The Psycho-Aesthetics of Combinatorial Sets

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Psycho-aesthetics is the concept of how the human mind observes, perceives and appreciates an achieved design, from the thrill of discovery through the pleasant contemplation of its beauty. The pursuit of knowledge, the wresting of order from chaos and the completion of a harmonious unity are the mind's most satisfying activities. Focusing these on the exploration of combinatorial sets can be a highly rewarding exercise that can carry over into more effective problem solving in the larger world.

Combinatorial sets are groups of geometric shapes formed by joining various multiples of the same basic building block along their unit edges, in all their possible relative positions. The simplest, best-known and most fascinating groups are those that consist of the three regular, convex polygons that can tile the plane. Here they are with their "family" names:

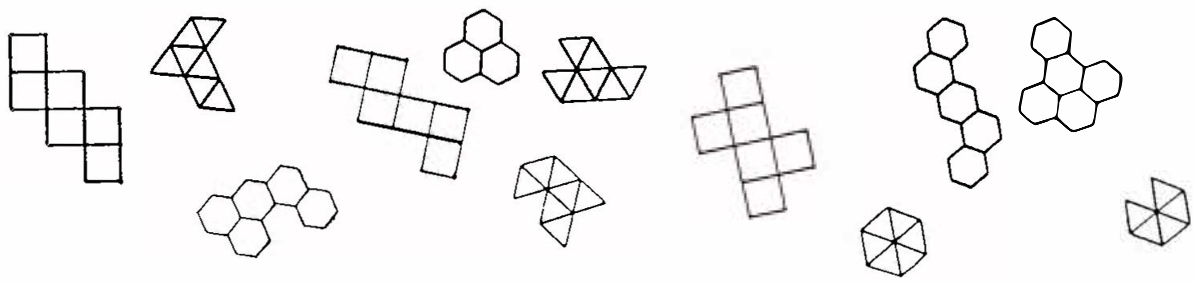
SQUARES—polyominoes
TRIANGLES—polyiamonds
HEXAGONS—polyhexes.

Combining ascending multiples of any one of these polygons yields progressively more intricate and more numerous distinct shapes. Figures 1a, 1b and 1c show the members of the three families, from unity up to the generally familiar levels or "orders" of size, together with their respective designations, as introduced in the landmark book, *Polyominoes,* by Solomon W. Golomb.

The amazing feature of all these shapes is that, for all their diversity and peculiarities, they can be fitted together with each other in coherent patterns, thus offering countless mathematical puzzles of surprising beauty and intellectual stimulation. Some unusual assemblies are shown in succeeding figures.

In the mind's eye, the rarer and more difficult to achieve are also the more precious and valued and the more attractive. Among the astronomical numbers of combinations possible with any polyform set, the most beautiful are those that have the elusive and special features of symmetry, regularity or self-replication. Symmetries are generally much more difficult to find and are more pleasing visually, sometimes to the point of approaching works of art.

Let it be borne in mind that the examples shown here are just as much the products of Nature's playfulness and ordered variety as are crystals, snowflakes, the DNA helix and the periodic table of elements, to name just a few. Readers are invited to judge whether their own minds don't resonate to these phenomena in a way best expressed by the colloquialism, "Oh wow!"



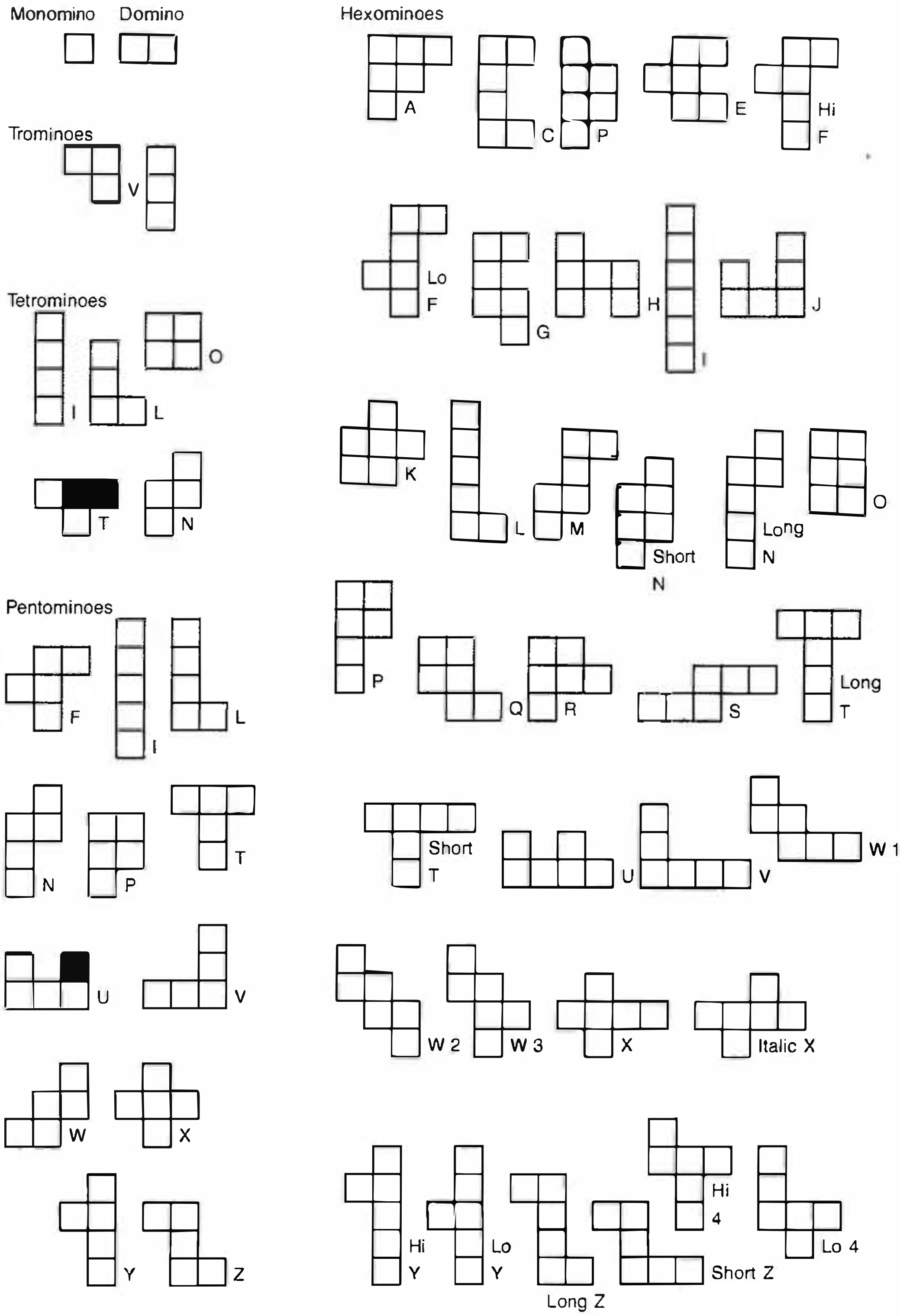


Figure 1a Polyominoes

Note—PENTOMINO is a registered trademark of Solomon W. Golomb.

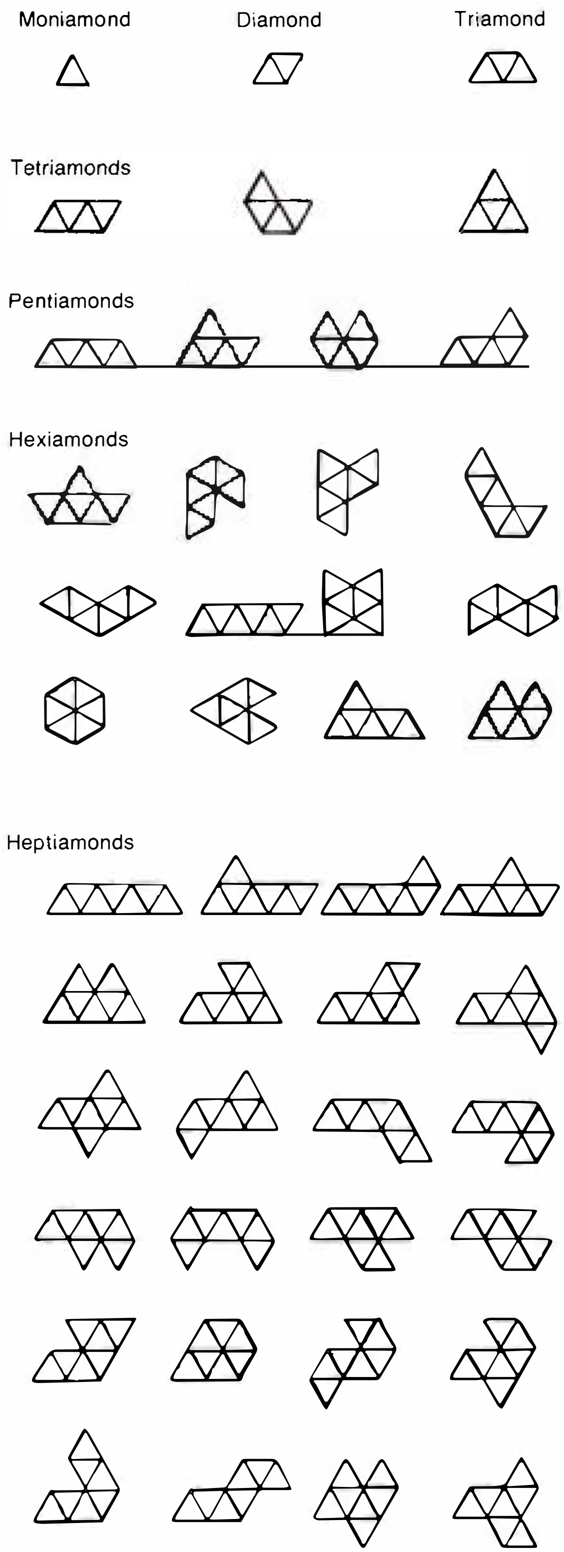


Figure 1b
Polyiamonds

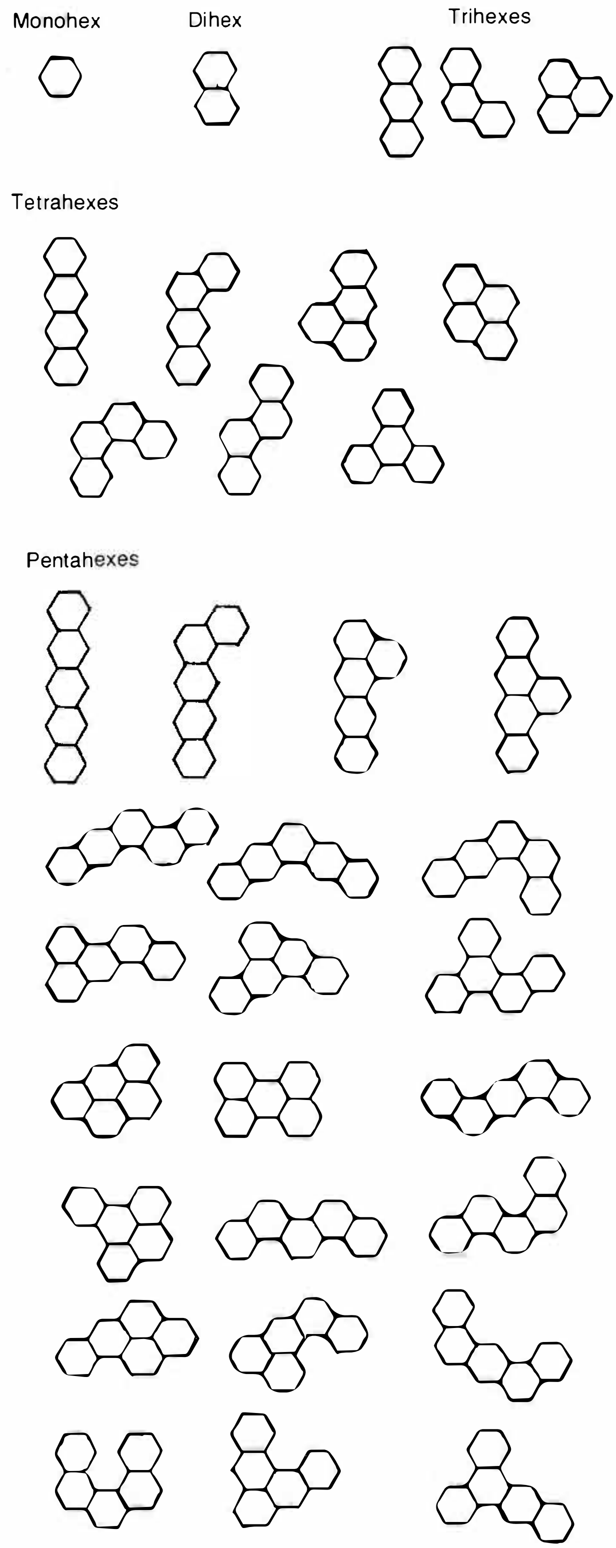


Figure 1c
Polyhexes

Figure 2 shows the most compact four-way symmetry with polyominoes of orders 1 through 4. We could ask that the monomino be in the centre or that a maximum number of pieces be totally enclosed.

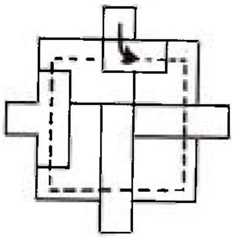


Figure 2

The line traces clockwise from the monomino around the outer row of the 5 x 5 square and passes through each polyomino in ascending order of sizes.

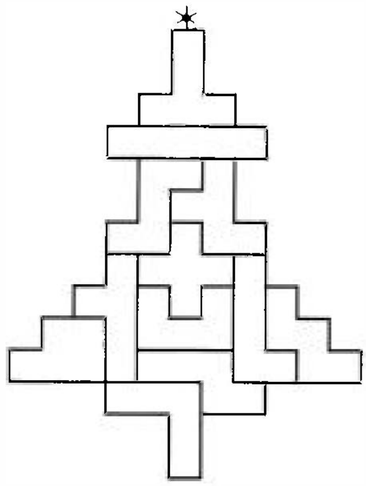


Figure 3

A symmetrical pentomino construction containing six symmetrical pairs. Can you spot them all?

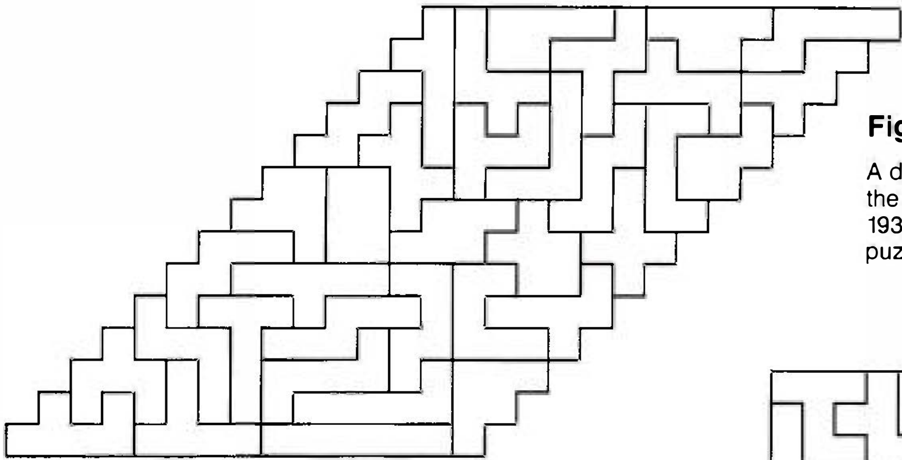


Figure 4

A design with rotational symmetry with the 35 hexominoes (first published in the 1930s in *Fairy Chess Review*, a British puzzle journal).

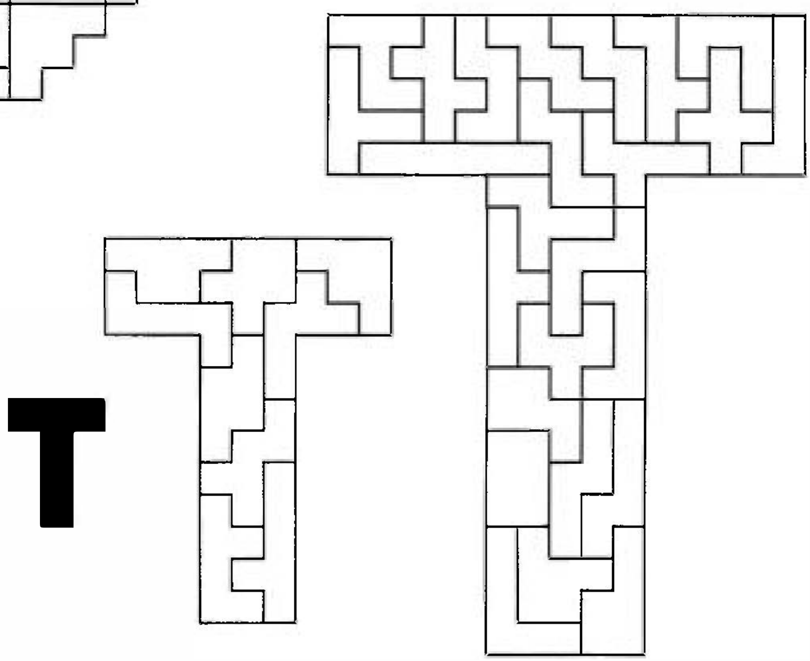


Figure 5

A replication solution of a hexomino, forming a triple and quintuple enlargement of the selected piece, thus involving all 35 pieces (*solution by the author*).

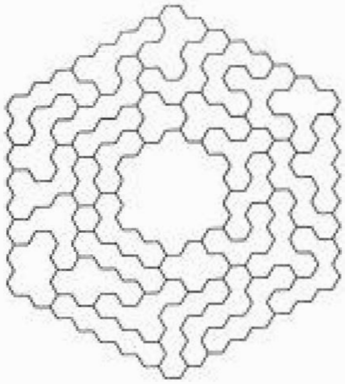


Figure 6a

A "Hex-Nut" arrangement of the polyhexes, sizes 1 through 5, in which the ascending sizes spiral from the inside to the outer ring, with the pentahexes forming a separate ring (*solution by Michael Keller*).

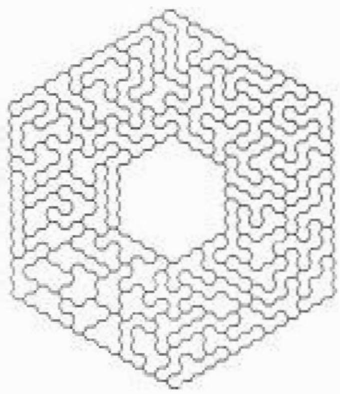


Figure 6b

A remarkable perfect ring formation of the 82 hexahexes (*solution by Michael Keller*).

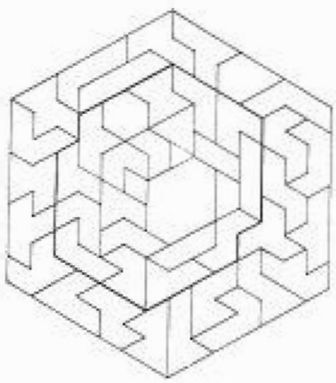


Figure 7

An "Iamond Ring" formation of the polyiamonds sizes 1 through 7, with the heptiamonds forming a symmetrical outer ring (*solution by the author*).

Combinatorial sets larger than those shown get somewhat unwieldy and overly complex for easy visualization and handling, although solutions have been found for rectangles with the 108 heptominoes (by David Klarner) and the 369 octominoes (by Michael Keller). Such enormous sets are no longer "user-friendly" for manipulation of the entire set except as *tours de force* by hardy investigators, such as the dazzling hexahex array in Figure 6b. Interesting explorations can, however, be done with selected subsets, perhaps in combination with smaller-order sets.

The restless search for greater challenges leads to problems with unique solutions. Computer programs and clever algorithms have made it possible to search for and verify unique solutions. Generally, the more pieces are used, the more solutions exist, so to tighten the problem the number of pieces is reduced or the shape of the region to be solved is constrained. The uniqueness of the letters in Figure 8 made with the 12 pentominoes was authenticated by Professor Yoshio Ohno of Tokyo. (The other 21 letters in this alphabet series have multiple solutions.)

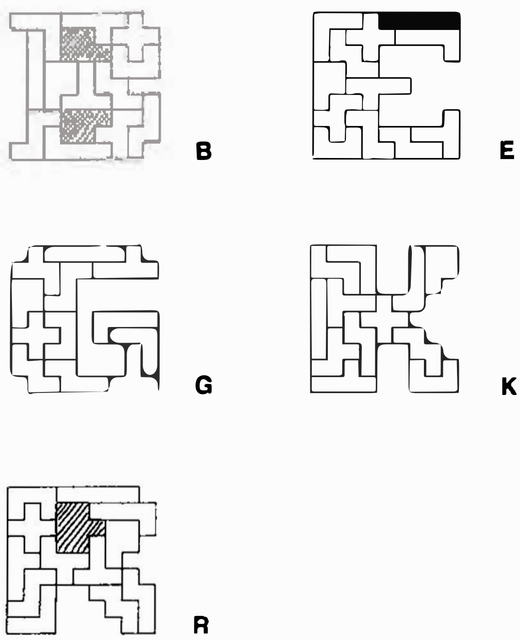


Figure 8

The B and E have unique solutions and the G, K and R have a minor variant (subsection flipped) in their otherwise unique arrangement.

Another approach to pursuing more difficult problems is to enter the third dimension. Solid pentominoes (made of cubes instead of squares) can form three different shapes of blocks: 3 x 4 x 5, 2 x 5 x 6 and 2 x 3 x 10. An astonishing discovery was that the latter two sizes can be accomplished by "folding" 6 x 10 rectangles, as shown in the following two figures.

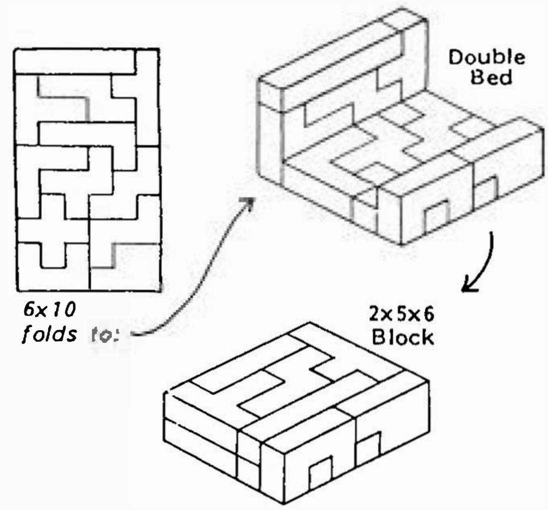


Figure 9
A foldable pentomino construction turns a 6 x 10 rectangle into a 2 x 5 x 6 block.

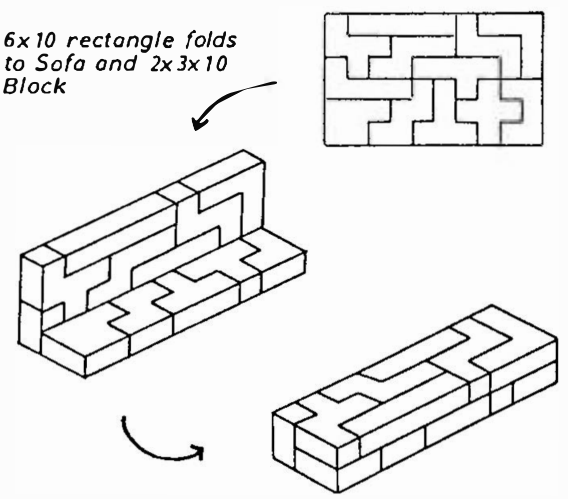


Figure 10
A unique solid pentomino conversion of a 6 x 10 rectangle into a 2 x 3 x 10 block.

In the three-dimensional realm, additional complexity can be introduced by joining five cubes in every 3-D combination, yielding 17 "pentacubes" (first constructed by David Klarner). These non-planar shapes, combined with the 12 solid pentominoes, provide inexhaustible possibilities for exploration and discovery.

Expanding the concept one more step, we find there are 166 hexacubes (the planar and non-planar joinings of six cubes). Supplemented by four single cubes, these 166 shapes will pack a 10 x 10 x 10 cube—one more exercise in heroic dimensions. Assaults on smaller subset problems, such as fitting 36 hexacubes into a 6 x 6 x 6 cube, are themselves thoroughly challenging.

It is actually in the smaller sets, which are easier to manage and for the mind to encompass, that we find the greatest popular appeal, accounting for the continuing charm of the now-classic Soma Cube, invented by Piet Hein many years ago. It consists of six tetracubes and one tricube. Its versatility and transformability have provided joy and delight to a generation of puzzlers.

The quest for novelty and new areas to conquer with combinatorial sets has inspired the creation of many variants on the theme of shape and color combinations. Ever since Major Percy MacMahon introduced his famous Three-Color Squares in the 1920s, the concept of color or contour adjacency has become an adjunct of groups defined by special characteristics. Figure 11 is one of the classic solutions with MacMahon Squares:

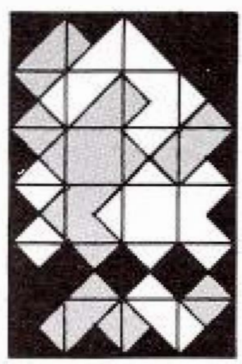


Figure 11
The 24 distinct three-color squares form a 4 x 6 rectangle where the border color is constant and every adjacent edge meets only a matching color.

MacMahon extended his research to four-color equilateral triangles (now known appropriately as MacMahon Triangles) and discovered that the 24 distinct pieces would indeed form a hexagon with constant border color *and* with matching color adjacency on touching edges. The solution he published is shown in Figure 12a and a variation by the author in Figure 12b.

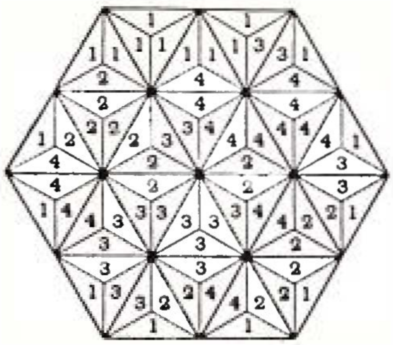


Figure 12a
The 24 distinct four-color triangles form an order-2 hexagon with constant border color and all adjacent edges touching only a matching color.

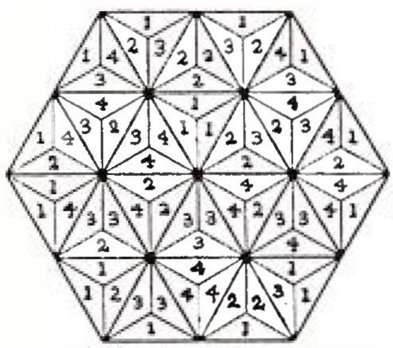


Figure 12b
This nearly symmetrical arrangement of the 24 MacMahon triangles has constant border color but *no* internal color adjacency.

Converting the edge coloring of MacMahon's Three-Color Squares to contours, where each square is a distinct combination of straight, convex and concave edges, we get 24 distinct shapes. Figure 13 is an array of them, supplemented by twelve duplicate pieces, forming a 6 x 6 square with a pronounced Escher flavor.

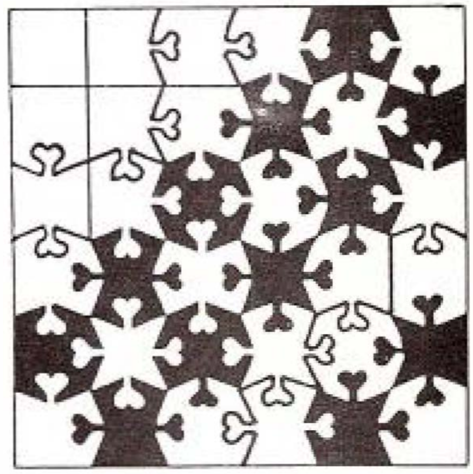


Figure 13
The topological equivalent of MacMahon squares (marketed under the product name, "Stockdale Super Square"); the dark pieces represent duplicates.

We can combine the concepts of combinatorial shapes and color permutations. One such set uses the shapes of right isosceles triangles joined (family name "polyaboloes" or, more commonly, "polytans"). Order-1 and order-2 polyaboloes are small triangle, double-size triangle, square, and left and right parallelogram. Permuted with two colors, the five shapes provide 15 distinct pieces that can tile an octagon and countless other symmetrical shapes with various color themes. Figure 14 is an array that has constant border color plus internal color symmetry.

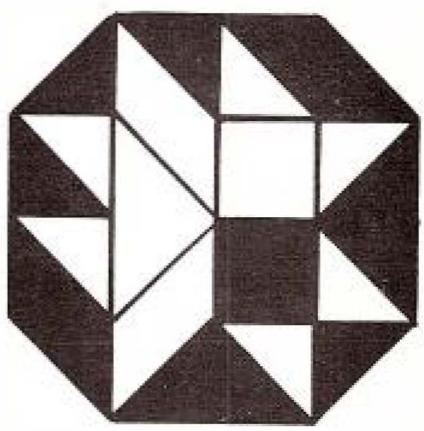


Figure 14
The fifteen order-1 and order-2 two-color polyaboloes with constant border color and overall color symmetry.

The next figure is a difficult arrangement of the same set as an octagon with all color-matched adjacent edges.

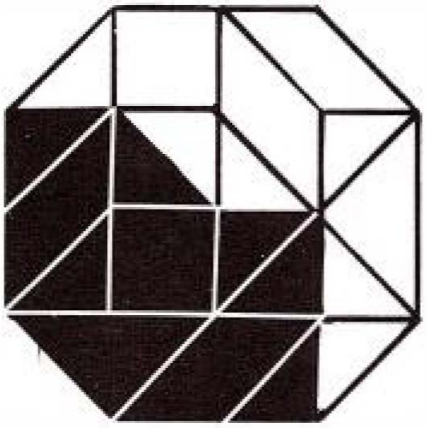


Figure 15
A color-matched solution of the 15 two-colored, order-1 and order-2 polyaboloes in their most compact form, an octagon.

Increasing the number of unit triangles per piece enlarges the polyabolo family to 4 triaboloes, 14 tetraboloes and 30 pentaboloes. These offer largely unexplored territory, but will allow themselves to be coaxed into two squares, supplemented by one unit triangle, as in Figures 16 and 17.

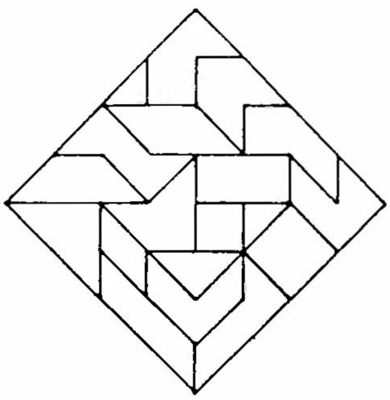


Figure 16
The polyaboloes, orders 1, 2 and 4, as a diagonal square (*solution by the author*).

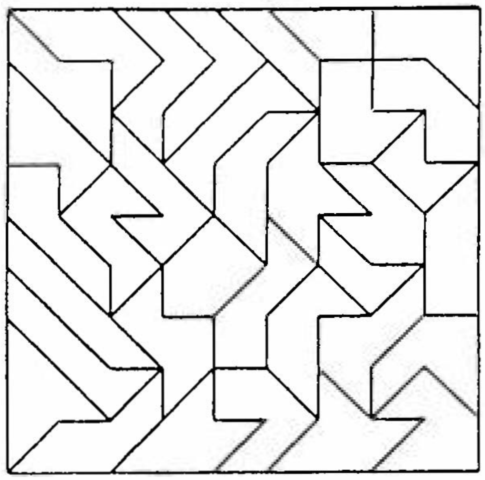


Figure 17
The polyaboloes, orders 3 and 5, as a 9 x 9 orthogonal square (*solution by Michael Keller*).

The 107 hexaboloes are still awaiting their conqueror.

It is easier to expand the multi-color order-2 set by adding more colors. There is a poetic progression of triangular numbers as the number of colors increases. Whereas the two-color set (shown in Figures 14 and 15) will form a 5 x 3 rectangle with color adjacency, a three-color set will tile a 5 x 6 rectangle; a four-color set will tile a 5 x 10 rectangle, and a five-color set will indeed fill a 5 x 15 rectangle—all with color adjacencies. Each new color added brings with it a single neutral triangle as a "filler" for the single triangle of each color. The neutral triangles act as wild cards that any color may touch. The 80 pieces of the five-color set skirt the limits of human endurance, yet the finished pattern is one of great abstract beauty, somewhat like a Vasarely painting. Figure 18 (overleaf) presents one solution found by the author. For best effects, it is recommended to color it in "by the numbers" with four colored pencils.

Moving further along the spectrum of shapes and colors, we enter the field of tilings and patterns, that is, completely covering a surface with multiple copies of the *same* shape or with combinations of two or more shapes. A wealth of research material is available, both historic and recent, and the subject of tessellations continues to fascinate with its kaleidoscopic variations and infinitely repeating patterns—inspirations for graphic artists and designers.

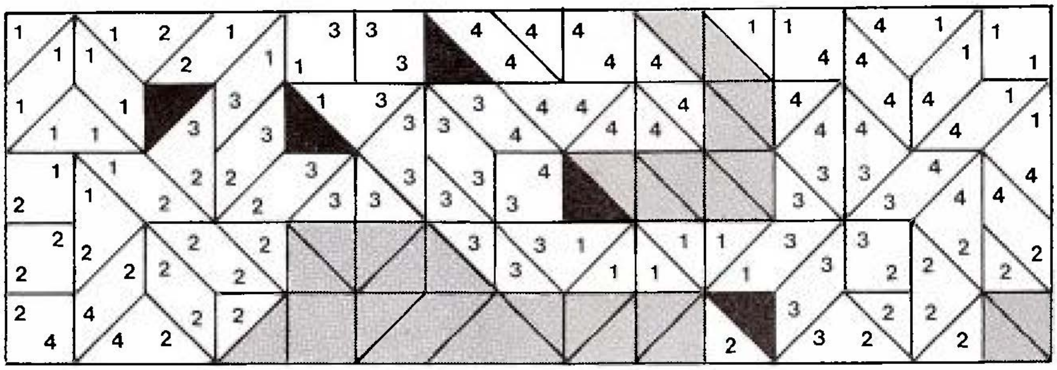


Figure 18

A five-color polyabolo set of orders 1 and 2, in a color-matched 5 x 15 rectangle.

A noteworthy tessellation problem concerns the pentominoes. Solomon Golomb has shown that any one of the 12 pentominoes can tile the plane with infinite copies of itself. A special case of this tiling ability is when multiples of a polyomino can form a square, and then multiples of this square can, in turn, form larger versions of the polyomino itself (Golomb calls them "rep-tiles"), and so on. It's a fine example of meta-thinking, like a box around a box around a box . . . . Combinatorics starts with a unit, a singularity; but there's no end in sight. A mini-version of rep-tiling can be seen in Figure 19.

A novel combination of tiling assembly and combinatorial shapes is the set shown in Figure 20. The shapes are formed by plotting circles on a square grid and using combinations of one, two and three circles and one, two and three "bridges", where the bridges are the connecting parts, somewhat like squares with curved-in sides, between the circles. The 7 x 7 grid is tiled with seven each of singles, doubles, triples and bridge pieces. The solution has the further interesting features that no two of the same shape touch each other and that there is a maximum amount of symmetry (solution found by Richard Grainger).

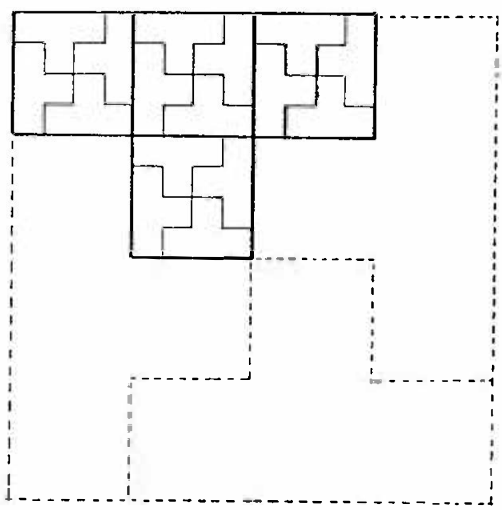


Figure 19

The T tetromino is replicated here in quadruple size. Four of *them* form a larger square, and four of those. . . .

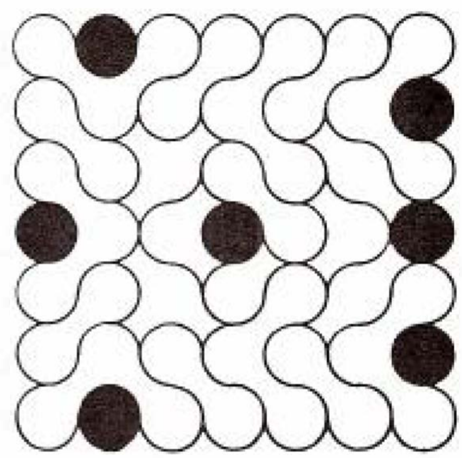


Figure 20

A lovely array of circle-based shapes on a square grid, with non-adjacency of similar shapes but with maximum symmetry.

The set when expanded to include combinations of four circles and four bridges will tile a 10 x 10 grid. The solution in Figure 21 has rotational symmetry as well as non-adjacency of similar shapes (solution by the author). The distribution of shapes makes for a dramatic visual impact.

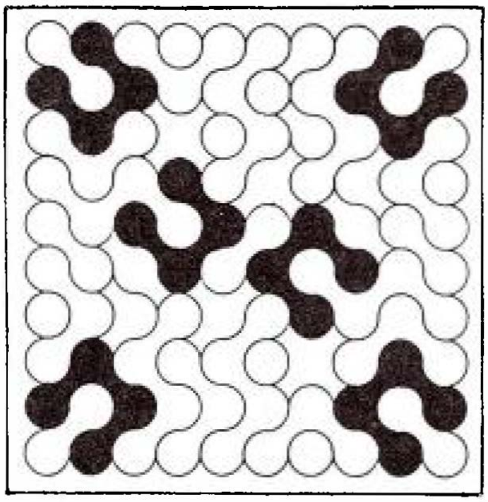


Figure 21

Multiples of 10 shapes of one, two, three and four rounds and connector bridges fill a 10 × 10 grid in a rare symmetry pattern.

Carrying the "round" idea into the third dimension leads us to the perennially popular ball pyramid puzzles. A close packing of spheres produces tetrahedra of various orders. The traditional simplest size is an order-4 pyramid with anywhere from four to seven pieces composed of various numbers of balls joined in a plane. If the balls are joined in a simple 60° or 120° relationship, the assembly is not usually difficult. When they are joined at 90° angles, the position of that piece within a triangular format becomes problematic and a real challenge for the puzzler. Len Gordon, today's foremost inventor of ball pyramid puzzles, has created three sizes of pyramids with computer-proven unique solutions. The component pieces range from two to four balls joined.

At the other end of the scale, Gordon has found thousands of solutions to an order-9 pyramid containing the 33 distinct planar shapes of five balls joined in every possible way, including those with 90° angles.

Figure 22 is a clever mini-pyramid using four copies of the same five-ball piece. It can be formed with two different arrangements, one of which allows the entire pyramid to be lifted by just its top. Can you visualize how the pieces must be joined?

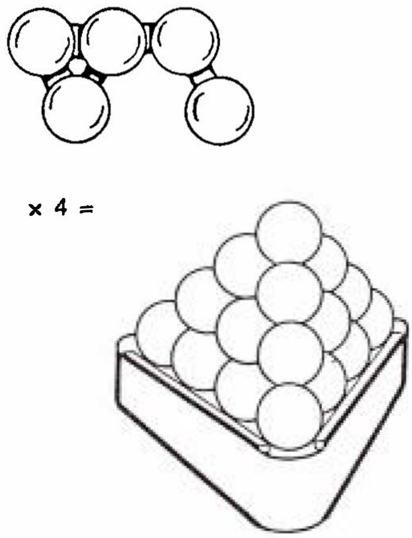


Figure 22

An order-4 tetrahedron formed with four of the pieces shown. They can interlock to hold together even when lifted by the top.

Some wonderful innovations in combinatorial sets involve hexagon-shaped tiles with special color patterns. One such set, sold under the tradename Kaliko and now sadly out of print, contains 85 distinct tiles with three colors permuted over the five distinct path patterns that join pairs of sides. Invented by Charles Titus and Craig Schenstedt, this magnificent set lends itself to the most wondrous and convoluted symmetric loop and color patterns. One is shown in Figure 23 (overleaf).

Another way to fit three colors on a hexagon was invented by Charles Butler. Here the tiles are divided into compartments of single and triple diamonds in each of three colors (the triples look like chevrons). The patterns give the impression of a perspective view of a 2 x 2 x 2 cube. Permuting the relative positions of the three colors produces 12 distinct patterns, which in turn combine into amazing symmetrical figures with color adjacency on all touching sides. Figure 24 is a triangle solution found by the author.

Figure 23
A Kaliko pattern with tri-color symmetry.

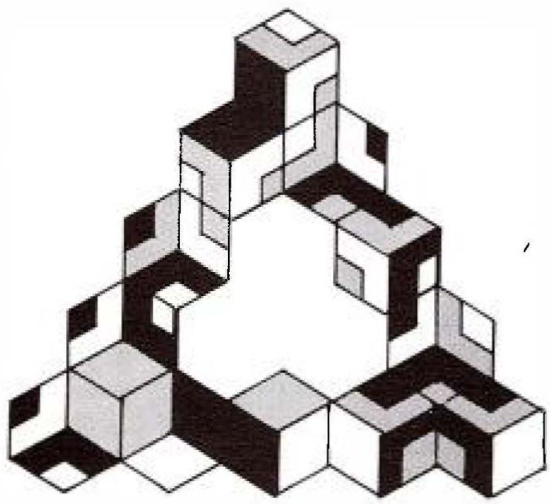


Figure 24
A color adjacency solution for the 12 tri-color Hexmozaix tiles.

John Horton Conway, England's great eccentric mathematician, discovered that regular pentagons, when edge-colored with five colors in every possible arrangement, produce 12 different tiles that will exactly cover a dodecahedron with color adjacency on every edge. We leave it to the nimble-fingered reader to construct and assemble this figure.

As human ingenuity knows no bounds, combinatorial sets can be derived from combinations of dissimilar unit shapes. A most exciting new multiform set was created by Dr. Andy Liu of the University of Alberta. Mapped onto the classic tiling pattern shown in Figure 25a, the members of the set are those that have not more than 4 adjoining cells and not more than 17 as the sum of their collective number of sides. The set so defined has 28 one-sided pieces and will exactly tile a triangular region. An elegant solution is shown in Figure 25b.

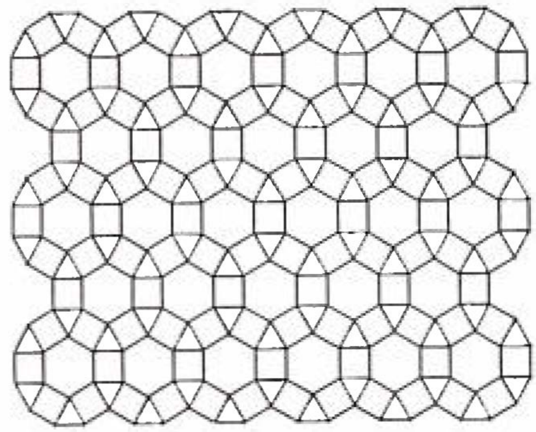


Figure 25a
A tessellation with the regular hexagons, squares and triangles.

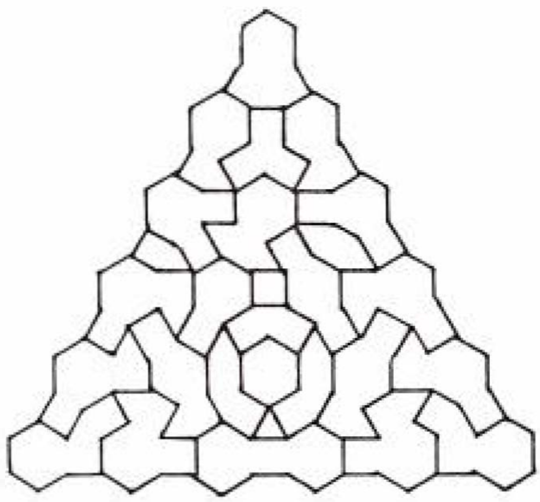


Figure 25b
The triangle formed with the 28 distinct tri-form shapes (*solution by Andy Liu*).

It is hoped that readers' scenic tour through some of the wonders of combinatorial diversity and harmony may have stimulated some appreciation for the aesthetic aspects of mathematical sets and kindled an interest in further study. Playing with combinatorial sets can develop an eye for spatial relationships, a facility for problem-solving and a deep pleasure in the process of research and discovery. The pride and satisfaction gained from successes in these pursuits, in turn, will serve as stimulus for future effort. The philosophical values gained from contemplating Nature's boundless variety of workable combinations are beyond measure.

An inherent logic and order in the unfolding of the genetic code, the processes at work in the interiors of stars, the passion of musical composition, the dynamics of economic striving—all are combinatorics. Only the level of complexity varies. Mathematics can model and make sense of a seemingly chaotic world—and keep things interesting with ever new challenges. For what is creativity but the combinatorics of the mind?

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