It All Depends What You Mean By . . .

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Defer reading for a few minutes and have a go at this sequence of problems. We'll discuss these problems later.

Problem 1
Find a prime number which is one less than a perfect square.

Problem 2
Find another prime number which is one less than a perfect square. How many other such primes are there?

Problem 3
Find a prime number which is one more than a perfect square.

Problem 4
Find another prime number which is one more than a perfect square. How many others are there?

Problem 5
Find a prime number which is one less than a perfect cube.

Problem 6
Find another prime number which is one less than a perfect cube. How many others are there?

Problem 7
Find a prime number which is one more than a perfect cube.

Problem 8
Find another prime number which is one more than a perfect cube. How many others are there?

Introduction

Question 1
What exactly do we mean by "mathematical talent"?

Question 2
How can we recognize it?

Question 3
What can we do to encourage its development?

If mathematics teaching were a science, it might be reasonable to try to answer these three questions in the given order. As things are, mathematics teaching is not (yet)[1] a science: it remains a craft. So one should not be surprised at the suggestion that the questions may be best tackled in the reverse order: first look for rich, challenging material that encourages mathematical thinking; while using such material, observe the different approaches used by individual students and try to assess their requirements and talents in the light of their performance; finally, make use of this experience to refine one's ideas of what does, and what does not, constitute "mathematical talent." Those who work with young children often have their own tried and trusted ways of nurturing whatever mathematical talent is present in their classes, but are far less sure how one can reliably assess the degree of talent present in any given individual and are usually most reluctant to define exactly what they mean by "mathematical talent."

I have seen this kind of pragmatism work extremely well in individual classrooms and

[1]This fact is reflected in the subtitle of Hans Frendenthal's thought-provoking book, *Weeding and sowing: preface to a science of mathematical education* (Reidel 1978).

schools. However, my thumbnail sketch has ignored the most basic question of all.

Question 3′
How do we distinguish between rich, challenging material that encourages mathematical thinking and material that is unsuitable?

Once this crucial question has been asked, it is clear that our response to the first three questions is bound to depend on our ideas of what mathematics itself is really about and of the kind of students we are inclined to call "talented."

Question 4 (a)
Which students are we mainly concerned about? That very select group, *la crême de la crême?* Or the much larger group of all those who belong to the "cream" and who crop up regularly in most high schools?

Question 4 (b)
What do we understand by "mathematics"? How do we decide whether an activity at a given level is or is not genuinely mathematical? How should the fact that one is working with youngsters affect the style and content of the mathematics?

One way forward?

The aspect of mathematics which appeals most strongly—perhaps at all ages—is the way in which elementary calculations and constructions can be used to resolve non-trivial problems.[2] All attempts to encourage students interested in mathematics must therefore exercise and extend students' ability to perform the relevant calculations or constructions. A good basic training in routine techniques is thus fundamental. Sadly, many of the talented students in our classes have only been expected to perform these routine techniques in the simplest imaginable contexts.

One cannot assume that "talented" students will somehow make up for our own limited expectations by making their own spontaneous generalizations. (For example, the resolution of the problem sequence above is entirely elementary, but seems to be totally unexpected—even for good college students majoring in mathematics. No one seems to have alerted them to even the most obvious connections, such as that between factorizing numbers and factorizing polynomials.) This immediately suggests one very simple way in which ordinary class teachers can make a significant contribution to the development of mathematical talent. We shall come back to this.

Our habit of teaching routine techniques in a very restricted way is one reason why problem competitions and enrichment materials which are officially aimed at mathematically talented students frequently turn out to be most unsuitable for our own talented students.[3] But the main reason for this mismatch seems to be that those

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[2]In his fascinating autobiography, *Disturbing the universe* (Harper and Row 1979) Freeman Dyson tells how, as a boy, he worked through the seven hundred or so problems in Piaggio's *Differential Equations* (G Bell 1920). Piaggio's book is quite different from modern texts. The author presents an absolute minimum of theory. Instead of waiting until a technique or method can be fully justified, he explains what he can and encourages the reader to "have a go"—the details of the complete picture becoming more clearly visible as one proceeds. Of course, one misses many important points first time through. But the book was important for Dyson not just as a way of mastering differential equations but also because it enabled him, through the joy of calculation, to fall in love with mathematics. Harold M. Edwards makes a related point in the Preface to his beautiful history of *Fermat's Last Theorem* (Springer 1977): "As even a superficial glance at history shows, Kummer and the other great innovators in number theory did vast amounts of computation and gained much of their insight in this way. I deplore the fact that contemporary mathematical education tends to give students the idea that computation is demeaning drudgery to be avoided at all costs."

[3]I am aware of two common strategies for stretching the most talented young mathematicians. Neither of these strategies seems to work very well with ordinary talented students. One approach involves presenting simplified treatments of selected topics from higher mathematics. But though the formal technical prerequisites may be kept well under control, such material often makes quite unreasonable assumptions about what is, and what is not, familiar, meaningful or interesting to bright high school students. Many such presentations have the additional weakness that the ratio of text to exercises is all wrong, as though talented students had less need of exercises! As a result these valiant efforts to make higher mathematics accessible are often best appreciated by adult mathophiles. The other approach involves competitions based on problems that are easy to state and whose content is elementary, but which are hard to solve (or hard to solve in the time allowed). I love these questions, but use them relatively rarely. They are a bit like those texts which claim to have no formal prerequisites other than a little "mathematical maturity." The trouble in both cases is that students with the necessary "maturity" are singularly hard to find.

who set the competition problems and write the booklets are chiefly interested in the "truly exceptional student." But this is, almost by definition, the kind of student most of us never see!

The answer of most teachers to Question 4(a) is likely to be determined as much by this simple fact of life as by any rigidly held educational principle: it is natural to be most concerned for the kind of talented students we come across regularly in our own institutions. Their talent may make little impression in competitions designed to identify *la crême de la crême*, but that does not lessen our responsibility for fostering the talent we know they have. If their talent is modest when compared with the very best, their numbers are so much greater that they present us with no less of a challenge. Moreover, given the right kind of material, such students are capable of some remarkable mathematics.

Most of the talented students I work with enjoy mathematics, but are not (perhaps never will be) ready to do battle with hard competition questions.[4] Instead they need lots of experience of tackling intuitively appealing problems which allow them to get started, which nevertheless remain strangely opaque, but slowly become first meaningful, then promising and finally transparent, as a result of intelligent groping.

The problem sequence at the beginning (taken from my book, *Mathematical Puzzling*, Oxford University Press 1987) illustrates one way of achieving this. Any student interested in mathematics immediately tries to answer the questions (especially if they are posed orally to a group of students). It is only when they fail to answer some of the harmless-looking questions (or when they discover that their own over-hasty answers are rejected by their peers), that they begin to realize there is something to explain— even if they are not at all sure what. By this point they are sufficiently committed not to back off in the face of a problem they would otherwise instinctively classify as being "too hard."

Mathematics is a much messier business than most textbooks are willing to admit. Basic techniques are important; so a part of mathematics instruction must certainly consist of applying standard procedures to solve familiar-looking problems or writing out solutions in a specified deductive form (as in geometry). But where did the solution come from in the first place? And how did one decide which standard algorithm to use?

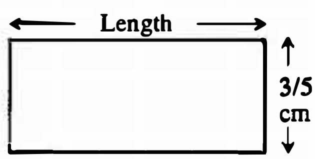
Much of the time we have very little idea how students find their solutions. As long as they continue to succeed one may argue that it does not matter. But once we find them beginning to struggle, it becomes all too tempting to cheat by making the route from the problem to its solution so short and direct that very little mathematical thinking is required. Students certainly need to master basic techniques, but mathematical thinking only really begins when the student has to select and coordinate a number of such basic steps to solve challenging multi-step problems.

A regular diet of even the simplest multi-step problems can have a dramatic effect on student perceptions and performance. As an indication of what happens when we fail to provide such a diet as part of our ordinary teaching, consider the following problems set to a large sample of 15-year-olds in the United Kingdom.

Question A

Area = 1/3 square centimetre

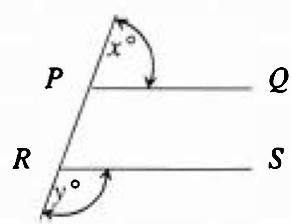
Length = ?



Question B

PQ is parallel to *RS*
$y = 2x$

What is the size of angle *PRS* in degrees?



Though each of these problems requires the student to identify an intermediate step, one

[4]Why not? The satisfaction which motivates those who do respond to hard competition problems stems from the prospect of occasional hard-won success. That, in its turn, presupposes extensive failure. Once the perceived prospect of success sinks below some personal threshold, the game loses its appeal. The talented student who has rarely been challenged by hard problems and who does not realize how important "failure" is in mathematics, only has to see a hard problem for the perceived prospect of success to sink way below his (inflated) threshold.

could scarcely call them hard! Yet success rates are abysmally low.[5] Such levels of incompetence are certainly not preordained: they are the result of years of systematic training in anti-mathematical thinking. Our persistent failure to set appropriately challenging problems ensures that many highly talented students simply lose interest in mathematics, while others perform so far below their potential level that their talent becomes almost invisible.

In my experience very few talented students—even those who have been well-taught—respond well to the kind of material that is often advocated for the most able.[6] These students need problems, or sequences of problems, which have a strong intuitive appeal, which make minimal technical demands while stretching students' own powers of calculation, and which above all force them to "think mathematically." (This kind of thinking is subtly different from the process of "seeing through" simple puzzles and generally requires extended periods of engagement.) My two books, *Mathematical Puzzling* and *Discovering Mathematics* (Oxford University Press 1987) represent two rather different responses to this challenge.

Discovering Mathematics is the more ambitious of the two in that it tries to convey to the talented high school student how one goes about exploring a substantial mathematical problem on one's own. It does this, not by talking about mathematics, but by involving the reader in extensive calculations, in making (often false!) conjectures and in checking and revising those initial guesses until the reader arrives at something requiring proof. Many teachers have enjoyed working through this material and have found the experience not only refreshing but also helpful in clarifying their own ideas in relation to Question 4(b).

Mathematical Puzzling has similar aims but a very different format. In spite of its emphasis on problems and on calculation, *Discovering Mathematics* presupposes a willingness to read "text." *Mathematical Puzzling* avoids "text" and consists largely of problems. The messages it seeks to convey (about the nature of mathematics, about the importance of looking for "connections," about being willing to experiment and explore, about the need to use one's judgment in making sense of a question, etc.) are therefore implicit in the choice and the wording of the problems and in the way they are grouped together. The material has been developed with various groups of 10- to 14-year-olds over the last ten years—though much of it has been used to good effect with much older students.

The opening example

At first sight, the opening example may look like a sequence of routine problems designed to test students' familiarity with prime numbers and powers. Though such an impression is superficial, it certainly reflects two very important general features of the problem sequence, namely,

—that each problem is accessible to and should evoke an immediate response from any student interested in mathematics, and

—that the content is entirely elementary and the initial demands on the student are restricted to calculating (preferably mentally) efficiently and accurately.

On a deeper level the problems emphasize the following points

- the importance of being willing to search systematically and intelligently for numbers with the required properties;
  (for example, Problem 2, is 8 prime? 15? 24? ...)
- the need for mental fluency;
  (what is 13^2? what is 6^3? how does one test quickly whether an unfamiliar number, like 143, is prime?)
- the inadequacy of merely guessing;
  (is 143 really prime? how about 217? or 511?)
- the virtue of reflecting on the results of one's own calculations.
  (why do powers of odd numbers obviously never work?)

However, there is far more to the problems than this, as the sting in the tail of the even-

[5]Only one student in 20 managed to answer the first question correctly and one in 10 managed the second. Yet the studies from which these examples are taken steadfastly refuse to draw the obvious conclusion that these statistics say far more about the way they have been taught than about children's inherent ability.

[6]See note 3.

numbered parts soon shows. Problems 2, 6 and 8 are meant to make students suspicious. How they respond will depend very much on their previous experience. Many behave uncritically and pick the first vaguely prime-looking number as "the answer" (for example, with Problem 2, many able students choose 143—or even 63); they then simply ignore the crucial question "How many are there?" Others are much more careful, but (perhaps because they have never been challenged to look for something which may not exist) remain totally unsuspecting—even after two or three similar searches (such as Problems 2, 6 and 8) draw a complete blank.[7]

Many teachers object to the wording of Problems 2, 6 and 8. If only one such question were asked, I would probably agree that it would be thoroughly unfair (unless, of course, one knew that the students were used to keeping their wits about them). I see little educational value in trick questions designed to catch people out.

But in this setting, the doomed searches generated by Problems 2, 6 and 8 and the group discussion to which they should give rise represent one of the many ways in which one can help able students

- to see that there is more to mathematics than simply getting the right answer.

In this case it is precisely the three "rogue" Problems 2, 6 and 8 and the unexplained contrast between these and Problem 4 that force students

- to begin to think about the mathematics behind the problem sequence as a whole and to look for a genuine explanation which distinguishes between the two kinds of observed behavior.

The habit of "wanting to explain," rather than being content just to "get the right answer," is far from natural. It has to be educated. Without it students never develop that independence and autonomy which allows them to take control of their own activity: they remain dependent on others to provide them with problems to solve and to validate or correct, the answers they come up with. It is for this reason, rather than because of some belief in the deductive character of "real" mathematics, that one of our prime objectives should be to cultivate the habit of wanting to explain in our talented students. I would therefore restrict the use of Problems 2, 6 and 8 to students who are familiar with the basic factorization[8]: $(*)\ x^2 - 1 = (x - 1)(x + 1)$.

This is not to say that one expects such students to spontaneously translate the verbiage of the first two problems into symbolic form. They won't! At least, not until their failure to answer Problems 2, 6 and 8 has left them with a puzzle which commonsense methods have failed to resolve. Once the relevance of the familiar identity $(*)$ is noticed, it is but a short step to suspect, and then to discover, the less familiar algebraic factorizations for $x^3 \pm 1$.

Students who get this far can then be challenged to formulate, and to try to resolve, the two general questions of which Problems 1, 2, 5, and 6 and 3, 4, 7 and 8 are special cases.

- For which values of m, n is $m^n - 1$ prime?
- For which values of m, n is $m^n + 1$ prime?

These questions are on a much higher level than the original problems; but the earlier, more simple-minded problems do seem to help students to respond appropriately. The trivial case "$n = 1$" has to be noticed and excluded (usually later rather than sooner). The questions could then be tackled by generalizing the elementary algebraic factorizations alluded to above. However, even very able students are likely to spend quite a long time experimenting with special cases before they realize this.

Like most mathematical problems, the questions as stated are ambiguous. The colloquial formulation leads one to think in terms of "sufficient" conditions on m and n which will guarantee the primeness of $m^n \pm 1$, whereas mathematically one can only hope to obtain "necessary" conditions (which may or may not turn out to be "sufficient"). It may take some time for this distinction to emerge, but it leads naturally to a discussion of the fundamental method of analysis, in which one supposes that

[7]Hundreds of keen 17-year-old students specializing in mathematics who were given ten days to tackle Problems 1, 2, 5, 6 enlisted the help of their home computers. Faced with a negative output, they merely reported, "The prime numbers must be very large." Barely a dozen of these able students smelt a rat.

[8]Some very able students are perfectly capable of seeing that $3^2 - 1 = (3 - 1)(3 + 1)$, $4^2 - 1 = (4 - 1)(4 + 1)$, etc. is part of a general pattern, whether or not they have any formal acquaintance with algebra. But most talented students need some fluency in algebra if they are to discover the factorizations for $x^3 \pm 1$, and for $m^n \pm 1$, for themselves.

one has an entity of the required kind—for example, a prime number of the form "$m^n - 1$"—and proceeds to analyze the possibilities for the numbers m and n.

For example, $m^n - 1 = (m - 1)(m^{n-1} + m^{n-2} + \ldots + m + 1)$, so if $m^n - 1$ is prime then we must obviously have $m - 1 = 1$. Moreover, if $n = a \times b$ is composite, then $2^{a\times b} - 1 = (2^a)^b - 1 = (2^a - 1)(2^{a(b-1)} + 2^{a(b-2)} + \ldots + 2^a + 1)$. Hence if $2^n - 1$ is prime, n must itself be prime.

The miracle of the method of analysis is that, if pushed far enough, the resulting "necessary" conditions often turn out to be "sufficient" as well (as in the classical analysis of primitive pythagorean triples: positive integers x, y, z with no common factors satisfying $x^2 + y^2 = z^2$). In our case, elementary algebra has led to the necessary condition that "if $m^n - 1$ is prime ($n \geq 2$), then $m = 2$ and n is prime." One naturally hopes that the condition "n is prime" may turn out to be sufficient to guarantee the primeness of $2^n - 1$. Well, does it?

A similar analysis of the second general question (When is $m^n + 1$ prime?) leads first to the observation that either $m = 1$ or m is even (Why?), and then to the observation that n must be a power of 2. (Suppose $n = a \times b$ with $b \geq 3$ odd, then $m^n + 1 = (m^a)^b + 1 = \ldots$.) Restricting to the simplest case where $m = 2$, we therefore know that, if $2^n + 1$ is to be prime, then n must be a power of 2. But is the condition $n = 2^k$ sufficient to guarantee the primeness of $2^n + 1$?

There is no need to stop there. A discussion of Mersenne and Fermat primes can lead on to more efficient ways of testing for primeness, based on *Fermat's Little Theorem* ($a^p \equiv a \pmod{p}$) or Lucas' test. All of this involves masses of calculation, but it is calculation that achieves results the students would have previously assumed to be beyond them. The limited vision of most talented students means that the initial phase of any activity needs to appear relatively straightforward; but if one is to broaden that vision, then one must somehow lead them on from these simple beginnings to higher things.