There Are No Holes Inside a Diamond

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A diamond is a girl's best friend! —Marilyn Monroe quote

There are twelve different ways of putting together five unit squares edge-to-edge. The results are the Pentominoes. (Pentomino is a registered trademark of Solomon W. Golomb who introduced them (Golomb 1954).) They are shown in Figure 1-1 with their single-letter names.

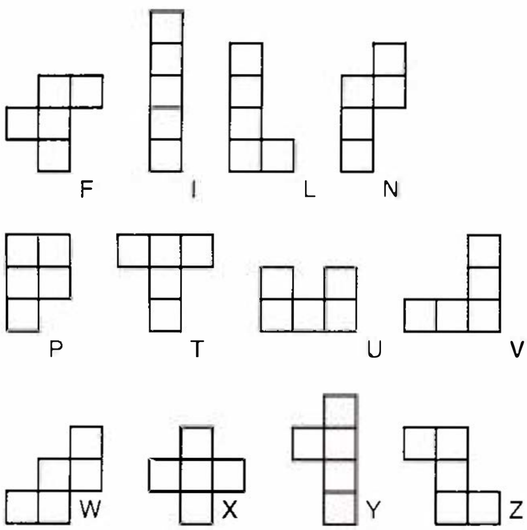

Figure 1-1

Note: Pentomino is a registered trademark of Solomon W. Golomb

The pentominoes can be used to construct many interesting shapes. On the other hand, there are shapes which are impossible to construct. The reason is sometimes obvious but often non-trivial. While some impossibility proofs are elegant, others are unavoidably complex.

One such shape is the diamond with a hole at its centre, as shown in Figure 1-2. Proposed by R. M. Robinson, it was proved by S. Earnshaw that its construction is impossible. The full proof was unpublished, but a four-page summary of the eight-step solution was presented by Golomb (1965, 69-73), who challenged his readers to find a simpler proof.

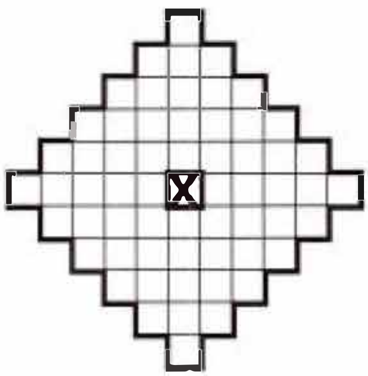

Figure 1-2

This paper presents a simpler argument which proves more—that the construction is impossible unless the hole is on the edge. In other words, there are no holes inside a diamond!

The approach is indirect. It will assume that a construction is possible with the hole inside and derive a contradiction.

Let us introduce some terminology. A pentomino is said to be *interior* if it is placed so that it does not cover any edge squares. A square is said to be *interior* if it is either the hole or covered by an interior pentomino. Note that the centre is always interior.

Acknowledgment—The author thanks the referee for valuable improvement to the presentation of this paper.

A pentomino is said to be *spectral* if it is placed so that it connects an edge square to one of the four squares adjacent to the centre. Only I, L, N, V, W and Z can be spectral. Finally, two spectral pentominoes are said to be *neighboring* if they cover at least one common point.

All 20 edge squares are to be covered. Each of F, W and X can cover three of them, each of N, P and Y two, and each of I, L, T, U, V and Z one. The total count is 21, so that there is no immediate contradiction, but there is some useful information.

☐ Observation 1.1. *There is at most one interior pentomino. There are at most six interior squares.*

☐ Observation 1.2. *If there is an interior pentomino, it is one of I, L, T, U, V, Z.*

We shall divide the proof of our main result into three parts, the first two devoted to proving the following auxiliary results.

☐ Theorem A. *There is exactly one interior pentomino.*

☐ Theorem B. *The interior pentomino is one of I, V, Z.*

Proof of Theorem A

Suppose there are no interior pentominoes. Then the hole must be at the centre and there will be four spectral pentominoes. Figure 2-1 illustrates a placing of L, N, V and W as the four spectral pentominoes.

Note that the remaining part of the diamond is partitioned into four regions. In order for them to be filled with the remaining pentominoes, the number of squares in each region must be divisible by five. While this is true for the regions between L and W, and between N and V, it is not the case for the regions between L and N, and between V and W.

Routine search reveals that only three pairs, L and W, N and V, and I and Z, can define regions with a number of squares divisible by five. It follows that no matter which four pentominoes are spectral, at most two of the regions can be filled with pentominoes.

Therefore, we must have at least one interior pentomino. By observation 1.1, there is at most one interior pentomino. Hence, there is exactly one interior pentomino and the proof of Theorem A is completed.

A most important result follows immediately from Theorem A.

☐ Corollary 2.1. *Each of F, W and X must cover three edge squares while each of N, P and Y must cover two edge squares.*

The technique used in proving Theorem A also yields an additional result.

☐ Lemma 2.2. *V and I cannot be neighboring spectral pentominoes, nor can V and Z.*

Proof: Routine search reveals that, no matter how V and I are placed as neighboring spectral pentominoes, the number of squares in the region between them is always two or three more than a multiple of five. Even if the hole is in this region, the rest of it still cannot be filled with pentominoes. The same holds for V and Z. Figure 2-2 shows two placings of V as spectral neighbors of I and Z, respectively.

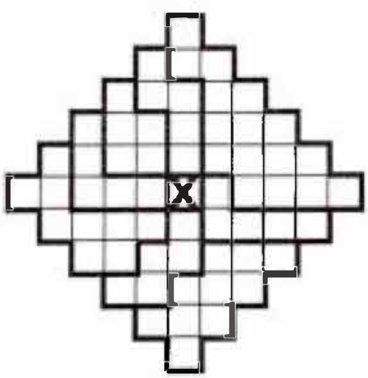

Figure 2-1

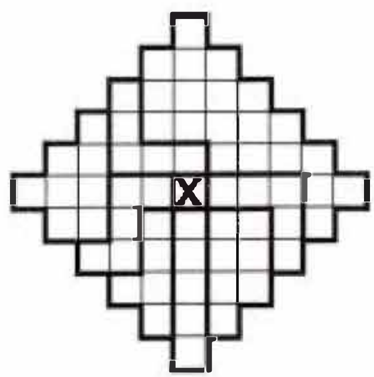

Figure 2-2

Proof of Theorem B

We shall prove that each of I, V and Z is either interior or spectral. It then follows that one of them must be interior. Otherwise, all three will be spectral and V must be a neighbor of I or Z. This is impossible by Lemma 2.2.

Let us now consider each of I, V and Z in turn. Z is the easiest to handle because it does not have any non-interior, non-spectral placings.

☐ Observation 3.1. *Z is either interior or spectral.*

☐ Lemma 3.2. *I is either interior or spectral.*

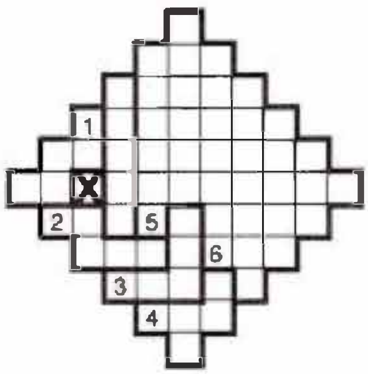

Figure 3-1

Proof: Figure 3-1 shows the only non-interior, non-spectral placing of I. Note that square 1 must be covered by W. We cannot cover square 2 with F by Corollary 2.1, so that N must be used. This isolates the square marked with a cross and it must be the hole. Square 3 must now be covered by V and square 4 by F. It is easy to see that square 5 is interior. The same holds for square 6 by Corollary 2.1. By Theorem A, exactly one pentomino is interior, but no pentomino can cover both squares 5 and 6. We have a contradiction to Observation 1.1.

It is much more difficult to prove that V is either interior or spectral. To do so, we have to find out more about how other pentominoes must be placed relative to one another.

In Figure 3-2, the shaded squares are called the "back" of W and the "back" of F, respectively.

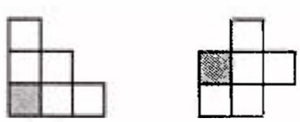

Figure 3-2

☐ Lemma 3.3. *The "back" of W is either interior or covered by L.*

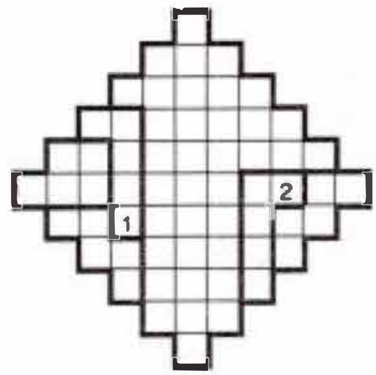

Figure 3-3

Proof: By Corollary 2.1, there are two placings of W, both shown in Figure 3-3. It is easy to see that, if square 1 is not interior, it must be covered by L. There are three possible ways, one of which is shown. On the other hand, Corollary 2.1 shows that, if square 2 is not interior, it must be covered by L as shown.

☐ Lemma 3.4. *The "back" of F must be covered by N.*

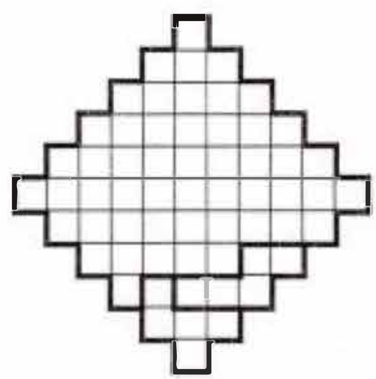

Figure 3-4

Proof: By Corollary 2.1, there is only one placing of F. If its "back" is to be covered by N, it can only be done in one way, as shown in Figure 3-4. On the other hand, if N is placed as shown, the placing of F is forced. To prove our result, we rule out other placings of N, of which there are two.

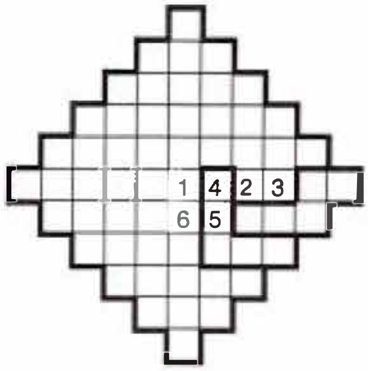

Figure 3-5

Suppose N occupies a corner as shown in Figure 3-5. It is easy to see that squares 1, 2 and 3 are interior. If square 4 is not, it must be covered by V. Note that V cannot fail to cover square 5, as otherwise it forces a placing of P contrary to Corollary 2.1. Now square 6 will also be interior. Moreover, no pentomino can cover three of squares 1, 2, 3 and 6. Hence square 4 is interior. It follows that square 5 is also.

Only Y can cover all five interior squares, 1, 2, 3, 4 and 5. However, this is forbidden by Observation 1.2. Hence the interior pentomino covers four of these five squares and only L, I and T can do so.

If L is interior, then square 5 must be the hole. However, by Lemma 3.3, the "back" of W will be a seventh interior square, contradicting Observation 1.1. If I or T is interior, then squares 5 or 1, respectively, will be the hole. Now square 6 cannot be interior and it is easy to see that it can only be covered by L. Once again, the "back" of W will be an impossible seventh interior square.

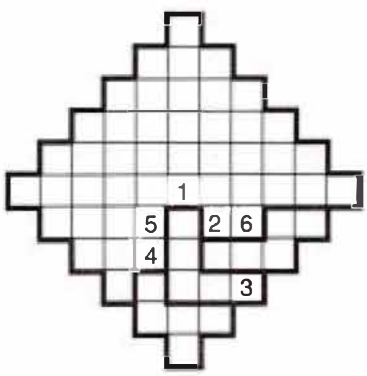

Figure 3-6

Figure 3-6 shows the other placing of N. It is easy to see that squares 1 and 2 are interior. Suppose square 3 is covered by W. By rearranging N and W, we can make them occupy the same region but with N occupying a corner. We have already shown that this is impossible.

Hence square 3 must be covered by V. Now the placing of F is forced and squares 4 and 5 become interior. The hole must either be square 2 or 4 as no pentomino can cover both. The interior pentomino must cover square 5 and cannot cover square 6. Now square 6 must be covered by L, and Lemma 3.3 furnishes a contradiction.

□ Lemma 3.5. *V is either interior or spectral.*

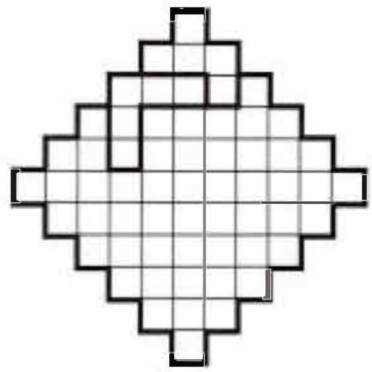

Figure 3-7

Proof: V has two non-interior, non-spectral placings. The one shown in Figure 3-7 can be ruled out as it forces the placing of F contrary to Lemma 3.4.

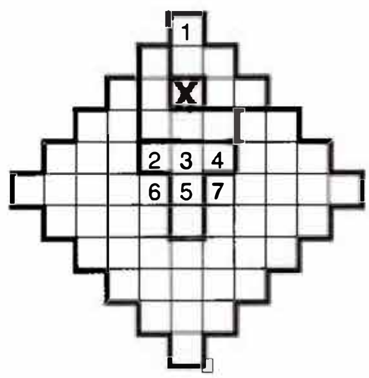

Figure 3-8

Figure 3-8 (previous page) shows the other possibility. Suppose square 1 is covered by W. Then the square marked by a cross must be the hole. Now squares 2, 3, 4 and 5 must all belong to the interior pentomino, and only T can cover all of them, P and Y being ruled out by Observation 1.2. By Observation 3.1 and Lemma 3.2, Z and I must cover squares 6 and 7 collectively. However, neither can cover square 6. We have a contradiction.

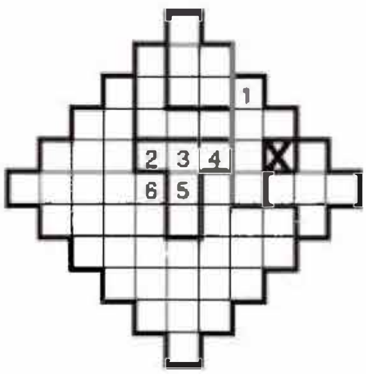

Figure 3-9

The only other pentomino that can be used in place of W is P. By Corollary 2.1, X must occupy a corner. Suppose it takes up its position as shown in Figure 3-9. Then square 1 must be covered by Y, creating a hole at the square marked with a cross. Now squares 2, 3, 4 and 5 are all interior and only T can cover all of them. Thus square 6 is not interior, and it can only be covered by L. By Lemma 3.3, the "back" of W will create an impossible seventh interior square.

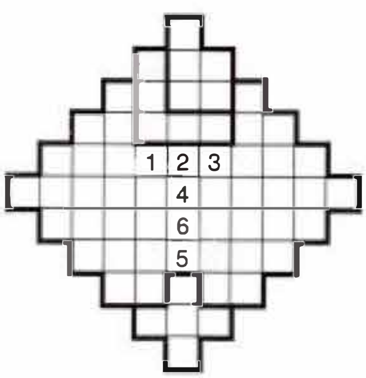

Figure 3-10

Now let X occupy the corner as shown in Figure 3-10. As before, squares 1, 2, 3 and 4 are all interior. Suppose square 5 is also interior. Then the interior pentomino will cover four of these five squares. Only T and L can do that and whichever one is interior will also cover square 6. By Lemma 3.2, I is spectral, but it is easily seen that it now has no spectral placings.

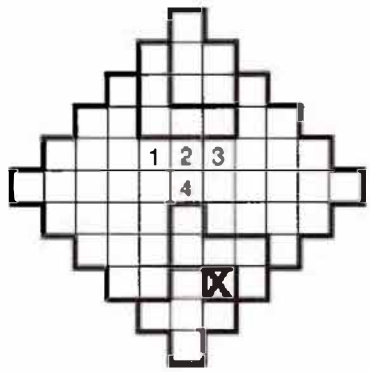

Figure 3-11

It follows that square 5 is not interior and it must be covered by either T or Z. Figure 3-11 shows Z in place, creating a hole at the square marked with a cross. Now no pentomino can cover all of squares 1, 2, 3 and 4 except Y, which cannot be interior. The same contradiction is arrived at if T covers square 5 instead. This completes the proof of Lemma 3.5 and hence of Theorem B.

Conclusion

By Theorem A, there is exactly one interior pentomino. By Theorem B, it is I, V or Z. Hence T is not interior. It has two non-interior placings. The first one, shown in Figure 4-1, can be ruled out since it forces a placing of F contrary to Lemma 3.4.

The second one is shown in Figure 4-2. Now square 1 can only be covered by W. We cannot use V as it again forces a placing of F contrary to Lemma 3.4. By Corollary 2.1, squares 2, 3, 4, 5 and 6 are all interior and the interior pentomino must cover at least four of them. This can only be I, and the hole is at square 4. By Lemma 3.5, V must be spectral, but it is easy to verify that it now has no spectral placings. This completes the proof of our main result.

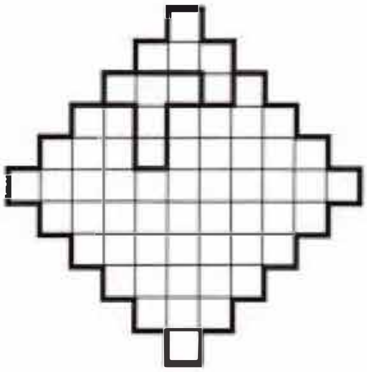

Figure 4-1

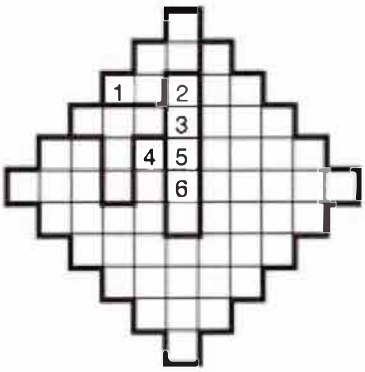

Figure 4-2

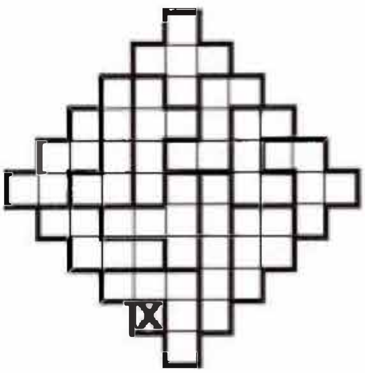

Figure 4-3

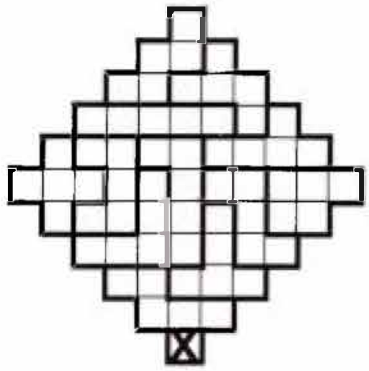

Figure 4-4

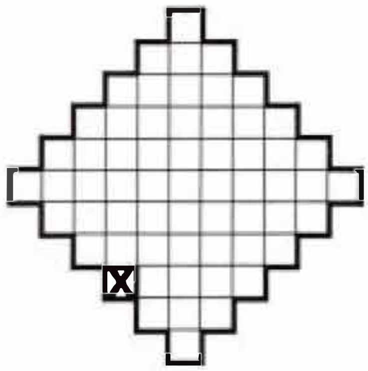

Figure 4-5

Now that we know there are no holes inside the diamond, the natural question is whether the hole can be on the edge. This is indeed possible, as shown in Figure 4-3, attributed to J.A. Lindon (Golomb 1965, 73). Figure 4-4 shows that the hole can be placed more aesthetically at a corner. It is left to the reader to decide whether Figure 4-5 can be constructed with a set of pentominoes and whether eleven pentominoes can fit into the diamond covering all 20 edge squares.

A rough diamond . . . A proverb

References

Golomb, Solomon W. "Checkerboard and Polyominoes," *American Mathematics Monthly* 61(1954):675-682.

Golomb, Solomon W. *Polyominoes,* New York: Charles Scribner's Sons, 1965.

Addendum to There Are No Holes Inside a Diamond

Diamonds are forever.—James Bond movie title

The diamond considered in the main part of this paper is but one member of an infinite family of diamonds. The first five are shown in Figure A-1, and our diamond is D_5, the next in line. If we denote by d_n the number of squares in D_n, it is an easy exercise to show that $d_n = 2n^2 + 2n + 1$.

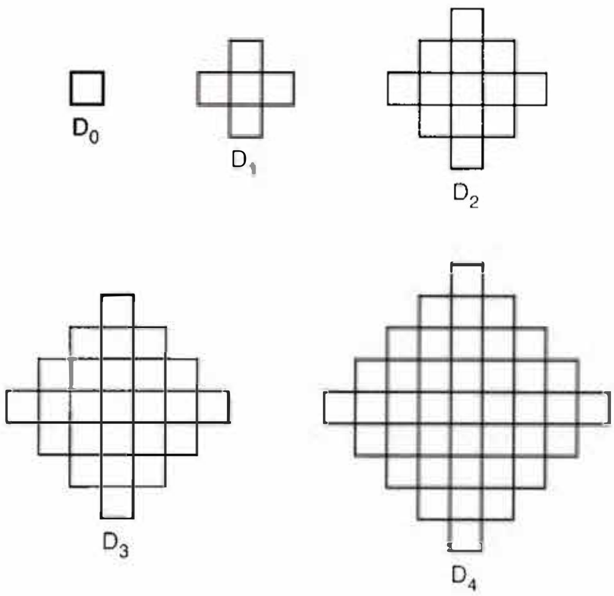

Figure A-1

If $n \equiv 1$ or $3 \pmod 5$, then $d_n \equiv 0 \pmod 5$ and D_n can be constructed with pentominoes. D_1 is trivial since it is just X. Figure A-2 shows a construction of D_3. We pose the following problems.

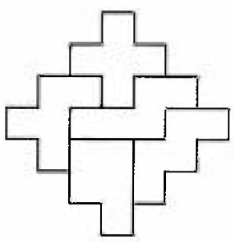

Figure A-2

Problem 1
Construct D_3 using F, W, X, N and Y.

Problem 2
Construct D_6 using a complete set of pentominoes plus F, W, X, P and Y.

Problem 3
Construct D_8 using two complete sets of pentominoes plus F, W, X, N and P.

Note that the best pentominoes for diamonds, F, W and X, are used in each problem. Each problem also uses two of the next best pentominoes, N, P and Y, and a different pair in each case.

If $n \equiv 2 \pmod 5$, then $d_n \equiv 3 \pmod 5$ and D_n cannot be constructed with pentominoes unless we leave three holes. This makes the problem too loose, but the reader may wish to explore for interesting designs.

If $n \equiv 0$ or $4 \pmod 5$, then $d_n \equiv 1 \pmod 5$ and D_n can be constructed with pentominoes if we leave only one hole. D_0 is trivial since no pentominoes are required. Figure A-3 shows a construction of D_4 with the hole inside, and the reader may try to decide if the hole can be at the centre. D_5 having already been dealt with, we pose one final problem.

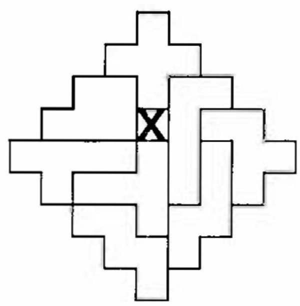

Figure A-3

Problem 4
Construct D_9 using three complete sets of pentominoes, leaving a hole at the centre.