sum the skipped numbers as well. Now compare the results of your computations.

The sums of Figure 1 and Figure 2 are both 3, 15, 42, 90, 165, 273 and 420. Since the two sets of sums are equal, we may write:

$$
\begin{aligned}
1 + 2 &= 3 \\
4 + 5 + 6 &= 7 + 8 \\
9 + 10 + 11 + 12 &= 13 + 14 + 15 \\
16 + 17 + 18 + 19 + 20 &= 21 + 22 + 23 + 24 \\
25 + 26 + 27 + 28 + 29 + 30 &= 31 + 32 + 33 + 34 + 35 \\
36 + 37 + 38 + 39 + 40 + 41 + 42 &= 43 + 44 + 45 + 46 + 47 + 48 \\
49 + 50 + 51 + 52 + 53 + 54 + 55 + 56 &= 57 + 58 + 59 + 60 + 61 + 62 + 63 \\
&\vdots
\end{aligned}
$$

Observe that the first term of each row is a perfect square and the last term of each row is one less than the next perfect square. Also, the number of terms to the left of the equal sign is one more than the number of terms to the right.

Activity 2

Find the differences of the sums of Figure 1. After how many subtractions will a constant emerge?

3			
	> 12		
15		> 15	
	> 27		> 6
42		> 21	
	> 48		> 6
90		> 27	
	> 75		> 6
165		> 33	
	> 108		> 6
273		> 39	
	> 147		
420			
•			
•			
•			

Note that after three subtractions the constant 6 emerges.

Activity 3

Write the first seven squares and subtract them from the sums of Figure 1. Now find differences. Will a constant emerge?

```
  3 -  1 =   2
               >  9
 15 -  4 =  11       > 13
               > 22       > 6
 42 -  9 =  33       > 19
               > 41       > 6
 90 - 16 =  74       > 25
               > 66       > 6
165 - 25 = 140       > 31
               > 97       > 6
273 - 36 = 237       > 37
               > 134
420 - 49 = 371
```

The constant 6 appears after three subtractions.

Activity 4

This time write the first seven cubes and subtract them from the sums of Figure 1. Again find differences. Does a constant appear?

```
  3 -   1 =  2
               >  5
 15 -   8 =  7       > 3
               >  8
 42 -  27 = 15       > 3
               > 11
 90 -  64 = 26       > 3
               > 14
165 - 125 = 40       > 3
               > 17
273 - 216 = 57       > 3
               > 20
420 - 343 = 77
```

After only two subtractions, the constant 3 appears.

Activity 5

Now write the first seven fourth powers. Then from the fourth powers subtract the sums of Figure 1. Find the constant differences. How many subtractions are needed to find the constant?

2401 − 420 = 1981				
	> 958			
1296 − 273 = 1023		> 395		
	> 563		> 126	
625 − 165 = 460		> 269		> 24
	> 294		> 102	
256 − 90 = 166		> 167		> 24
	> 127		> 78	
81 − 42 = 39		> 89		> 24
	> 38		> 54	
16 − 15 = 1		> 35		
	> 3			
1 − 3 = −2				

The constant 24 appears after four differences are computed.

Challenge

Consider summing sets of consecutive odd numbers or consecutive even numbers. What patterns can you generate?

Casino Gambling: The Best Strategy

Dennis Connolly

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Though the old problem of "Gambler's ruin" has long been solved (see W. Feller's "Introduction to Probability Theory"), a new study of it offers a strategy for the best odds in gambling at a casino.

We enter a casino with \$n and agree to play (repeatedly) at some game (craps, roulette, 21, etc.) for \$1 a play initially—where the probability of winning \$1 on one play is p and the probability of losing \$1 on one play is q, $(p + q = 1)$—until we either end up with \$N or with \$0 (ruined!).

The question that must be answered is, "What is the probability of being ruined (in terms of n, N, p and q)?"

Let R_n denote the probability of being ruined when you have \$n (n = 0,1,2, . . ., N). Clearly, $R_0 = 1$, for if we have \$0, ruin is certain. $R_N = 0$, for if we have \$N, we leave the casino.

With \$n, we make a play and we could be ruined in two mutually exclusive ways: we could "win the play and then be ruined" *or* we could "lose the play and then be ruined." It follows that

$$R_n = pR_n + 1 + q\,R_n - 1$$

The solution to this equation (with the boundary conditions $R_0 = 1$ and $R_N = 0$) is

$$R_n = 1 - n/N \quad \text{if } p = q = \tfrac{1}{2}$$

$$= \frac{(q/p)^N - (q/p)^n}{(q/p)^N - 1} \quad \text{if } p \neq q.$$

While it is not easy to come up with these solutions, unless we are familiar with difference equations, it is a simple matter to verify that they are indeed solutions to the given problem.

The effect of halving the bet is to replace n and N by 2n and 2N so that the probability of ruin with bets halved is equal to:

$$\frac{(q/p)^{2N} - (q/p)^{2n}}{(q/p)^2 - 1} = R_n \frac{(q/p)^N + (q/p)^n}{(q/p)^N + 1}$$

If $q > p$, then $(q/p)^n > 1$ and the right hand side is larger than R_n.

That is, if $q > p$, as is the case in a casino, the probability of being ruined increases if we halve the bets. It is immediately clear that we should bet big to lessen our chances of ruin.

Note that the probability of ruin remains constant if $p = q$, since $R_n = 1 - 2n/2N$.

The following is a list of some examples of various bets:

Example F1. Enter Fairyland Casino (where $p = q = ½$ for all games) with $9,000 and agree to leave when you have $10,000 or when you are ruined.

Bets	n	N	P(Ruin)	Expected Gain	Expected Duration
$1	9,000	10,000	0.1	0	27 yrs @ 15 hrs/day
$10	900	1000	0.1	0	100 days @ 15 hrs/day
$100	90	100	0.1	0	15 hours
$1000	9	10	0.1	0	9 minutes (9 plays)

Example F2. Enter Fairyland Casino with $100 and agree to leave when you have $200 or when you are ruined.

Bets	n	N	P(Ruin)	Expected Gain	Expected Duration
$1	100	200	0.5	$0	11 days @ 15 hrs/day
$10	10	20	0.5	$0	100 minutes
$100	1	2	0.5	$0	1 minute (1 play)

Example F3. Enter Fairyland Casino with $1 and agree to leave when you have $100 or when you are ruined.

Bets	n	N	P(Ruin)	Expected Gain	Expected Duration
$1	1	100	0.99	$0	99 minutes

Summary for gambling in Fairyland Casinos:

1. The expected gain is zero for all games in Fairyland!
2. The size of the bet (the "action") has no effect on the probability of ruin or on the expected gain.
3. The closer n is to N, the lower the probability of ruin.
4. The expected (average) duration is inversely proportional to the square of the action. The expected duration is determined by solving another difference equation (in another paper). For further discussion of the Classical Ruin Problem, see Chapter 14 of W. Feller's *An Introduction to Probability Theory and Applications,* published by John Wiley and Sons.

The following is a list of examples of playing the game of craps in a real casino:

Example R1. Enter any casino with $9,000 and agree to leave when you have $10,000 or you are ruined. For the game of craps: $p = 0.492\overline{9}29$.

Bets	n	N	P(ruin)	Expected Gain	Expected Duration
$1	9,000	10,000	$.5 \times 10^{10}$	−$9000	2 years @ 15 hours/day
$10	900	1,000	0.94089	−$8409	66 days @ 15 hours/day
$100	90	100	0.262	−$1620	19 hours
$200	45	50	0.174	−$740	4.37 hours
$500	18	20	0.127	−$270	38.6 minutes
$1000	9	10	0.113	−$130	9.3 minutes
Ultimate Strategy			0.1053	−$53	1.9 minutes
Fairyland			0.1000	$0	9 minutes

Example R2. Enter any casino with $100 and agree to leave when you have $200 or when you are ruined. Play the game of craps.

Bets	n	N	P(ruin)	Expected Gain	Expected Duration
$1	100	200	0.9442	−$88.62	104.70 hours
$10	10	20	0.5702	−$14.04	99.34 minutes
$50	2	4	0.5141	−$2.82	4 minutes
$100	1	2	0.507	−$1.42	1 minute

Example R3. Enter with $150 and agree to leave with eight $200 or $0.

Bets	n	N	P(ruin)	Expected Gain	Expected Duration
$1	150	200	0.7595	−$101.90	5 days @ 15 hours/day
$10	15	20	0.3053	−$11.06	78.14 minutes
$50	3	4	0.2607	−$2.50	3.03 minutes
Ultimate Strategy			0.25712	−$1.60	1.50707 minutes

Final Example R4. Enter a real casino with $1 and play on the craps table (‘‘come line’’) until you have $128 or you are ruined!

Action	n	N	P(ruin)	Expected Gain	Expected Duration
$1	1	128	0.9992	−89.90 cents	63.55 minutes
Ultimate Strategy			0.9929	−9.12 cents	2.17 minutes

The Ultimate Strategy

The Ultimate Strategy is simply to bet the maximum that your resources will allow, to achieve your stated goal.

For Example R1: Bet #1: $1000. If you win, you leave; if you lose then
bet #2: $2000. If you win, you leave; if you lose then
bet #3: $4000. If you win, you leave; if you lose then
bet #4: $2000. (All the cash you have left. A stiff upper lip is needed!)
If you lose, you leave; if you win then
bet #5: $4000. And so on.

For Example R2, the Ultimate Strategy is to simply bet the $100. You then either leave with $200 or leave ruined.

For Example R3, the Ultimate Strategy is simple again, comprising at most two bets. Bet #1: $50. If you win, you leave; if you lose, then bet #2: $100. You either leave with $200 or you leave ruined.

For Example R4, the Ultimate Strategy is to bet $1, then $2, then $4, then $8, then $16 and so on, as long as you keep winning, until you reach your preset goal of $128. You must win seven times running to avoid ruin.

Appendix A

The Game of CRAPS (Betting on the "Come Line")

Throw two dice; if the sum is 7 or 11, you win (even money); if the sum is 2, 3 or 12, you lose. If you shoot 4, 5, 6, 8, 9 or 10, this becomes your "mark" and you continue to throw the two dice until you shoot your "mark" and win (even money). If you shoot 7, you lose.

This is the very best (no decisions) game in any casino in that the probability of winning any one game is $p = 0.4929$; the probability of losing any one game is $q = 0.507$. This is as close to Fairyland odds ($p = q = 0.50$) as you can get for a "no-decisions game." Blackjack or 21 are "decisions games," and p can exceed 0.50 if you make the correct decisions (see E.O. Thorp's *Beat the Dealer*). The expected duration for a game of craps is 3.3757 throws.

Snooker Sam Gets Rich

A. Craig Loewen

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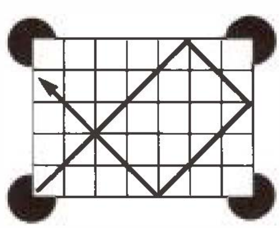
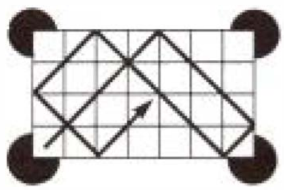
Snooker Sam was a sneaky sort
with that old snooker show.
He'd pack up his tables of varied size,
and off to work he would go.
On one sunny day, Sam rolled into town,
and set up his show in the mall.
And as the townsfolk gathered around,
they heard old Snooker Sam call:
"You see," said Snooker Sam, laying the bait,
"I'm a simple man, and honest, no less.
But if you'll give me a problem, with these tables here,
I'll bet I'm right whenever I guess.
If I set a ball at this corner here,"
and he pointed to the lower left with a grin,
"And I hit the ball diagonally, I wonder,
Can you guess which pocket it will fall in?"



5 x 7

Now the table Sam used was five foot by seven,
and at first the crowd didn't dare guess.
"You don't know either" came a cry from the back.
"Well," said Sam, "lets run this little test."
And with those words Sam struck the ball, and
it rolled from one edge to the other.
And as it rolled, he called "upper right,"
"Sure enough," cried the crowd, "Do another!"

"I'll tell you what," said the sneaky chap,
"Lets try this other table of mine.
It is four foot by seven, and the same rules apply, but
the ball will drop upper left, this time."



4 x 7

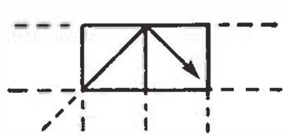
"Uh-uh," said one. "You're wrong," said another.
Said Sam, "Lets just try it and see," and
As I'm sure you have guessed, Sam was right once again
and it silenced them all quite effectively.
"How does he know? How does it work?"
Rang the incredulous cry through the crowd.
"Listen to me, I know!" said one little boy,
though he said it not very loud.
"Well, I'll do another, but it will cost you a dollar.
If I guess right, the dollar is mine.
If I am wrong," said the sneak, "The money is yours
and I'll not take up more of your time."

So the afternoon went, and on into the night
Sam continued to take money from the folk.
If only they had listened to that one little boy,
They'd have known it was just a cruel joke.
Now if you would like to be rich like Sam,
then you will have to know what he knows.
But, as for Sam, he's retired now;
one little boy put a stop to his shows.

How could Sam possibly predict the pocket that the ball would fall into regardless of the length of the sides of the table?

Here is another look at the rules:

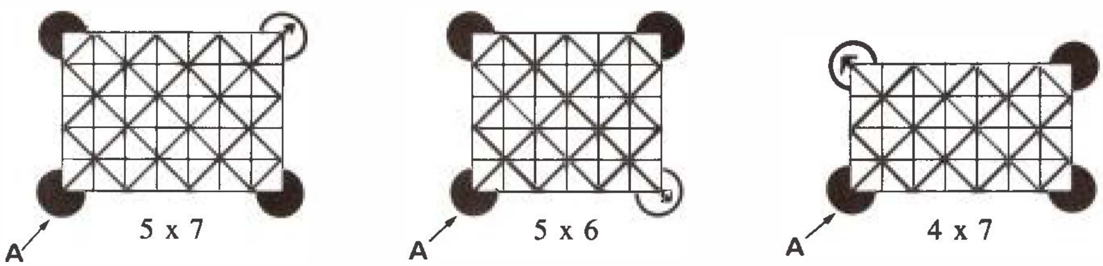
1. Each table is flat and rectangular.
2. Each table has only four pockets, one in each corner.
3. The ball always starts in the lower left.
4. The ball is hit diagonally at 45 degrees to the bottom wall.
5. When the ball collides with a wall, it rebounds to form a 90-degree angle.



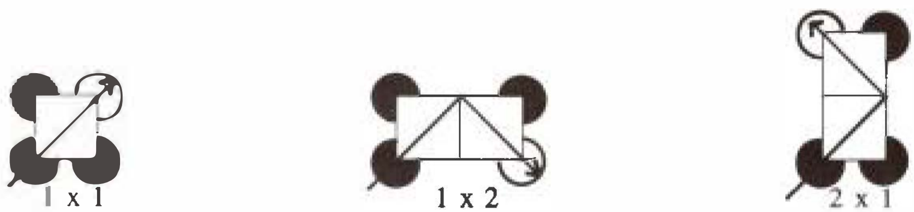
6. The ball continues traveling until it falls into one of the four pockets.

Solution

To solve this problem, consider some of the same table dimensions that Sam used.



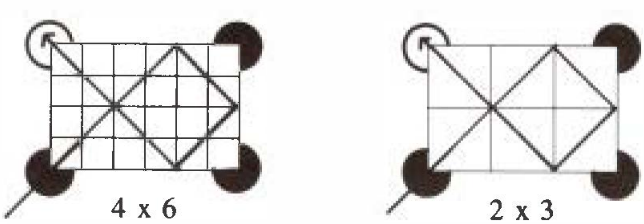
The ball always originates at Point A, so leave the bottom and left edges alone. If you remove the right-most column from the 5 x 7 table, you have an identical pattern to that of the 5 x 6 table. Likewise, if you take the top row off of the 5 x 7 table you are left with 4 x 7 table. In this manner, you can continue to strip off rows and columns until such time as you have one of these three trivial cases:



From these cases, it can be seen that any "odd by odd" table will land the ball in the upper right pocket while any "odd by even" table lands the ball in the lower right pocket, and any "even by odd" table lands the ball in the upper left pocket.

Each table is actually built upon the smaller tables which precede it in the series, and it is this property which dictates the pocket into which the ball will fall.

Be careful! Tables must be expressed in their lowest terms. For example, a 4 x 6 table is really just a 2 x 3 table.



Equal scaling horizontally and vertically does not change the pocket in which the ball will fall.

Can you think of other variations or solutions to this problem?

Erratum

In R. Scott Erickson's article ''Escher Revisited: Modeling Gradual Deformations Using Logo,'' published in *delta-K,* volume 27, number 1, June 1988, the three designs appearing on page 20 are improperly credited. ''Razor Blades'' was created at Carnegie-Mellon University, ''Consternation'' by Scott Grady and ''Fylfot Flipflop'' by Fred Watts are designs that originated at the Basic Design Studio of William S. Huff, professor of architectural design at the State University of New York at Buffalo. The designs originally appeared in ''Metamagical Themas'' written by Douglas R. Hofstadter and published in *Scientific American,* volume 249, number 1, July 1983.

R. Scott Erickson, John Percevault (editor of *delta-K*) and the Mathematics Council of The Alberta Teachers' Association apologize to William S. Huff and students at the Basic Design Studio and Carnegie-Mellon University.

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ISSN 0319-8367
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